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An Extended Construction of Hopfian Free-torsion Abelian Groups

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Abstract The study of Hopficity in Abelian groups has been largely motivated by the fundamental results of Baumslag, who proved that torsion groups are always Hopfian regardless of their cardinality, but left several questions open concerning torsion-free groups. Later, Corner addressed some of these questions by providing counterexamples showing that a direct sum of two Hopfian groups can be non-Hopfian, and that a group with an automorphism group of order two does not guarantee Hopficity. These results highlighted the need for new constructions to explore Hopficity in torsion-free Abelian groups. Our work introduces a new approach based on divisibility techniques, as it contributes to the understanding of free-torsion groups with respect to the Hopficity property, providing new insights into their structural properties and implications within group theory. Our analysis also demonstrates how divisibility properties, as well as the introduction of totally invariant subgroups and homomorphisms, can be used to establish Hopficity in specific Abelian groups, particularly those that are free-torsion. In order to reach all of this, we start by taking a group defined as an infinite direct sum of cyclic groups; then we construct a specific subgroup generated by two particular families of elements; and finally we show that this group is Hopfian through results from the theory of divisible subgroups.

Keywords Hopfian Groups, Free Torsion Groups, Abelian Groups, p-divisible Groups, Maximal Divisible Groups, Fully Invariant Subgroups

1 Introduction

The concept of Hopficity plays a pivotal role in the study of algebraic structures, particularly in group and module theory. A group G is called *Hopfian* if every surjective endomorphism $\alpha : G \to G$ is an automorphism. This concept has been crucial in understanding the rigidity of algebraic structures and has been explored extensively in various mathematical contexts.

The study of Hopficity dates back to Nielsen [1], who used an algebraic approach to show that a finitely generated free group cannot be isomorphic to one of its proper quotients. Later, Hopf [2] provided a topological proof of the same result, thus formalizing the notion. In 1944, Baer [3] extended the study of Hopfian groups through the introduction of Q-groups and S-groups, offering new perspectives on the structure of Hopfian and non-Hopfian groups.

In the context of Abelian groups, Baumslag [4] proved that torsion groups are Hopfian, regardless of their cardinality, but this property does not always hold for free-torsion groups. This result left several open questions concerning the structure of Hopfian Abelian groups.

In 1965, Corner [5] addressed some of these questions by providing explicit examples of free-torsion non-Hopfian Abelian groups, showing that the property is not necessarily preserved under particular direct sums or automorphism constraints. Later, in 1969, Irwin and Takashi [6] examined quasi-decomposable Abelian groups, thus revealing additional complexities in the behavior of Hopficity in infinite groups.

More recently, the concept of Hopficity has been extended beyond groups. In 1999, Haghany [7] explored Hopficity and co-Hopficity in Morita contexts, a central notion in category theory. In 2005, Wang [8] introduced new classes of Hopfian modules, showing that certain weakly Hopfian modules are not necessarily Hopfian. In 2015, Abdelalim [9] studied strongly Hopfian Abelian groups, proving that the torsion part of a strongly Hopfian Abelian group is not necessarily strongly Hopfian.

Further developments have been made in the study of Hopficity. In 2025, Bouzendaga *et al.* [10] introduced the notion of generalized Hopficity groups, extending existing theorems about the behavior of Hopficity under certain algebraic conditions. By using hereditary constructions, several approaches have been proposed to illustrate and analyze Hopficity.

Beyond Abelian groups, Rhemtulla [11] constructed finitely generated non-Hopfian groups using amalgamations, while Haghany [7] explored the concept of Hopficity and co-Hopficity within Morita contexts, and Wang [8] introduced new classes of Hopfian and non-Hopfian modules.

In this paper, we introduce a new construction of Hopfian Abelian groups. Our approach investigates free-torsion Hopfian Abelian groups by applying results based on divisibility properties in Abelian groups, thus offering a new pathway to achieving Hopficity and enriching the understanding of its role in the structure of infinite Abelian groups.

We consider a group T, defined as an infinite direct sum of cyclic groups, and construct a specific subgroup H, generated by two particular families of elements. We then show that this group H is Hopfian by applying results from the theory of divisible subgroups. Our approach relies on introducing a subgroup H_1 of T containing H, and analyzing two p-divisible subgroups, which enable us to establish precise constraints on the endomorphisms of H.

This paper is divided into three sections. Section 2 is devoted to the definitions and preliminary results necessary for our study. Section 3 presents the proof of our main construction (Theorem). Finally, in Section 4, we conclude our work.

To ensure clarity and readability, we now introduce the notations and conventions used throughout the text:

- The term "group" refers to an abelian additive group, and we will use the additive notation.
- p_i denotes a prime number indexed by an integer i.
- $\mathbb{N}$ represents the set of natural numbers.
- $\mathbb{Z}$ denotes the group of integers.
- $\oplus$ denotes a direct sum of groups.
- φ represents a group homomorphism.

2 Preliminary results

In this section, we provide the necessary definitions and key propositions that form the foundation for the construction of Hopfian groups presented in the following section. These definitions are essential for understanding the algebraic properties that will be used in our proof, with references given in [2, 3, 12, 13].

Definition 2.1.
- A group G is said to be Hopfian if every surjective endomorphism $\alpha : G \to G$ is an automorphism.
- A group G is said to be free-torsion if, for all $x \in G$, the order of x, denoted by $\circ(x)$, is infinite.
- A group G is said to be divisible if, for all $n \in \mathbb{N}$, $G = nG$.
- A group G is said to be divisible p-groups if, for all $n \in \mathbb{N}$, $G = p^n G$.
- A group H is said to be maximal divisible subgroup of G if there is no subgroup divisible K, of G such that, $H \subset K$, or every subgroup divisible of G is containing in H.
- We say that a subgroup H of a group G is fully invariant when every endomorphism of G leaves H stable, that is, maps H into itself

Remark 2.2. If we define $T = \bigoplus_{n\in\mathbb{N}} \langle a_n \rangle$ with $\circ(a_n) = \infty$.

$\{q; p_0; p_1; p_2; p_3 \dots p_n \dots\}$ a set of distinct prime numbers.

$$T_n = \left\langle \left\{ m_n \tfrac{a_n}{q^{k'_n} p_n^{k_n}} / m_n \in \mathbb{Z}, k_n \in \mathbb{N}, k'_n \in \mathbb{N} \right\} \right\rangle.$$
$$T'_n = \left\langle \left\{ m_n \tfrac{a_n}{p_n^{k_n}} / m_n \in \mathbb{Z}, k_n \in \mathbb{N} \right\} \right\rangle.$$

$$H = \left\langle \begin{array}{c} \left\{ \bigoplus\limits_{n\in\mathbb{N}} \left\langle \frac{m_n a_n}{p_n^{k_n}} \right\rangle \,\middle|\, m_n \in \mathbb{Z},\ k_n \in \mathbb{N} \right\} \\ \cup \left\{ \bigoplus\limits_{n\in\mathbb{N}} \left\langle \frac{m'_n (a_{n+1}+a_0)}{q^{k'_n}} \right\rangle \,\middle|\, m'_n \in \mathbb{Z},\ k'_n \in \mathbb{N} \right\} \end{array} \right\rangle$$

$$H_1 = \left\{ \bigoplus_{n\in\mathbb{N}} \left\langle m_n \frac{a_n}{p_n^{k_n} q^{k'_n}} \right\rangle / m_n \in \mathbb{Z}, k_n \in \mathbb{N}, k'_n \in \mathbb{N} \right\},$$

then we get H as a subgroup of H_1. In fact, we have H is not an empty set since $0 \in H$.
Let $x, y \in H$.
Then, $x = \sum\limits_{n=0}^{s} m_n \frac{a_n}{p_n^{k_n}} + \sum\limits_{n=0}^{s} m'_n \frac{(a_{n+1}+a_0)}{q^{k'_n}}$
and also, $y = \sum\limits_{n=0}^{s} \alpha_n \frac{a_n}{p_n^{k'_n}} + \sum\limits_{n=0}^{s} \alpha'_n \frac{(a_{n+1}+a_0)}{q^{k''_n}}$.
Thus,

$$x - y = \sum_{n=0}^{s} \frac{m_n p_n^{k'_n} - \alpha_n p_n^{k_n}}{p_n^{k_n+k'_n}} a_n + \sum_{n=0}^{s} \frac{m'_n q^{k''_n} - \alpha'_n q^{k'_n}}{q^{k'_n+k''_n}} (a_{n+1}+a_0).$$

Which implies that $x - y \in H$. Hence, $H \leqslant H_1$.

The following proposition is essential for the use of divisibility in our construction.

Proposition 2.3. If

$$T_n = \left\langle \left\{ m_n \frac{a_n}{q^{k'_n} p_n^{k_n}} / m_n \in \mathbb{Z}, k_n \in \mathbb{N}, k'_n \in \mathbb{N} \right\} \right\rangle,$$

then T_n is the maximal p_n-divisible subgroup of H_1.

Proof. First we show that T_n is p_n-divisible subgroup.

For this, we need only to show $T_n \subset p_n T_n$.
Let $t \in T_n$, then $t = \frac{m_n}{p_n^{k_n} q^{k_n'}} a_n$, and we can write, $t = \frac{p_n m_n}{p_n^{k_n+1} q^{k_n'}} a_n$,

Hence $t \in p_n T_n$, and consequently T_n is a p_n-divisible subgroup.
Now, we show that T_n is maximal, assume that T_n is not the maximal p_n-divisible subgroup, then there exists $C \leqslant H_1$ such that $T_n \subsetneq C$.
Thus, there exists $t \in C \backslash T_n$ such that $t = \sum_{j=0}^{s} \frac{m_j}{q^{\beta_j} p_j^{\beta_j}} a_j$ because we have : $t = \frac{m_n}{q^{\beta_n} p_n^{\beta_n}} a_n + \sum_{j=0}^{s} \frac{m_j}{q^{\beta_j} p_j^{\beta_j}} a_j \; j \neq n$. (1)

Now we assume that, for $0 \leq j \leq s$ and $j \neq n : m_j \neq 0$.
There exists then $r \in \mathbb{N}^*$ such that p_n^r does not divide the product $m_0 \times m_1 \times m_2 \times ... \times m_s$.

Therefore, $t = p_n^r \cdot t_1$ where $t_1 \in C$.

Thus, $t = p_n^r \left(\frac{m_n'}{q^{\beta_n'} p_n^{\beta_n'}} a_n + \sum_{j=0, j\neq n}^{s'} \frac{m_j'}{q^{\beta_j'} p_j^{\beta_j'}} a_j \right)$.

Then, $t = \frac{p_n^r m_n'}{q^{\beta_n'} p_n^{\beta_n'}} a_n + \sum_{j=0, j\neq n}^{s'} \frac{p_n^r m_j'}{q^{\beta_j'} p_j^{\beta_j'}} a_j$. (2)

From (1) and (2), we deduce the following cases:

- For $j = n : \frac{m_n}{q^{\beta_n} p_n^{\beta_n}} = \frac{p_n^r m_n'}{q^{\beta_n'} p_n^{\beta_n'}}$.
- For $0 \leq j \leq s$ and $j \neq n : \frac{m_j}{q^{\beta_j} p^{\beta_j}} = \frac{p_n^r m_j'}{q^{\beta_j'} p_j^{\beta_j'}}$.

From this last case, we get $p_n^r m_j' q^{\beta_j} p_j^{\beta_j} = m_j q^{\beta_j'} p_j^{\beta_j'}$, then $p_n^r / m_j q^{\beta_j'} p^{\beta_j'}$.

Thus, p_n^r / m_j for $j \in \{0, 1, , , , , , , s\}$ and $j \neq n$ (because $p_n^r \wedge p = 1$, and $p_n^r \wedge q = 1$), and also $p_n^r / m_0 m_1 m_2 .....m_s$ which is absurd.

Hence, T_n is maximal p_n-divisible subgroup. □

In the result above, we showed that T_n is p_n maximal divisible subgroup.

As for now,we demonstrate that a_n is not divisible by q in H for all $n \in \mathbb{N}^*$.

The following proposition is very important as it will be used to prove the lemma thereafter as well as the fundamental property of the construction.

Proposition 2.4. Let q be a prime number, then a_n is not divisible by q in H for all $n \in \mathbb{N}^*$.

Proof. We assume that for all $n \in \mathbb{N}^*$, a_n is divisible by q in H, then there exists $t \in H$ such that $a_n = qt$, where

$$\begin{aligned} t &= \sum_{k=0}^{m_0} \frac{r_k}{p_k^{\alpha_k}} a_k + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta}} (a_k + a_0) \\ &= \frac{r_0}{p_0^{\alpha_0}} a_0 + \sum_{k=1}^{m_0} \frac{r_k}{p_k^{\alpha_k}} a_k + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta}} (a_k + a_0) \\ &= \frac{r_0}{p_0^{\alpha_0}} a_0 + \sum_{k=1}^{m_0} \frac{r_k}{p_k^{\alpha_k}} a_k + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta}} a_k + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta}} a_0 \\ &= \frac{r_0}{p_0^{\alpha_0}} a_0 + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta}} a_0 + \sum_{k=1}^{m_0} \frac{\left(q^{\beta} r_k + s_k p_k^{\alpha_k}\right)}{p_k^{\alpha_k} q^{\beta}} a_k \end{aligned}$$

Thus, $qt = \frac{q r_0}{p_0^{\alpha_0}} a_0 + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta-1}} a_0 + \sum_{k=1}^{m_0} \frac{\left(q^{\beta} r_k + s_k p_k^{\alpha_k}\right)}{p_k^{\alpha_k} q^{\beta-1}} a_k$
Hence, $a_n = \left(\frac{q r_0}{p_0^{\alpha_0}} + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta-1}} \right) a_0 + \sum_{k=1}^{m_0} \frac{\left(q^{\beta} r_k + s_k p_k^{\alpha_k}\right)}{p_k^{\alpha_k} q^{\beta-1}} a_k$.

Now we deduce that :

$$\begin{cases} \frac{q r_0}{p_0^{\alpha_0}} + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta-1}} = 0 \\ \frac{q^{\beta} r_k + s_k p_k^{\alpha_k}}{p_k^{\alpha_k} q^{\beta-1}} = 0, for \; 1 \leq k \leq m_0 \, and \, k \neq n \\ \frac{q^{\beta} r_n + s_n p_n^{\alpha_n}}{p_n^{\alpha_n} q^{\beta-1}} = 1, for \; k = n \end{cases}$$

$$\Longleftrightarrow \begin{cases} q^{\beta} r_0 = -p_0^{\alpha_0} \sum_{k=1}^{m_0} s_k \, then \; q^{\beta} / \sum_{k=1}^{m_0} s_k \, because \, q \wedge p_0 = 1 \\ q^{\beta} r_k = -s_k p_k^{\alpha_k}, for \; 1 \leq k \leq m_0 \, and \, k \neq n, \\ q^{\beta} r_n + s_n p_n^{\alpha_n} = p_n^{\alpha_n} q^{\beta-1}, for \; k = n \end{cases}$$

$$\Longleftrightarrow \begin{cases} \sum\limits_{k=1}^{m_0} s_k = q^{\beta} q^{'} \\ s_k = q^{\beta} s_k^{'} \, because \, q \wedge p_k = 1, \; for \, 1 \leq k \leq m_0 \, and \, k \neq n \\ q^{\beta} r_n = p_n^{\alpha_n} \left(q^{\beta-1} - s_n \right), for \; k = n \end{cases}$$

$$\Longleftrightarrow \begin{cases} s_n + \sum\limits_{k=1, k\neq n}^{m_0} s_k = q^{\beta} q^{'} \\ s_k = q^{\beta} s_k^{'}, for \, 1 \leq k \leq m_0 \, and \, k \neq n \\ q^{\beta} r_n = p_n^{\alpha_n} \left(q^{\beta-1} - s_n \right), for \; k = n \end{cases}$$

$$\Longleftrightarrow \begin{cases} s_n = q^{\beta} q^{'} - \sum\limits_{k=1, k\neq n}^{m_0} q^{\beta} s_k^{'} \\ s_k = q^{\beta} s_k^{'}, for 1 \leq k \leq m_0 \, and \, k \neq n \\ q^{\beta} r_n = p_n^{\alpha_n} \left(q^{\beta-1} - s_n \right), for \; k = n \end{cases}$$

$$\Longleftrightarrow \begin{cases} s_n = q^{\beta} \left(q^{'} - \sum\limits_{k=1, k\neq n}^{m_0} s_k^{'} \right) = q^{\beta} s^{"}, \\ s_k = q^{\beta} s_k^{'}, for \, 1 \leq k \leq m_0 \, and \, k \neq n \\ q^{\beta} r_n = p_n^{\alpha_n} \left(q^{\beta-1} - q^{\beta} s^{"} \right), for \; k = n \;\; (*) \end{cases}$$

□

According to $(*)$, we have

$$\frac{q^{\beta}}{q^{\beta-1} - q^{\beta} s''},$$

and since $q \wedge p_n = 1$, it follows that

$$\frac{q}{1 - q s''},$$

which leads to a contradiction.
Therefore, a_n is not divisible by q for every $n \in \mathbb{N}^*$.

According to $(*)$, we have $q^{\beta} / \left(q^{\beta-1} - q^{\beta} s^{"} \right)$ because $q \wedge p_n = 1$, then $q/1 - qs^{"}$, which leads to a contradiction. Therefore, a_n is not divisible by q for $n \in \mathbb{N}^*$

Proposition 2.5. Let q be a prime number, then a_0 is not divisible by q in H.

Proof. Assume that a_0 is divisible by q in H, then there exists $t \in H$ such that $a_0 = qt$, and

$$\begin{aligned} t &= \sum_{k=0}^{m_0} \frac{r_k}{p_k^{\alpha_k}} a_k + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta}} \left(a_k + a_0 \right) \\ &= \left(\frac{r_0}{p_0^{\alpha_0}} + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta}} \right) a_0 + \sum_{k=1}^{m_0} \frac{\left(q^{\beta} r_k + s_k p_k^{\alpha_k} \right)}{p_k^{\alpha_k} q^{\beta}} a_k. \end{aligned}$$

Then,

$$a_0 = \left(\frac{q r_0}{p_0^{\alpha_0}} + \sum_{k=1}^{m_0} \frac{s_k}{q^{\beta-1}} \right) a_0 + \sum_{k=1}^{m_0} \frac{\left(q^{\beta} r_k + s_k p_k^{\alpha_k} \right)}{p_k^{\alpha_k} q^{\beta-1}} a_k.$$

Hence, we deduce $\frac{q r_0}{p_0^{\alpha_0}} + \sum\limits_{k=1}^{m_0} \frac{s_k}{q^{\beta-1}} = 1 \; (**)$, and $\frac{\left(q^{\beta} r_k + s_k p_k^{\alpha_k} \right)}{p_k^{\alpha_k} q^{\beta-1}} = 0$ for $1 \leq k \leq m_0$ and $k \neq 0$.

So $\left(q^{\beta} r_k + s_k p_k^{\alpha_k} \right) = 0$, then $q^{\beta} r_k = -s_k p_k^{\alpha_k}$.

Thus q^{β} / s_k, and hence $s_k = q^{\beta} q^{'}$ and by $(**)$, we deduce $\frac{q r_0}{p_0^{\alpha_0}} + \sum\limits_{k=1}^{m_0} q q^{'} = 1$ or equivalently $\frac{q r_0}{p_0^{\alpha_0}} + m_0 q q^{'} = 1$, and also $q r_0 + m_0 q q^{'} p_0^{\alpha_0} = p_0^{\alpha_0}$ which implies $q r_0 = p_0^{\alpha_0} \left(1 - m_0 q q^{'} \right)$, hence $q / \left(1 - m_0 q q^{'} \right)$, so $q/1$ which is absurd thus a_0 is not divisible by q. □

3 Construction of Hopfian Free Torsion Groups

This section is devoted to the formal proof of the main theorem, which demonstrates the construction of new Hopfian Abelian groups. We present the key steps involved in establishing the result and provide a detailed explanation of the construction process.

Proposition 3.1. Let T be a maximal p_n-divisible subgroup of A. Then every p_n-divisible subgroup B of A is contained in T.

Proof. Suppose that

$$T_1 = T + B.$$

Let $a \in T_1$. Then, there exists $(t, b) \in T \times B$ such that

$$a = t + b.$$

Moreover, since T and B are p_n-divisible, we can write

$$a = p_n t' + p_n b', \quad t' \in T, \; b' \in B.$$

Hence,

$$a = p_n(t' + b'), \quad \Rightarrow \quad a \in p_n T_1.$$

Therefore, T_1 is a p_n-divisible subgroup of A.

Since $T \subset T_1$ and T is maximal p_n-divisible, we conclude that

$$T_1 = T.$$

□

Lemma 3.2. Let

$$T'_n = \left\langle \tfrac{m_n}{p_n^{k_n}} a_n \;\middle|\; k_n \in \mathbb{N},\ m_n \in \mathbb{Z},\ n \in \mathbb{N} \right\rangle$$

be a p_n-divisible subgroup in H. Then

$$T'_n = T_n \cap H.$$

Proof. We have that T'_n is a p_n-divisible subgroup in H, and since $H \leqslant H_1$ (as defined in Remark 2.2), it follows that T'_n is also p_n-divisible in H_1.

As T_n is maximal p_n-divisible in H_1, by Proposition 3.1 we deduce

$$T'_n \subset T_n.$$

Hence,

$$T'_n \subset T_n \cap H.$$

Now, the remaining task is to show that

$$T_n \cap H \subset T'_n.$$

Let $t \in T_n \cap H$. Then

$$t = \frac{m_n a_n}{p_n^{\alpha_n} q^{s_n}},$$

with $m_n \wedge q = 1$. We will show that $s_n = 0$.

Assume that $s_n \geq 1$. Since $m_n \wedge q = 1$, there exist $(u, v) \in \mathbb{Z}^2$ such that

$$u m_n + v q^{s_n} = 1.$$

Additionally,

$$\frac{u m_n a_n}{p_n^{\alpha_n} q^{s_n}} + \frac{v q^{s_n} a_n}{p_n^{\alpha_n} q^{s_n}} = \frac{a_n}{p_n^{\alpha_n} q^{s_n}} = t'.$$

Therefore,

$$\frac{u m_n a_n}{p_n^{\alpha_n} q^{s_n}} \in H, \quad \frac{v q^{s_n} a_n}{p_n^{\alpha_n} q^{s_n}} \in H, \quad \Rightarrow \quad t' \in H.$$

Hence,

$$a_n = t' p_n^{\alpha_n} q^{s_n} \quad \Longrightarrow \quad q^{s_n} \mid a_n.$$

Since $t' p_n^{\alpha_n} \in H$ and by Proposition 2.4, q does not divide a_n in H. This implies $s_n = 0$.

Therefore,

$$t = \frac{m_n a_n}{p_n^{\alpha_n}}, \qquad t \in T'_n.$$

Consequently,

$$T_n \cap H \subset T'_n \quad \Longrightarrow \quad T_n \cap H = T'_n.$$

□

Finally, in the following theorem, we will show that H is a Hopfian group.

Theorem 3.3. Let $T = \bigoplus_{n\in\mathbb{N}} \langle a_n \rangle$ with $\circ(a_n) = \infty$.

$$T_n = \left\langle \left\{ m_n \frac{a_n}{q^{k'_n} p_n^{k_n}} / m_n \in \mathbb{Z}, k_n \in \mathbb{N}, k'_n \in \mathbb{N} \right\} \right\rangle.$$

$$T'_n = \left\langle \left\{ m_n \frac{a_n}{p_n^{k_n}} / m_n \in \mathbb{Z}, k_n \in \mathbb{N} \right\} \right\rangle.$$

and

$$\{q; p_0; p_1; p_2; p_3 \ldots p_n \ldots\}$$

a set of distinct prime numbers.

If H is a group defined as

$$H = \left\langle \begin{array}{c} \left\{ \bigoplus_{n\in\mathbb{N}} \left\langle \frac{m_n a_n}{p_n^{k_n}} \right\rangle \;\middle|\; m_n \in \mathbb{Z},\ k_n \in \mathbb{N} \right\} \\ \cup \left\{ \bigoplus_{n\in\mathbb{N}} \left\langle \frac{m'_n (a_{n+1} + a_0)}{q^{k'_n}} \right\rangle \;\middle|\; m'_n \in \mathbb{Z},\ k'_n \in \mathbb{N} \right\} \end{array} \right\rangle$$

then H is Hopfian group.

Proof. Let $\varphi \in \mathrm{End}(H)$ be an epimorphism.
Since T'_n is maximal p_n-divisible in H, then by Property (D) - page 98 in [12], $\varphi(T'_n)$ is p_n-divisible in H.
Hence, by Proposition 3.1 and previous Lemma 3.2, we have

$$\varphi(T'_n) \subset T'_n = T_n \cap H,$$

then T'_n is fully invariant subgroup of H, which implies $\varphi(T'_n) \subset T_n$. Therefore, $\varphi(a_n) \in T_n$ for $a_n \in T'_n$. Now, let

$$\varphi(a_n) = \frac{m_n a_n}{p_n^m q^n},$$

assume that $n \geq 1$ and $m_n \wedge q^n = 1$, so there exists $u \in \mathbb{Z}$ and $v \in \mathbb{Z}$ such that $u m_n + v q^n = 1$, which implies

$$u m_n + v q^n = 1.$$

Also,

$$\frac{u m_n a_n}{p_n^m q^n} + \frac{v q^n a_n}{p_n^m q^n} = \frac{a_n}{p_n^m q^n}$$

and since

$$\frac{um_na_n}{p_n^m q^n} \in H, \qquad \text{and} \qquad \frac{vq^na_n}{p_n^m q^n} \in H,$$

then

$$\frac{a_n}{p_n^m q^n} \in H.$$

Hence,

$$\frac{a_n}{p_n^m q^n} = t'' \in H$$

implies that

$$a_n = t''\, p_n^m q^n.$$

Since a_n is not divisible by q in H, we have $n = 0$. Therefore,

$$\varphi(a_n) = \frac{m_na_n}{p_n^m},$$

where

$$\varphi(a_i) = \frac{m_ia_i}{p_i^{\alpha_i}}, \quad \text{for every } i \in \mathbb{N}.$$

Also, for every $i \in \mathbb{N}$ we have

$$\begin{aligned} p_i^{\alpha_i}\varphi(a_i + a_0) &= p_i^{\alpha_i}\varphi(a_i) + p_i^{\alpha_i}\varphi(a_0) \\ &= p_i^{\alpha_i}\frac{m_ia_i}{p_i^{\alpha_i}} + p_i^{\alpha_i}\frac{m_0a_0}{p_0^{\alpha_0}}. \\ &= m_ia_i + m_ia_0 - m_ia_0 + p_i^{\alpha_i}\frac{m_0a_0}{p_0^{\alpha_0}} \\ &= m_i(a_i + a_0) + \left(\frac{m_0p_i^{\alpha_i} - m_ip_0^{\alpha_0}}{p_0^{\alpha_0}}\right) a_0. \end{aligned}$$

Let T_q be the maximal q-divisible subgroup of H. Then $(a_i + a_0) \in T_q$ implies that

$$p_i^{\alpha_i}\,\varphi(a_i + a_0) \in T_q \quad \text{and} \quad m_i(a_i + a_0) \in T_q.$$

Therefore,

$$\left(\frac{m_0p_i^{\alpha_i} - m_ip_0^{\alpha_0}}{p_0^{\alpha_0}}\right) a_0 \in T_q.$$

For every $i \in \mathbb{N}$, let

$$M_0 = m_0p_i^{\alpha_i} - m_ip_0^{\alpha_0}.$$

Assume that $M_0 \neq 0$. Then there exists $n_0 \in \mathbb{N}^*$ such that q^{n_0} does not divide M_0. Hence,

$$\frac{M_0a_0}{p_0^{\alpha_0}} = q^m t \quad \text{for some } t \in T_q.$$

Therefore,

$$\begin{aligned} \frac{M_0a_0}{p_0^{\alpha_0}} &= q^m\left(\sum_{j=0}^{s}\frac{m_ja_j}{p_j^{\alpha_j}} + \sum_{j=1}^{s}\frac{n_j^{'}(a_j + a_0)}{q^{\beta}}\right) \\ &= q^m\left(\frac{m_0a_0}{p_0^{\alpha_0}} + \sum_{j=1}^{s}\frac{m_ja_j}{p_j^{\alpha_j}} + \sum_{j=1}^{s}\frac{n_j^{'}a_j}{q^{\beta}} + \sum_{j=1}^{s}\frac{n_j^{'}\{a_0\}}{q^{\beta}}\right) \\ &= q^m\left(\frac{m_0a_0}{p_0^{\alpha_0}} + \sum_{j=1}^{s}\frac{n_j^{'}\{a_0\}}{q^{\beta}} + \sum_{j=1}^{s}\frac{m_jq^{\beta} + n_j^{'}p_j^{\alpha_j}}{q^{\beta}p_j^{\alpha_j}}a_j\right) \end{aligned}$$

Then, for $1 \leq j \leq s$ we have : $\frac{M_0}{p_0^{\alpha_0}} = q^m\left(\frac{m_0}{p_0^{\alpha_0}} + \sum\limits_{j=1}^{s}\frac{n_j^{'}}{q^{\beta}}\right)$ (1), and $m_jq^{\beta} + n_j^{'}p_j^{k_j} = 0$.

Therefore $q^{\beta}/n_j^{'}p_j^{k_j}$, and hence $q^{\beta}/n_j^{'}$, thus $n_j^{'} = q^{\beta}n_j^{"}$ and also $\frac{M_0}{p_0^{\alpha_0}} = q^m\left(\frac{m_0}{p_0^{\alpha_0}} + \sum\limits_{j=1}^{s} n_j^{"}\right)$.

Therefore $M_0 = q^m\left(\frac{m_0}{p_0^{\alpha_0}} + \sum\limits_{j=1}^{s} n_j^{"}\right)p_0^{\alpha_0}$, and hence q^m/M_0 which is absurd.
Then, $M_0 = 0$ which implies $m_0p_i^{\alpha_i} = m_ip_0^{\alpha_0}$ for every $i \in \mathbb{N}$ (2).
According to (1), we have

$$\frac{m_0}{p_0^{\alpha_0}} = -\frac{1}{q^{\beta}}\sum_{j=1}^{s} n_j',$$

which implies

$$q^{\beta}m_0 = -p_0^{\alpha_0}\sum_{j=1}^{s} n_j'.$$

Moreover, since $\frac{p_0^{\alpha_0}}{q^{\beta}}m_0$ is an integer multiple of $\frac{p_0^{\alpha_0}}{m_0}$, we deduce that

$$m_0 = p_0^{\alpha_0}m_0'$$

for some integer m_0'.

From (2), we have for every $i \in \mathbb{N}$:

$$p_0^{\alpha_0}p_i^{\alpha_i}m_0' = m_ip_0^{\alpha_0},$$

hence

$$m_i = p_i^{\alpha_i}m_0'.$$

Therefore,

$$\varphi(a_i) = \frac{m_ia_i}{p_i^{\alpha_i}} = m_0'a_i.$$

Thus, $\varphi = m_0'\,\mathrm{id}$ for every $i \in \mathbb{N}$. Since H is a torsion-free group, φ is injective, and consequently φ is an isomorphism.

Finally, H is a Hopfian group. □

4 Conclusions

The first major construction in the context of Abelian groups was proposed by Baumslag, who proved that torsion Abelian groups are always Hopfian, regardless of their cardinality. However, the author also pointed out that in the case of free-torsion groups, a more nuanced analysis is required, as Hopficity is not always guaranteed.

A significant breakthrough occurred with Corner, who introduced explicit examples demonstrating the complexity of Hopficity in free-torsion Abelian groups. For instance, the author showed that the direct sum of two Hopfian groups can be non-Hopfian while the square of a non-Hopfian group can be Hopfian.

In this paper, we suggested a new type of group construction which realizes the Hopficity property in the category of Abelian groups. For this, we stated a result about the Hopficity through the construction of a subgroup of a free-torsion group. In addition, we introduced several results on p-divisible groups, fully invariant subgroups and homomorphisms by relying on structural properties of infinite Abelian groups, and thereby we finally reached the proof of our statement.

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