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Neumann Problem for a Smooth Bounded Domain in the Heisenberg Group $\mathbb{H}_n$

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Abstract Boundary value problems arise while studying differential equations and play a fundamental role in diverse areas of scientific, medical, and engineering disciplines, such as medical sciences involving diffusion processes of drugs, neuroscience, environmental studies, modelling in economics and finance, and simulations for computer graphics. Consequently, their study becomes essential in real-world applications. Two boundary value problems, namely the Dirichlet and Neumann problems associated with the Laplace equation, are of substantial significance in the discipline of partial differential equations. The Dirichlet problem involves finding a harmonic function within a domain, subject to the condition that its values coincide with a given continuous function on the boundary. On the other hand, the Neumann problem demands a solution in the form of a harmonic function whose normal derivative equals a specified function on the boundary of the domain. These problems acquire increased significance when the regularity of the associated differential operator is degraded. The Heisenberg group, a non-abelian and a non-compact Lie group, becomes a nice object to study these boundary value problems as being the simplest example having said properties in association with a subelliptic Laplace like operator called the Kohn-Laplacian. Gaveau was the first to discuss the Dirichlet problem for the Kohn–Laplacian on the Heisenberg groups in 1977. Later, Jerison further discussed it by calculating estimates in the Dirichlet problem in a smooth domain D, along with the regularity of the solution. The Neumann problem for the Kohn-Laplacian on the Koranyi ball in the Heisenberg group was initially addressed by Kumar, Dubey and Mishra in 2016, which was further generalized to H-type groups by Pandey and Mishra for certain gauge balls in H-type groups. We further generalize the existence and uniqueness results of the Neumann problem for the Kohn-Laplacian for bounded domains with smooth boundary that have no characteristic points in the Heisenberg group. We have established certain estimates of the derivatives of the fundamental solution and obtained the necessary and sufficient condition for the solvability of the interior Neumann problem for the same.

Keywords Heisenberg Group, Neumann Problem, Kohn-Laplacian

1 Introduction

Poisson equation along with Neumann boundary condition is highly significant pertaining to the field of partial differential equations. Let Δ denote the Laplacian operator on the Euclidean space $\mathbb{R}^n$ and consider the following problem

$$\begin{cases} \Delta v = 0 \text{ in } D, \\ \frac{\partial v}{\partial n} = h \text{ on } \partial D, \end{cases} \tag{1.1}$$

where D represents a smooth domain in $\mathbb{R}^n$, h is a specified function on ∂D and n is the outward normal at ∂D, the boundary of domain D (see, for instance [1]). Then the Neumann problem for D in $\mathbb{R}^n$ is to determine a $v \in C^2(D) \cap C^1(\overline{D})$ satisfying (1.1).

In the context of a unit ball in $\mathbb{R}^n$, the essential condition for the solvability of the Neumann problem is that the integral of the values allotted to the normal derivative, reduces to zero over the boundary surface ∂D i.e.

$$\int_{\partial D} h ds = 0.$$

In Euclidean space, the existence, uniqueness, and boundary regularity of solution of Dirichlet and Neumann boundary value problems for the Laplacian operator are well established. Now it becomes a natural objective to generalize these outcomes in a non-trivial geometric setting which motivates to work on the Heisenberg group as a prototype of sub-Riemannian geometry, where the Laplacian operator has been displaced by the Kohn–Laplacian operator. The Laplace operator is an elliptic operator that remains invariant under translations and rotations and demonstrates homogeneity of degree two. A similar operator can be defined on non-Euclidean spaces such as Lie groups. In reference [2], the wellposedness of the Neumann boundary value problem for the subelliptic sublaplacian operator $\mathcal{L}_0$ has been analyzed in the case of a gauge ball on the Heisenberg group, while the same problem in the framework of an unbounded domain, particularly for a half-space ($\Omega = \{(\xi, t) \in \mathbb{H}_n : t > 0\}$) has been discussed in [3]. In the latter case, there is a supplementary condition on boundary data which ensures the convergence of integrals near infinity and the existence of the necessary and also sufficient conditions for the solvability of the Neumann problem which is as follows: $h \in C(\partial\Omega)$ is such that $h(\alpha) = O(\frac{1}{\zeta^k})$ as $\alpha(= (\xi, t))$ approaches infinity, for some $k \geq 2$. Then the necessary and also sufficient condition for the solvability of the interior Neumann problem is that the integral of the values assigned to the normal derivative turns to zero over $\partial\Omega$, the boundary surface i.e.

$$\int_{\partial\Omega} h d\sigma.$$

Moreover, in [4], Neumann type boundary value problem has been formulated for the canonical sub-Laplacian operator on a gauge ball in groups of type H, and existence alongwith the uniqueness of its solution has been vastly discussed. Ruzhansky in [5], found an integral boundary condition of the Newton potential(volume potential) v on the boundary $b\Omega$ of a domain Ω which is bounded and characterized by a smooth boundary, such that with this boundary condition, the boundary value problem for the Kohn-Laplacian operator on $b\Omega$ has a unique solution that is the Newton potential itself. We discuss the solvability of the Neumann problem for Kohn-Laplacian on arbitrary bounded domains with smooth boundaries without characteristic points in the Heisenberg group. This paper fills the gap in consolidating the geometric elaborateness of the Heisenberg group with the structural challenge of Neumann boundary conditions and the technical difficulty of arbitrary smooth bounded domains. The techniques and ideas of this paper can be developed and generalized further for the study of Neumann boundary value problem and other similar problems for domains with characteristic points, in more general Carnot groups, or in domains having more complex geometrical structure.

2 Neumann Problem on the Heisenberg Group

The Heisenberg groups $\mathbb{H}_n$ are Lie groups of odd dimensions with the underlying real manifold $\mathbb{C}^n \times \mathbb{R}(n \geq 1)$, and the group law

$$[w, t]o[w', t'] = [w + w', t + t' + 2\Im(w\bar{w'})],$$

where $w\bar{w'} = \sum_{k=1}^{n} w_k \bar{w'_k}$ and $w = (w_1, w_2, ..., w_n), \quad w_k = x_k + iy_k$.

We write $x + iy$ for w so that

$$\begin{aligned} X_k &= \frac{\partial}{\partial x_k} + 2y_k \frac{\partial}{\partial t}, \\ Y_k &= \frac{\partial}{\partial y_k} - 2x_k \frac{\partial}{\partial t}, \\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

form a basis for left invariant vector fields on $\mathbb{H}_n$. The following complex vector fields are often more handy with the calculations

$$\begin{aligned} Z_k &= \frac{1}{2}(X_k - iY_k) = \frac{\partial}{\partial w_k} + i\bar{w}_k \frac{\partial}{\partial t}, \quad k = 1, 2, ...n, \\ \bar{Z}_k &= \frac{1}{2}(X_k + iY_k) = \frac{\partial}{\partial \bar{w}_k} - iw_k \frac{\partial}{\partial t}, \quad k = 1, 2, ...n. \end{aligned}$$

The Kohn-Laplacian operator is explicitly denoted by

$$\begin{aligned} \Delta_{\mathbb{H}_n} &= \sum_{k=1}^{n} (X_k^2 + Y_k^2), \qquad (2.1) \\ &= 2\sum_{k=1}^{n} (\bar{Z}_k Z_k + Z_k \bar{Z}_k). \end{aligned}$$

For more details about the Heisenberg group and Kohn-Laplacian, one may refer to [6] and [7].

A positive constant c can be found such that the function

$$g(\alpha) := cN(\alpha)^{-2n},$$

where $N(\zeta, t) = (|\zeta|^4 + t^2)^{\frac{1}{4}}$ is a homogeneous norm on $\mathbb{H}_n$, satisfies

$$\Delta_{\mathbb{H}_n}(g(\alpha)) = -\delta.$$

The existence of such a constant c is guaranteed by [8].

Following the notations of [9, 10], we define $C(\beta, \alpha) = |\zeta|^2 + |\zeta'|^2 + i(t' - t)$ and $Q(\beta, \alpha) = 2\zeta\bar{\zeta}'$, where $\alpha = (\zeta, t)$ and $\beta = (\zeta', t')$, so that the fundamental solution with pole at β can be expressed as $g_\beta(\alpha) = c|C(\beta, \alpha) - Q(\beta, \alpha)|^{-n}$.

Consider $W = \text{Span}\{X_j, Y_j : 1 \leq j \leq n\}$. The vectors that belong to W are termed as horizontal. If we define an inner product on the space of horizontal vectors by declaring W as the orthonormal spanning set, we get a sub-Riemannian metric on $\mathbb{H}_n$. More about the origin and relevance of this metric can be found in [11]. In what follows, an $\langle ., . \rangle_{hor}$ will denote this inner product on W.

By Riesz representation theorem, for a smooth function ρ on $\mathbb{H}_n$, there exists a unique horizontal vector $y(\rho)$ such that

$\langle y(\rho), v\rangle_{hor} = v(\rho)$ for all horizontal vectors v. We call this $y(\rho)$ the horizontal gradient of ρ and denote it by $\nabla_{hor}\rho$. Equivalently,

$$\nabla_{hor}\rho = \sum_{j=1}^{n}\{a_jX_j + b_jY_j\},$$

where $a_j = X_j\rho$ and $b_j = Y_j\rho$

$$= 2\sum_{j=1}^{n}\left\{(\bar{Z_j}\rho)Z_j + (Z_j\rho)\bar{Z_j}\right\}.$$

A horizontal normal unit vector pointing outwards for a bounded domain D with smooth boundary $F = 0$ on $\mathbb{H}_n$ is defined as

$$\partial^\dagger = \frac{\nabla_{hor}F}{\|\nabla_{hor}F\|_{hor}}.$$

A point where $\nabla_{hor}F$ vanishes is called a characteristic point. For $\partial^\dagger$ to be well-defined we assume that D is a domain without characteristic points.

We consider the Neumann problem as formulated in [2] but for an arbitrary bounded domain with smooth boundary given by $F = 0$ that has no characteristic points.

This interior Neumann problem on $\mathbb{H}_n$ is given as

$$\begin{cases}\Delta_{\mathbb{H}_n}u = 0 \text{ in } D,\\ \partial^\dagger u = q \text{ on } .\end{cases} \tag{2.2}$$

where $u \in C^2(D)\cap C^1(\overline{D})$ and $q \in C(\partial D)$.

If D is a bounded domain having a smooth boundary, then $\mathbb{H}_n\backslash D$ is an unbounded domain with the boundary same as ∂D but oriented in the reverse. A boundary value problem for $\mathbb{H}_n\backslash D$ is termed as an exterior boundary value problem. In particular, Dirichlet and Neumann boundary value problems for $\mathbb{H}_n\backslash D$ are called exterior Dirichlet and exterior Neumann boundary value problems for D, respectively.

3 Solvability of the Neumann Problem

We seek a solution of the interior Neumann problem (2.2) in the form of the integral given below

$$u(\beta) = \int_{\partial D}\theta(\alpha)g_\beta(\alpha)d\sigma_\alpha \ , \ \beta \in D, \tag{3.1}$$

with continuous function θ where θ will be a solution of the following integral equation

$$\theta(\beta) + \int_{\partial D}\theta(\alpha)\partial^\dagger g_\beta(\alpha) = q(\beta) \ , \ \beta \in \partial D. \tag{3.2}$$

For that purpose, we define integral operators $\mathcal{T}, \mathcal{T}' : C(\partial D) \to C(\partial D)$ as

$$(\mathcal{T}\varphi)(\beta) := \int_{\partial D}\varphi(\alpha)\left\{\partial^\dagger g_\beta(\alpha)\right\}_\alpha d\sigma_\xi, \ \ \beta \in \partial D, \tag{3.3}$$

$$(\mathcal{T}'\theta)(\beta) := \int_{\partial D}\theta(\alpha)\left\{\partial^\dagger g_\beta(\alpha)\right\}_\beta d\sigma_\alpha, \ \ \beta \in \partial D. \tag{3.4}$$

Now it is observed that with respect to the dual system $< C(\partial\Omega), C(\partial\Omega) >$ defined by

$$\langle\varphi, \theta\rangle := \int_{\partial\Omega}\varphi\theta d\sigma \quad, \varphi, \theta \in C(\partial\Omega),$$

the operators $\mathcal{T}$, $\mathcal{T}'$ are adjoint.

Fredholm's alternative says that if T is a compact operator on a Hilbert space, then (cf. [12])

$$\text{Range}(T) = (\text{Null}(T^*))^\perp.$$

From (3.3), it is evident that the existence of θ in (3.2) is guaranteed if and only if q lies in the range space of $I + \mathcal{T}$. If $\mathcal{T}$ is compact, so is $I + \mathcal{T}$. Fredholm alternative then indicates that a function q belongs to the range of $I + \mathcal{T}$ if and only if q is orthogonal to the null space of $(I + \mathcal{T})^*$, which is $I + \mathcal{T}'$ in this case.

If $\varphi(\neq 0)$ belongs to the null space of $I + \mathcal{T}'$, a standard technique (see [2], [4]) may be applied to show that φ must be a constant function. Hence, the necessary and sufficient condition for q to lie in the range space of $I + \mathcal{T}$ is that for every non-zero constant k,

$$\int_{\partial D}k.qd\sigma = 0.$$

i.e. if and only if $\int_{\partial D}qd\sigma = 0$. If $I + \mathcal{T}$ and $I + \mathcal{T}'$ have nullity 1, then the above statement is equivalent to saying that $\int_{\partial D}qd\sigma = 0$ (c.f. [2]). We first prove that $\mathcal{T}$ is compact. Towards this end, we first establish certain estimates of derivatives of $g_\beta(\alpha)$.

Lemma 3.1. *Let K denote a compact neighbourhood of $\beta \in \partial D$. Then*

$$\left|\partial^\dagger g_\beta(\alpha)\right| \le c_K(N(\alpha^{-1}\beta))^{-2n-4} \ \ on \ \ K,$$

where the constant c_K depends on K.

Proof. In $\mathbb{H}_n$, if $\alpha = (\mathbf{x}, \mathbf{y}, t)$ and $\beta = (\mathbf{x}', \mathbf{y}', t')$, we have

$$\begin{aligned} g_\beta(\alpha) &= c\left|(|\mathbf{x}|^2 + |\mathbf{y}|^2) + (|\mathbf{x}|^2 + |\mathbf{y}|^2) + i(t - t')\right. \\ &\quad \left. -2\mathbf{x}.\mathbf{x}' - 2\mathbf{y}.\mathbf{y}' - 2i(\mathbf{y}.\mathbf{x}' - \mathbf{x}.\mathbf{y}')\right|^{-n}, \end{aligned}$$

and

$$\begin{aligned} \partial^\dagger g_\beta(\alpha) &= \frac{\nabla_{hor}F}{\|\nabla_{hor}F\|_{hor}}g_\beta(\alpha), \\ &= -cnN(\alpha^{-1}\beta)^{-2n-4} \times \\ &\quad \frac{\sum_{j=1}^{n}\left\{(X_jF)A + (Y_jF)B\right\}}{\|C\|_{hor}}, \end{aligned}$$

where A is given by,

$$\begin{aligned} \textstyle\sum_{j=1}^{n} &\left[2(x_j - x_j')\left\{(x_j - x_j')^2 + (y_j - y_j')^2\right\}\right. \\ &+2y_j'\left\{t - t' - 2(y_jx_j' - x_jy_j')\right\} \\ &\left.+y_j\{(t - t') - 2(y_jx_j' - x_jy_j')\}\right], \end{aligned}$$

and B is given by,

$$\begin{aligned}\textstyle\sum_{j=1}^{n} \big[2(y_j - y_j')\left\{(x_j - x_j')^2 + (y_j - y_j')^2\right\} \\ -2x_j'\left\{t - t' - 2(y_j x_j' - x_j y_j')\right\} \\ -x_j\{(t-t') - 2(y_j x_j' - x_j y_j')\}\big],\end{aligned}$$

$$C = \sum_{j=1}^{n}\left(\left\{\frac{\partial F}{\partial x_j} + 2y_j\frac{\partial F}{\partial t}\right\}^2 + \left\{\frac{\partial F}{\partial y_j} - 2x_j\frac{\partial F}{\partial t}\right\}^2\right).$$

As A and B are continuous functions on the bounded set K, these are bounded functions. Moreover, their coefficients are smaller than 1. Therefore, we have the following:

$$\left|\partial^\dagger g_\beta(\alpha)\right| \le cnN(\alpha^{-1}\beta)^{-2n-4}(|A| + |B|), \quad \beta \ne \alpha,$$

$$\left|\partial^\dagger g_\beta(\alpha)\right| \le c_K N(\alpha^{-1}\beta)^{-2n-4}, \quad \beta \ne \alpha, \tag{3.5}$$

where the constant c_K depends on the neighbourhood K. □

Lemma 3.2. *For each boundary point β and for $\epsilon > 0$, suppose $\Omega_\beta(\epsilon) = \{\alpha \in \partial D : N(\beta^{-1}\alpha) \le \epsilon\}$. Then the integral*

$$\int_{\Omega_\beta(\epsilon)} \varphi(\alpha) g_\beta(\alpha) d\sigma_\alpha,$$

exists for sufficiently small ϵ.

Proof. We have $g_\beta(\alpha) = cN(\alpha^{-1}\beta)^{-2n}$. As $\Omega_\beta(\epsilon) \subseteq \partial D$ and $\varphi \in L^\infty(\partial D)$, we have

$$\left|\int_{\Omega_\beta(\epsilon)} \varphi(\alpha) g_\beta(\alpha) d\sigma_\alpha\right| \le c \sup_{\alpha \in \Omega_\beta(\epsilon)} |\varphi(\alpha)| \int_{\Omega_\beta(\epsilon)} N(\alpha^{-1}\beta)^{-2n} d\sigma_\alpha.$$

Let $T_\beta(\partial D)$ denote the tangent space to the boundary surface ∂D at the point β. We define $F : \Omega_\beta(\epsilon) \to T_\beta(\partial D)(\alpha \mapsto \alpha - \langle\alpha, \mathbf{n}\rangle\mathbf{n})$, where $\alpha \in \Omega_\beta(\epsilon)$ and n denotes the inward unit normal to the surface ∂D at the point β. Range of this map F lies inside $D_\beta := F(\Omega_\beta(\epsilon)) \cap T_\beta$ and for sufficiently small ϵ, this map F defines a bijective mapping on its range.

By suitable change of coordinates, integral over $\Omega_\beta(\epsilon)$ can be transformed into integral over D_β. Again suitable translation can be used to shift β to origin so that T_β passes through origin. Furthermore, this translated plane intersects the hyperplane $t = 0$ in a line l (say) passing through origin and suitable rotation about this line l coincides the plane with $t = 0$ and the image of $\Omega_\beta(\epsilon)$ after these transformations is a domain of the form $|z| < \epsilon_1$ in the plane $t = 0$. Using polar coordinates of $\mathbb{H}_3$, the existence of the integral can be easily confirmed. □

Theorem 3.3. *For a bounded domain D, $\varphi \in C(\partial D)$ and for $\beta \in \mathbb{H}_n$, the integral*

$$V(\beta) = \int_{\partial D} \varphi(\alpha) g_\beta(\alpha) d\sigma_\alpha,$$

exists. Moreover, V defines a continuous function on $\mathbb{H}_n$.

Proof. As $g_\beta(\alpha)$ admits a pole at $\alpha = \beta$, we shall compute the integral over $\Omega_\beta(\epsilon)$ and its complement. By Lemma 3.2 we can say that the integral exists on $\Omega_\beta(\epsilon)$. From [10], $d\sigma = \dfrac{\nabla_{hor}F}{\|\nabla_{hor}F\|_{hor}} ds$, with ds being the Euclidean surface element on ∂D. Set $D' = \partial D \setminus \Omega_\beta(\epsilon)$, then

$$V(\beta) = c\int_{D'} \frac{\varphi(\alpha)}{||\zeta|^2 + |\zeta'|^2 + i(t'-t) - 2\zeta\bar{\zeta}'|^n} \frac{\|\nabla_{hor}F\|_{hor}}{\|\nabla F\|} ds. \tag{3.6}$$

As D is bounded, ∂D is compact; hence so is D' and $\|\nabla F\|$ being a continuous function on ∂D must attain a positive minimum value, say c_2 over $D' = \partial D \setminus \Omega_\beta(\epsilon)$. Thus $\|\nabla F\| \ge c_2 > 0$ on D'.

Furthermore, $\|\nabla_{hor}F\|_{hor}$ being continuous function on D' is bounded over there, hence

$$\|\nabla_{hor}F\|_{hor} < c_1 \text{ on } D',$$

where c_1 is a positive constant. Therefore $\dfrac{\|\nabla_{hor}F\|_{hor}}{\|\nabla F\|} < \dfrac{c_1}{c_2}$ on D'. The remaining part of the integrand on the right hand side of (3.6) is bounded on D' and the denominator part of it i.e. $||\zeta|^2 + |\zeta'|^2 + i(t'-t) - 2\zeta\bar{\zeta}'|$ is away from zero because $\alpha \ne \beta$.

Therefore, $V(\beta)$ exists. To check the continuity of $V(\beta)$, we consider the following two cases:

Case 1: When β_0 belongs to interior or exterior of D, we have

$$\begin{aligned}\lim_{\beta\to\beta_0} V(\beta) &= \int_{\partial D} \varphi(\alpha) \lim_{\beta\to\beta_0} g_\beta(\alpha) d\sigma_\alpha \\ &\quad \text{(using bounded convergence theorem)}, \\ &= \int_{\partial D} \varphi(\alpha) g_{\beta_0}(\alpha) d\sigma_\alpha, \\ &= V(\beta_0).\end{aligned}$$

Case 2: When $\beta_0 \in \partial D$, we have, $g_\beta(\alpha) = cN(\alpha^{-1}\beta)^{-2n}$ and $\varphi(\alpha)$ has an upper bound; therefore using dominated convergence theorem,

$$\begin{aligned}\lim_{\beta\to\beta_0} V(\beta) &= \int_{\partial D} \varphi(\alpha) \lim_{\beta\to\beta_0} g_\beta(\alpha) d\sigma_\alpha, \\ &= \int_{\partial D} \varphi(\alpha) g_{\beta_0}(\alpha) d\sigma_\alpha, \\ &= V(\beta_0).\end{aligned}$$

Thus, $V(\beta)$ is continuous throughout $\mathbb{H}_n$.

□

Lemma 3.4. *On the boundary ∂D, $g_\beta(\alpha)$ satisfies*

$$\int_{\partial D} \partial^\dagger g_\beta(\alpha) d\sigma_\alpha = \begin{cases} -1, & \beta \in D, \\ -\frac{1}{2}, & \beta \in \partial D, \\ 0, & \beta \in \mathbb{H}_n \setminus \overline{D}. \end{cases}$$

Proof. We shall use Gaveau-Green's Formula[10]:

$$\int_\Omega (u\Delta_{\mathbb{H}_n} v - v\Delta_{\mathbb{H}_n} u) d\mu = \int_{\partial\Omega} \left\{u\partial^\dagger v - v\partial^\dagger u\right\} d\sigma, \tag{3.7}$$

where $d\sigma = \frac{\|\nabla_{hor}F\|_{hor}}{\|\nabla F\|}ds$, with ds being the Euclidean surface element on the boundary $\partial\Omega$.

In case 1, when $\beta \in D$ and case 3 when $\beta \in \mathbb{H}_n \setminus \bar{D}$, taking $v = 1$ and $u = g_\beta(\alpha)$ in Gaveau-Green's formula, gives the desired result. Now we consider the case 2 when $\beta \in \partial D$. Let us take $r > 0$ and consider $\Omega_\beta(r) = \{\alpha \in \mathbb{H}_n : N(\beta^{-1}\alpha) < r\}$.

Now taking $u = 1$ and $v = g_\beta(\alpha)$ on $D \setminus \Omega_\beta(r)$ in Green's formula, we have the following:

$$\int_{\partial(D\setminus\Omega_\beta(r))} \partial^\dagger g_\beta(\alpha)d\sigma = 0.$$

Now, $\int_{\partial(D\setminus\Omega_\beta(r))} \partial^\dagger g_\beta(\alpha)d\sigma = \int_{\partial D\setminus\Omega_\beta(r)} \partial^\dagger g_\beta(\alpha)d\sigma + \int_{D\cap\partial\Omega_\beta(r)} (-1)\partial^\dagger g_\beta(\alpha)d\sigma.$

Making use of the above two equations, we have

$$\int_{\partial D\setminus\Omega_\beta(r)} \partial^\dagger g_\beta(\alpha)d\sigma = \int_{D\cap\partial\Omega_\beta(r)} \partial^\dagger g_\beta(\alpha)d\sigma.$$

Therefore, $\lim_{r\to 0}\int_{\partial D\setminus\Omega_\beta(r)} \partial^\dagger g_\beta(\alpha)d\sigma = \frac{1}{2}\lim_{r\to 0}\int_{\partial\Omega_\beta(r)} \partial^\dagger g_\beta(\alpha)d\sigma$. As $\beta \in \Omega_\beta(r)$, so we can use case 1 here to get

$$\int_{\partial\Omega_\beta(r)} \partial^\dagger g_\beta(\alpha)d\sigma = -1,$$

so that,

$$\lim_{r\to 0}\int_{\partial D\setminus\Omega_\beta(r)} \partial^\dagger g_\beta(\alpha)d\sigma = -\frac{1}{2}.$$

Thus, $\int_{\partial D} \partial^\dagger g_\beta(\alpha)d\sigma = -\frac{1}{2}$ when $\beta \in \partial D$. □

Theorem 3.5. *The function* $v(\beta) := \int_{\partial D} \varphi(\alpha)\partial^\dagger g_\beta(\alpha)d\sigma_\alpha$, *for any boundary point* β *satisfies*

$$\lim_{\zeta\to\beta-} v(\zeta) = \int_{\partial D} \varphi(\alpha)\partial^\dagger g_\beta(\alpha)d\sigma_\alpha - \frac{1}{2}\varphi(\beta), \qquad (3.8)$$

and

$$\lim_{\zeta\to\beta+} v(\zeta) = \int_{\partial D} \varphi(\alpha)\partial^\dagger g_\beta(\alpha)d\sigma_\alpha + \frac{1}{2}\varphi(\beta), \qquad (3.9)$$

where the limits have been taken along the normal direction at point β.

Proof. We shall prove (3.8) only, the proof of (3.9) follows on similar lines. Consider a point ζ on boundary of $D, \partial D$ and let $\beta = \zeta + h\widehat{\zeta}$ where $h > 0$ and $\widehat{\zeta}$ represents outward normal at ζ. Limit on left hand side in (3.8) is same as the limit of $v(\beta)$ as $h \to 0^+$.

Now we write,

$$v(\beta) = \varphi(\zeta)w(\beta) + u(\beta),$$

where,

$$w(\beta) = \int_{\partial D} \partial^\dagger g_\beta(\alpha)d\sigma_\alpha, \qquad (3.10)$$

$$u(\beta) = \int_{\partial D} (\varphi(\alpha) - \varphi(\zeta))\partial^\dagger g_\beta(\alpha)d\sigma_\alpha. \qquad (3.11)$$

For $\beta \in \partial D$, the integral $u(\beta)$ exists and it represents a function which is continuous on ∂D. By lemma (3.4) , $w(\beta) = -\frac{1}{2}$, for $\beta \in \partial D$, hence to build the theorem, it is sufficient to prove that

$$\lim_{h\to 0} u(\zeta + hn_0(\zeta)) = u(\zeta), \qquad \zeta \in \partial D,$$

where the limit is uniform on subsets of ∂D that are relatively compact. Denote $\partial D(\zeta : \delta) = \partial D \cap \partial D[\zeta : \delta], D[\zeta : \delta] = \{\alpha \in \mathbb{H}_n : N(\alpha^{-1}\zeta) \le \delta\}$. Take $\delta < N(\beta^{-1}\zeta) = \gamma$ (say) for δ sufficiently small, $N(\alpha^{-1}\beta)$ is bounded below by $\gamma - \delta$.

Now, we consider the following inequality,

$$\begin{aligned}\int_{\partial D(\zeta:\delta)} \left|\partial^\dagger g_\beta(\alpha)\right| d\sigma_\alpha &\le c_1\int_{\partial D(\zeta:\delta)} N(\alpha^{-1}\beta)^{-2n-4}d\sigma_\alpha\\ &\quad where\ \beta \ne \alpha,\\ &\le c_1\int_{\partial D(\zeta:\delta)} \frac{1}{(\gamma-\delta)^{2n+4}}d\sigma_\alpha,\\ &\le c_1\frac{1}{(\gamma-1)^{2n+4}}|\partial D(\zeta:\delta)|,\\ &= c_2 \text{ say.}\end{aligned}$$

As $\left|\partial^\dagger g_\eta(\xi)\right| \le c_K N(\xi^{-1}\eta)^{-2n-4}$ on every compact neighbourhood K of η for domain D, we have the following:

$$\begin{aligned}\left|\partial^\dagger g_\beta(\alpha) - \partial^\dagger g_\zeta(\alpha)\right| &\le c_3 N(\beta\zeta^{-1})\nabla_\zeta\left(\partial^\dagger g_\zeta(\alpha)\right),\\ &\le c_4\frac{N(\beta\zeta^{-1})}{(N(\zeta\alpha^{-1}))^{2n+4}}.\\ \int_{\partial D\setminus\partial D(\zeta:\delta)} \left|\partial^\dagger g_\beta(\alpha)\right| d\sigma_\alpha &\le c_5\frac{N(\beta\zeta^{-1})}{\delta^{2n+4}}.\end{aligned}$$

Therefore there exists $\delta_0 > 0$ and a constant $\mathcal{K} > 0$ such that

$$|u(\beta) - u(\zeta)| \le \mathcal{K}\left\{\max_{\alpha\in D[\zeta:\delta]} |\varphi(\alpha) - \varphi(\zeta)| + \frac{N(\beta\zeta^{-1})}{(\delta)^{2n+4}}\right\},$$

for all $\delta < \delta_0$. Since φ is continuous on ∂D (and hence uniformly continuous), there exists $\delta_1 > 0$ such that

$$\sup_{\alpha\in D[\zeta:\delta]} |\varphi(\alpha) - \varphi(\zeta)| \le \frac{\epsilon}{2\mathcal{K}},$$

for all $\delta < \delta_1$. Now if $\delta' < \frac{\epsilon(min\{\delta_0, \delta_1\})^{2n+4}}{2\mathcal{K}}$, so that $|u(\beta) - u(\zeta)| < \epsilon$ for all $N(\beta\zeta^{-1}) < \delta'$. This establishes our claim. □

Theorem 3.6. *In the case of the integral* $m(\beta) := \int_{\partial D} \varphi(\alpha) g_\beta(\alpha) d\sigma_\alpha$, *the following equation holds true:*

$$\partial^\dagger m_\pm(\beta) = \int_{\partial D} \varphi(\alpha)\partial^\dagger g_\beta(\alpha) d\sigma_\alpha \pm \frac{1}{2}\varphi(\beta) \ , \ \beta \in \partial D.$$

Proof. For $\beta = \zeta + h\widehat{\zeta} \in U \setminus \partial D$, we obtain

$$\widehat{\zeta}.\nabla(m(\beta)) + v(\beta) = \int_{\partial D} \{\widehat{\zeta} + \widehat{\beta}\}.\nabla_\alpha(g_\beta(\alpha))\varphi(\alpha) d\sigma_\alpha,$$

where $\nabla_\beta(g_\beta(\alpha)) = \nabla_\alpha(g_\beta(\alpha))$ has been used. □

Theorem 3.7. *The equation* $\lim_{h\to 0+} \widehat{\beta}.\{\nabla v(\beta + h\widehat{\beta}) - \nabla v(\beta - h\widehat{\beta})\} = 0$, *is satisfied by the integral*

$$v(\beta) := \int_{\partial D} \varphi(\alpha)\partial^\dagger g_\beta(\alpha) d\sigma_\alpha,$$

uniformly for all $\beta \in \partial D$.

Proof. Similar to proof of Thm 3.6. □

4 An Example of a Singular Domain

The Korányi ball plays the role of a model domain to discuss boundary value problems in the Heisenberg group. Dirichlet problem has also been discussed for a half-Korányi, the domain obtained by intersecting the Korányi ball with the upper half plane in [2]. Here, we shall look at this half-ball as a domain that has a "difficult" set of characteristic points, and see a method of finding a solution that works as an approximation. The method demonstrated, however, does not provide a general rule to deal with the characteristic set.

Let us take $\Omega = \{(z,t) \in \mathbb{H}_n : |z|^4 + t^2 \le 1, t \ge 0\}$. This domain Ω has the set $J = \{(z,0) \in \mathbb{H}_n$ such that $|z| = 1\} \cup \{(\mathbf{0},1)\}$ as the set of characteristic points. The characteristic point $(\mathbf{0},1)$ can easily be handled using the techniques of [2] but the same cannot be applied to the rest of the characteristic points as normal derivatives of $g_\beta(\alpha)$ along different directions do not agree on it; this has been shown in Figure 1 for the case of $\mathbb{H}_1$.

We offer a convergent approximate solution for the following problem for Ω

$$\begin{cases} \Delta_{\mathbb{H}_n} u = 0 \text{ in } \Omega, \\ \partial^\dagger u = f \text{ on } \partial\Omega \setminus J, \end{cases} \tag{A1}$$

where f represents a continuous function on $\partial\Omega$. Consider the domain Ω_ϵ which is bounded by $\{|z|^4 + t^2 = 1$ when $t \ge \epsilon\}$; $\{F(z,t) = 0$ for $0 \le t \le \epsilon\}$ and $\{(z,0) \in \mathbb{H}_n : |z| = 1 - \epsilon\}$, where the function F shall be chosen later. Consider the Dirichlet problem

$$\begin{cases} \Delta_{\mathbb{H}_n} v = 0 \text{ in } \Omega, \\ v = f \text{ on } \partial\Omega. \end{cases} \tag{A2}$$

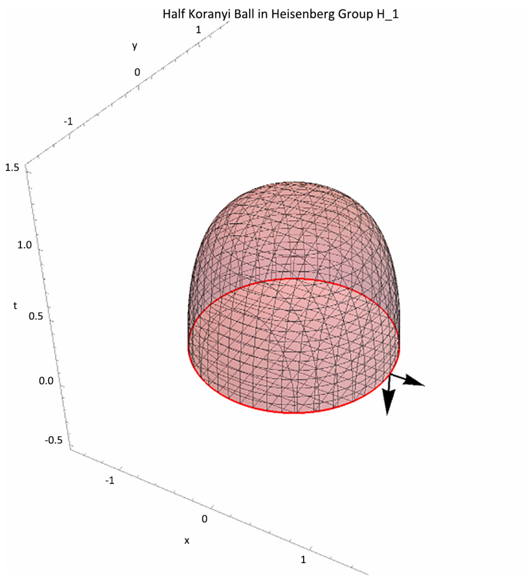

Figure 1. Half Koranyi ball in $\mathbb{H}_1$

This problem admits a solution that is continuous upto boundary(for detailed existence results see [13] and [14]). Now we consider a point p in Ω and an approximate solution in a neighbourhood of p of (A1) will be a solution of the following problem

$$\begin{cases} \Delta_{\mathbb{H}_n} u = 0 \text{ in } \Omega_\epsilon, \\ \partial^\dagger u = v \text{ on } \partial\Omega_\epsilon. \end{cases} \tag{A3}$$

Here we choose ϵ small enough so that p is an interior point of Ω_ϵ and choose F in such a way that on the boundary $\partial\Omega_\epsilon$ the normal derivative is given by

$$\left(\partial^\dagger\right)_\epsilon = \left(\frac{1 - |z|^4 - t^2}{1 - |z|^4}\right)\left(\partial^\dagger\right)_2 + \left(\frac{t}{\sqrt{(1 - |z|^4)}}\right)\left(\partial^\dagger\right)_1,$$

where $\left(\partial^\dagger\right)_1$, the unit horizontal normal derivative on boundary of half space $t = 0$ is given by

$$\left(\partial^\dagger\right)_1 = \frac{i}{|z|}(E - \bar{E}) \text{ where } E = \textstyle\sum_{j=1}^n z_j Z_j.$$

And $\left(\partial^\dagger\right)_2$,the unit horizontal normal derivative on part of the boundary where $N = 1$ is given by

$$\left(\partial^\dagger\right)_2 = \frac{1}{|z|}(\bar{A}E + A\bar{E}) \text{ where } E = \textstyle\sum_{j=1}^n z_j Z_j \text{ and } A = |z|^2 + it.$$

Existence of such an F is an implication of the existence of integral curves for the horizontal vector field $\left(\partial^\dagger\right)_\epsilon$ through every point of $\partial\Omega$. Also,

$$\left(\partial^\dagger g_\eta\right)_\epsilon(\xi) = -2nc\frac{N^{-2n-4}}{|z|}(S_1 + S_2 + S_3). \tag{4.1}$$

where S_1, S_2 and S_3 are rational functions in the coordinates of the boundary point with denominator bounded away from origin. For the sake of brevity, we demonstrate the expressions for the 3-dimensional Heisenberg group $\mathbb{H}_1$:

$$S_1 = \left\{ \frac{-y(1-|z|^4-t^2)}{1-|z|^4} + \frac{t}{\sqrt{1-|z|^4}}(x(x^2+y^2+ty)) \right\} \times$$
$$\left\{((x-x')^2+(y-y')^2)(x-x')+(2xy'-2yx'+t-t')y'\right\},$$

$$S_2 = \left\{ \frac{x(1-|z|^4-t^2)}{1-|z|^4} + \frac{t}{\sqrt{1-|z|^4}}(y(x^2+y^2-tx)) \right\} \times$$
$$\left\{((x-x')^2+(y-y')^2)(y-y')+(2xy'-2yx'+t-t')(-x')\right\}$$

and

$$S_3 = \left\{ \frac{tx}{\sqrt{1-|z|^4}}(tx-y(x^2+y^2))(2xy'-2yx'+t-t') \right\},$$

where $N = N(\eta, \xi^{-1}) = [\{(x-x')^2+(y-y')^2\}^2 + \{2xy'-2yx'+t-t'\}^2]^{1/4}$. From (4.1), we conclude that $\left|\left(\partial^\dagger g_\eta\right)_\epsilon(\xi)\right| \leq c_K(N(\eta^{-1}\xi))^{-2n-4}$ on any compact subset K of $\partial\Omega_\epsilon$ that does not contain $(\mathbf{0}, 1)$, which is in accordance with Lemma 3.1.

This Ω_ϵ is a bounded domain having a smooth boundary with only one isolated characteristic point, and it can be seen that the solution of the interior Neumann problem exists for it. This is an approximation method which gives an approximate numerical solution of the Neumann boundary value problem (A1) at any particular interior point of such kind of a domain.

5 Final Results

Theorem 5.1. *A solution of the interior Neumann problem for D, if it exists, is unique upto additive constants.*

Proof. Since ∂D is smooth, it is an $\mathbb{H}$-Caccioppoli set [15]. The proof of uniqueness of the solution for the Koranyi ball as in [2], now straightaway generalizes to D. □

Theorem 5.2. *Each of the operators $I + \mathcal{T}$ and $I + \mathcal{T}'$ has nullity 1.*

Proof. Using estimate(3.5), we can easily conclude that $\mathcal{T}$ and $\mathcal{T}'$ are compact. These are adjoint also. Now the proof proceeds in a similar manner as that of [2, Theorem 3.9]. □

Theorem 5.3. *For $\theta \in C(\partial D)$,*

$$m(\beta) = \int_{\partial D} \theta(\alpha) g_\beta(\alpha) d\sigma_\alpha \ \ , \beta \in D,$$

is a solution of the interior Neumann problem provided θ solves the following integral equation:

$$\theta(\beta) + \int_{\partial D} \theta(\alpha)\partial^\dagger g_\beta(\alpha) d\sigma_\alpha = q(\beta) \ \ , \beta \in \partial D.$$

Proof. This can be derived using Theorem 3.6. □

Theorem 5.4. *Let D be a bounded domain having smooth boundary without characteristic points. Then the necessary and sufficient condition for the solvability of the interior Neumann problem (2.2) is given by the following condition:*

$$\int_{\partial D} q d\sigma = 0.$$

Proof. The necessary part can be established using Gaveau-Green's formula (3.7) and the sufficient part derives from Fredholm's theorem (see [12]) and Theorem (5.3). □

6 Conclusions

Existence and uniqueness results have been established for the Neumann boundary-value problem in case of subelliptic Kohn-Laplacian operator on the Heisenberg group $\mathbb{H}_n$, for a bounded domain with a smooth boundary that is free of characteristic points. The necessary and sufficient condition for the solvability of the homogeneous interior Neumann problem is that the integral of the values assigned to the normal derivative vanishes over the boundary surface of the given domain.

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