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On the Summatory Function of Ω(B(N))

Sinyavsky O. V.

Military Academy (Odessa), 65009, Odessa, str. Fontansky road, 10, Ukraine

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Abstract An asymptotic formula is derived for the summatory function $\sum\limits_{n\le x}\Omega(b(n))$, where $\Omega(n)$ denotes the total number of prime factors of n counted with multiplicity, and $b(n)$ is a multiplicative arithmetical function satisfying $b(p^k)=g(k)$ for primes p and non-negative integers k, where $g(0)=1$ and $g(k)\in\mathbb{N}$ for $k\ge 1$. The study builds on a rich history in analytic number theory, including classical results by Dirichlet on the divisor function $d(n)$, and refinements using zeta-function estimates, as well as probabilistic approaches like the Erdős–Kac theorem extended to $\Omega(n)$ (distinct primes) over h-free and h-full numbers. However, prior research has largely overlooked the multiplicity in $\Omega(n)$ and its twisting by broad classes of multiplicative functions beyond divisors, particularly for square-full integers. The analysis covers three distinct cases: when $b(n)$ belongs to the subclass B_1 (where $b(p)=1$ for all primes p); when $b(n)$ is in the broader class B but not in B_1; and when n is square-full with $b(n)\in B$. Examples of such functions include the number of non-isomorphic Abelian groups of order n, the number of square-full divisors of n, the divisor function $\tau(n)$, and the k-fold divisor function $\tau_k(n)$. The results are obtained using Dirichlet series $D(s,z)=\sum\limits_{n=1}^{\infty}\frac{z^{\Omega(b(n))}}{n^s}$, which admit an Euler product decomposition due to multiplicativity, enabling analytic continuation via differentiation with respect to an auxiliary parameter z, contour integration, and estimates for the Riemann zeta function, as well as analytic continuation techniques. The following results were obtained: for case (i), $\sum_{n\le x}\Omega\big(b(n)\big)=x\cdot c_1+O\left(x^{1/2}\log^{3/2}x\log\log x\right)$; for case (ii), $\sum_{n\le x}\Omega\big(b(n)\big)=xc_2+O\left(x^{1/2+\varepsilon}\right)$; and for square-full n, $\sum_{n\le x}\Omega\big(b(n)\big)=x^{1/2}c_3+O\left(x^{1/3}\log^2 x\right)$. The work is theoretical in nature. The results of this study can be applied in further research in number theory, group theory, and discrete mathematics, with potential applications in algorithmic number theory (e.g., efficient computation of group orders) and cryptographic protocols relying on prime factorizations.

Keywords Multiplicative Functions, Asymptotic Formulas Divisor Function, Trigonometric Sums, Distribution of Values

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1. Introduction

The study of additive and multiplicative arithmetic functions has long occupied a central position in analytic number theory. Among the most classical quantities is the function $\Omega(n)=\sum\limits_{p^k\|n}k$, which counts prime factors of n with multiplicity. Beginning with the foundational works of Hardy–Ramanujan and Erdős–Kac, the distribution of $\Omega(n)$ and its summatory functions has been deeply explored, revealing a rich interplay between probabilistic and analytic methods. However, considerably less is known when $\Omega(\cdot)$ is composed with a nontrivial multiplicative transformation. In particular, let B denote the class of multiplicative functions $b(n)$ with the condition $b\left(p^k\right)=g(k)$, where p is a prime, k is a non-negative integer, and $g(0)=1$, $g(\alpha)\in \mathrm{N}$ for $\alpha\geq 1$. Further, B_1 denotes the functions from B that satisfy the condition $b(p)=1$ for any prime p. The following are examples of such functions: $a(n)$ is the number of non-isomorphic Abelian groups; $\beta(n)$ is the number of divisors of n divisible by the square of a prime; $\tau(n)$ is the number of divisors of n; $\tau_k(n)$ is the number of ways in which a natural number n can be represented as a product of k factors. It is natural to ask how the classical behavior of Ω(n) extends to the generalized summatory function $S(x)=\sum\limits_{n\leq x}\Omega(b(n))$.

There are several related papers in the literature. For instance, Das et al. [1] examine the distribution of $\omega(n)$, the number of distinct prime factors, over h-free and h-full numbers, extending classical Erdős–Kac results to specialized integer subsets. Their approach, however, is probabilistic and limited to $\omega(n)$, which counts only distinct prime factors, excluding multiplicity. Another relevant recent investigation is that of Jakimczuk [2] who examines the parity of Ω(n) restricted to square-full integers, showing that among such numbers the subset with Ω(n) even has larger density than those with Ω(n) odd. Though the setting is specialized, the techniques and insights into behavior of Ω on constrained sets are informative in the broader study of weighted summatory functions.

Classical results concerning the distribution of the function $\Omega(n)$ were obtained by Hardy and Ramanujan, and later refined in the probabilistic framework of Erdős and Kac. More recent developments can be found in the analytic approach of Granville and Soundararajan [3], as well as in the mean-value estimates for multiplicative functions over smooth numbers due to Bhowmik and Schlage-Puchta [4]. A systematic exposition of analytic techniques related to multiplicative functions is also given in the monograph of Montgomery and Vaughan [5].

More recent advances on divisor-bounded and almost multiplicative functions [6], and Ω-type lower bounds for products of multiplicative functions [7] provide tools for estimating mean values, but none of them address the specific structure of $\Omega(b(n))$.

In contrast to these studies, our work is the first to systematically investigate the summatory function $\sum\limits_{n\leq x}\Omega(b(n))$ for a broad class of multiplicative functions $b(n)$, taking into account both the multiplicity of prime divisors in $\Omega(n)$ and the structural properties of $b(n)$ through the condition $b(p^k)=g(k)$. Our analytical approach enables us to obtain precise asymptotic formulas with explicit remainder terms for three fundamentally different cases.

This approach generalizes classical results for $\Omega(n)$ and divisor functions. Thus, our results are more general than the probabilistic approach of [1]. The purpose of this paper is to obtain an asymptotic formula for $\sum\limits_{n\leq x}\Omega(b(n))$.

2. Research Methods

Many problems in analytic number theory are related to the summation of multiplicative functions, i.e., to the study of the asymptotic behavior (as $x\to\infty$) of sums of the form

$$S(x)=\sum_{n\leq x} g(n), \tag{1}$$

where $g(n)$, as a rule, is a multiplicative function of a natural argument. In the case where $g(n)=c(n)f(n)$, ($f(n)$ is continuously differentiable) and an asymptotic formula for $\sum_{n\leq x}c(n)$ is known, the asymptotic estimate for $S(x)$ can be obtained using Abel's lemma on partial summation [8].

In analytic number theory, the method of Dirichlet generating series is widely used for evaluating sums (1). To this end, we define a series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{2}$$

where a_n are complex coefficients, $0<|\lambda_n|\to\infty$, and $s=\sigma+it$ is a complex variable.

When $\lambda_n=\log n$, this yields the so-called ordinary Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}. \tag{3}$$

If this series converges absolutely in the half-plane $\Re s>\sigma_0$, (and thus defines an analytic function $f(s)$ there), then the asymptotic behavior of the summatory function

$$\Phi(x)=\sum_{n\leq x} a_n \left(x\to\infty\right)$$

can be studied using the Perron formula.

Dirichlet showed that

$$\sum_{n\leq x}\tau(n)=x\log x+(2\gamma-1)x+\Delta(x),$$

where $\tau(n)$ is the number of divisors of n. Moreover, he proved elementarily that $\Delta(x)=O\left(x^{1/2}\right)$.

Using new tools from analytic number theory, the error terms in previously obtained asymptotic formulas were refined, and new asymptotic formulas were constructed.

To obtain the main results, analytic methods from number theory were employed, focused on the investigation of summatory functions depending on the number of prime factors, accounting for multiplicity. The applied approach is based on the use of Dirichlet series, methods of complex analysis, and estimates for the Riemann zeta function.

For the studied function $\Omega(b(n))$, the corresponding Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{\Omega(b(n))}{n^s},$$

is considered, which, due to the multiplicativity of the function $b(n)$, admits an Euler product decomposition. This allows for the analysis of the analytic properties of the series and its continuation to the required regions of the complex plane.

The summatory function is represented in integral form

$$S(x)=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}F(s)\frac{x^s}{s}ds.$$

After establishing this representation, we shift the contour of integration to the left. This leads to the appearance of new integrals along several segments of the modified contour, which are then estimated individually.

The main contribution to the asymptotic formula comes from the residue of the integrand $F(s)x^s/s$ at $s=1$. Once this residue is computed, the contour is shifted further to the left to capture the remaining parts of the integral. The integrals along the horizontal and vertical segments of the new contour are then bounded by standard techniques. These estimates together yield the error term in the final expression.

In analyzing convergence and estimating error terms, well-known results for $\zeta(s)$ and its derivatives are applied, including estimates of the second moment and behavior on the critical line. These estimates allow for the strict bounding of the contributions from integrals along vertical lines when shifting the contour of integration.

The Euler product for $F(s)$ is considered as a function of an auxiliary parameter *z*, which enables the expression of the dependence on $\Omega(n)$ through derivatives with respect to *z*. Differentiating the power series and subsequently substituting $z=0$ provides a means to isolate components related to the number of prime factors.

3. Results

We consider three cases:

a. when $b(n)\in B_1$;
b. when $b(n)\in B\setminus B_1$ and $b(n)\notin B_1$;
c. when *n* is square-full and $b(n)\in B$.

For the first case, when $b(n)\in B_1$,

Theorem 1. Let $b(n)\in B_1$, $n\in\mathbb{N}$. Then

$$\sum_{n\le x}\Omega\big(b(n)\big)=x\left[\Omega\big(g(2)\big)\log\zeta(2)+\frac{\partial}{\partial z}H(1,1)\right]+$$
$$+O\left(x^{1/2}\log^{3/2}x\log\log x\right),$$

where $$H(s,z)=\prod_p\left(1+\frac{z^{\Omega(g(3))}-z^{\Omega(g(2))}}{p^{3s}}+\right.$$
$$\left.+\frac{\left(z^{2\Omega(g(2))}-z^{\Omega(g(2))}\right)+z^{\Omega(g(4))}-z^{\Omega(g(3))}}{p^{4s}}+\ldots\right),$$

$|z|\le 1$, $H(s,z)$ is regular in the region $\operatorname{Re}s>1/3$.

Proof. Consider the sum $\sum_{n=1}^{\infty}\frac{z^{\Omega(b(n))}}{n^s}$, where $0\leqslant|z|\leqslant 1$ and $\operatorname{Re}s>1$. By the multiplicativity of the function $z^{\Omega(b(n))}$, we have:

$$\sum_{n=1}^{\infty}\frac{z^{\Omega(b(n))}}{n^s}=$$
$$=\prod_p\left(1+\frac{1}{p^s}+\frac{z^{\Omega(g(2))}}{p^{2s}}+\ldots+\frac{z^{\Omega(g(\alpha))}}{p^{\alpha s}}+\ldots\right)=\zeta(s)\times$$
$$\times\prod_p\left(\begin{array}{l}1+\frac{z^{\Omega(g(2))}-1}{p^{2s}}+\ldots+\frac{z^{\Omega(g(3))}-z^{\Omega(g(2))}}{p^{3s}}+\ldots\\ +\frac{z^{\Omega(g(\alpha))}-z^{\Omega(g(\alpha-1))}}{p^{\alpha s}}+\ldots\end{array}\right)=$$
$$=\zeta(s)H(s,z).$$

Since $0\le z\le 1$, then $|H(s,z)|\le\prod\left(1+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\ldots\right)$.

The latter product converges for $\operatorname{Re}s>1/2$, hence the product $H(s,z)$ also converges in this region. Then $H(s,z)$ can be represented as a power series in *z*, and differentiated with respect to *z* in the region of convergence of this series $|z|\le 1$, $\operatorname{Re}s<1/2$.

Furthermore, we will need the following representation:

$$H(s,z)=\zeta(2s)^{z^{\Omega(g(2))}-1}\times$$
$$\times\prod_{p}\left(1+\frac{z^{\Omega(g(3))}-z^{\Omega(g(2))}}{p^{3s}}+\right.$$
$$\left.+\frac{\left(z^{2\Omega(g(2))}-z^{\Omega(g(2))}\right)+z^{\Omega(g(4))}-z^{\Omega(g(3))}}{p^{4s}}+\dots\right)=$$
$$=\zeta(2s)^{z^{\Omega(g(2))}-1}H_1(s,z).$$

Note that $H_1(s,z)$ can be represented as a power series in *z* and differentiated in the region of convergence $|z|\le 1$, $\mathrm{Re}s>\frac{1}{3}$. Next, we differentiate the series $\sum_{n=1}^{\infty}\frac{z^{\Omega(b(n))}}{n^s}$ with respect to *z* and then set $z=1$ to obtain:

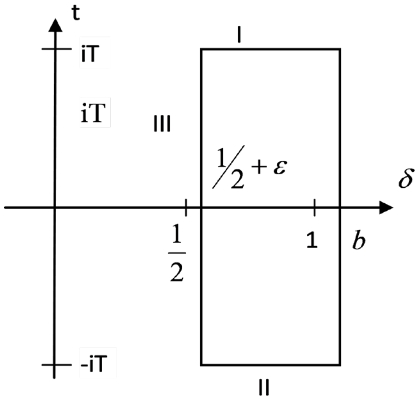
$$\sum_{n=1}^{\infty}\frac{\Omega(b(n))}{n^s}=\zeta(s)H'(s,1),$$

where $H'(s,1):=\left.\frac{\partial H(s,z)}{\partial z}\right|_{z=1}$. Hence, by the theorem on partial sums of Dirichlet series (see [9]), we have:

$$\sum_{n\le x}\Omega(b(n))=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta(s)H'(s,1)\frac{x^s}{s}ds+$$
$$+O\left(\frac{x^b}{T(b-1)^{\alpha}}\right)+O\left(\frac{xA(2x)\log x}{T}\right),$$

where $b=1+\varepsilon$, $\varepsilon>0$, $\alpha=1$, $\Omega(b(2x))\le A(2x)\le x^{\varepsilon}$.

Let us compute the last integral. For this, consider the contour:



By the residue theorem, we have:

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta(s)H'(s,1)\frac{x^s}{s}ds=\operatorname*{res}_{s=1}\left(\zeta(s)H'(s,1)\frac{x^s}{s}\right)+$$
$$+\frac{1}{2\pi i}\left(\int_{I}+\int_{II}+\int_{III}\right)\zeta(s)H'(s,1)\frac{x^s}{s}ds.$$

To compute the residue, note that:

$$\left.\frac{\partial H(s,z)}{\partial z}\right|_{z=1}=\Omega(g(2))\log\zeta(2s)H_1(s,1)+H_1'(s,1).$$

Therefore,

$$\operatorname*{res}_{s=1}\left(\zeta(s)H'(s,1)\frac{x^s}{s}\right)=$$
$$=x\Omega(g(2))\log\zeta(2)H_1(1,1)+xH_1'(1,1),$$

where $H_1'(1,1)=\left.\frac{\partial H_1(1,z)}{\partial z}\right|_{z=1}$. To evaluate the integrals along the lines I and II, we use the estimates $\zeta(\sigma+it)<<t^{(1-\sigma)/4}\log t$, $\frac{1}{2}<\sigma<1$, (see [10]).

Then, we obtain:

$$\frac{1}{2\pi i}\left(\int_{I}+\int_{II}\right)\zeta(s)H'(s,1)\frac{x^s}{s}ds<<$$
$$<<\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}\zeta(\sigma+iT)\frac{x^{\sigma}}{T}d\sigma<<\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}T^{\frac{1-\sigma}{4}}\log T\frac{x^{\sigma}}{T}d\sigma=$$
$$=\frac{\log T}{T^{3/4}\log\frac{x}{T^{1/4}}}\left.\left(\frac{x}{T^{1/4}}\right)\right|_{1/2+\varepsilon}^{1+\varepsilon}<<\frac{x^{1+\varepsilon}}{T}.$$

Next, we estimate the integral along line III:

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta(s)H'(s,1)\frac{x^s}{s}ds<<$$
$$<<\frac{1}{2\pi i}\int_{III}\zeta(s)\log\zeta(2s)\frac{x^s}{s}ds+\frac{1}{2\pi i}\int_{III}\zeta(s)\frac{x^s}{s}ds<<$$

$$<< \frac{1}{2\pi i} \int\limits_{III} \zeta(s) \log\zeta(2s) \frac{x^s}{s} ds <<$$

$$<< x^{1/2+\varepsilon} \left(\int\limits_1^T \left| \zeta\left(1/2 + \varepsilon + it \right) \right|^2 \frac{dt}{t} \right)^{1/2} \times$$

$$\times \left(\int\limits_1^T \left| \log\zeta(1+\varepsilon+it) \right|^2 \frac{dt}{t} \right)^{1/2} << x^{1/2+\varepsilon} \log^{3/2} T \log\log T$$

.

Here, we used the estimate for the second moments for $\zeta(s)$ and the fact that $\left|\log\zeta(1+\varepsilon+it)\right| << \varepsilon^{-1}$ (see [11]), and that $H_1'(s,1)$ converges in the region $\mathrm{Re}s > \frac{1}{2}$. Now, combining these estimates and setting $T = x$, $\varepsilon = \frac{1}{\log x}$ (noting that $H_1(1,1) = 1$ because $H(1,1) = 1$), we obtain:

$$\sum_{n \le x} \Omega(b(n)) = x\Big(\Omega(g(2)) \log\zeta(2) H_1(1,1) + H_1'(1,1) \Big) +$$

$$+ O\left(x^{1/2} \log^{3/2} x \log\log x \right),$$

.

and that completes the proof of the theorem.

Example 1. Number of Non-Isomorphic Abelian Groups

Let $a(n)$ denote the number of non-isomorphic Abelian groups of order n.

Then

$$a(p^k) = P(k),$$

where $P(k)$ is the number of integer partitions of k.

Thus $g(k) = P(k)$ and

$$\Omega(a(p^k)) = \Omega(P(k)).$$

Applying Theorem 1, we obtain

$$\sum_{n \le x} \Omega(a(n)) = x\left[\Omega(P(2)) \log\zeta(2) + \frac{\partial}{\partial z} H(1,1) \right] +$$

$$+ O\left(x^{1/2} \log^{3/2} x \log\log x \right).$$

Although $a(n)$ fluctuates irregularly at prime powers, its global average behavior is linear, with a fully explicit main term determined by $P(1), P(2), P(3), \ldots$.

Now consider the second case, when $b(n) \in B$ and $b(n) \notin B_1$.

Theorem 2. Let $b(n) \in B$, but $b(n) \notin B_1$, $n \in \mathbb{N}$. Then

$$\sum_{n \le x} \Omega(b(n)) = x \frac{\partial}{\partial z} G(1,1) + O\left(x^{1/2+\varepsilon} \right),$$

where

$$G(s,z) = \prod_p \left(1 + \frac{z^{\Omega(g(2))} - \frac{1}{2} z^{2\Omega(g(1))} - \frac{1}{2} z^{\Omega(g(1))}}{p^{2s}} + \ldots \right),$$

$|z| \le 1$, $G(s,z)$ is regular in the region $\mathrm{Re}s > 1/2$.

Proof. Similar to the proof of Theorem 1, we have:

$$\sum_{n=1}^{\infty} \frac{z^{\Omega(b(n))}}{n^s} =$$

$$= \prod_p \left(1 + \frac{z^{\Omega(g(1))}}{p^s} + \ldots + \frac{z^{\Omega(g(\alpha))}}{p^{\alpha s}} + \ldots \right) = \zeta(s)^{z^{\Omega(g(1))}} \times$$

$$\times \prod_p \left(1 + \frac{z^{\Omega(g(2))} - \frac{1}{2} z^{2\Omega(g(1))} - \frac{1}{2} z^{\Omega(g(1))}}{p^{2s}} + \ldots \right) =$$

$$= \zeta(s)^{z^{\Omega(g(1))}} G(s,z) . \tag{4}$$

The product $G(s,z)$ converges for $0 \le z \le 1$, $\mathrm{Re}s > \frac{1}{2}$, since $\left| G(s,z) \right| \le \prod \left(1 + \frac{2}{p^{2s}} + \ldots \right)$. Hence, the product $G(s,z)$ converges in this region. Then $G(s,z)$ can be represented as a power series in *z* and differentiated with respect to *z* in the region of convergence of this series $|z| \le 1$, $\mathrm{Re}s > \frac{1}{2}$.

Thus, differentiating the series $\sum_{n=1}^{\infty} \frac{z^{\Omega(b(n))}}{n^s}$ with respect to *z* and setting $z = 1$, we obtain:

$$\sum_{n=1}^{\infty} \frac{\Omega(b(n))}{n^s} = \Omega(g(1)) \zeta(s) \log\zeta(s) G(s,1) + \zeta(s) G'(s,1),$$

here $G'(s,1) := \left. \frac{\partial G(s,z)}{\partial z} \right|_{z=1}$, and $G(s,1)$ is easily computed. Since $\sum_{n=1}^{\infty} \frac{z^{\Omega(b(n))}}{n^s} = \zeta(s)$ at $z = 1$, from (4) we obtain $G(s,1) = 1$.

Hence, as in the previous case, we have:

$$\sum_{n\le x}\Omega(b(n))=\frac{\Omega(g(1))}{2\pi i}\int_{b-iT}^{b+iT}\zeta(s)\log\zeta(s)G(s,1)\frac{x^s}{s}ds+$$

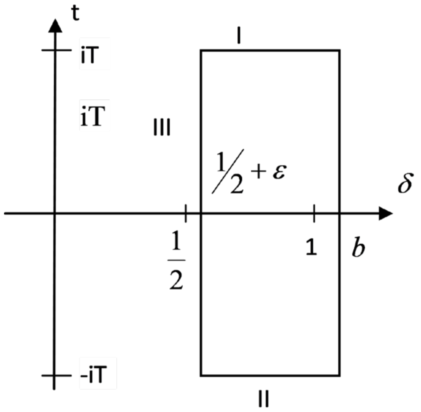
$$+\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta(s)G'(s,1)\frac{x^s}{s}ds+O\left(\frac{x^b}{T(b-1)^{\alpha}}\right)+$$

$$+O\left(\frac{xA(2x)\log x}{T}\right)=J_1+J_2+$$

$$+O\left(\frac{x^b}{T(b-1)^{\alpha}}\right)+O\left(\frac{xA(2x)\log x}{T}\right),$$

where $\alpha=2$, $b=1+\varepsilon$, $A(2x)\ll x^{\varepsilon}$.

Consider the integral J_2:



By the residue theorem, we have:

$$J_2=xG'(1,1)+\frac{1}{2\pi i}\left(\int_I+\int_{II}+\int_{III}\right)\zeta(s)G'(s,1)\frac{x^s}{s}ds.$$

Similarly, to the previous case:

$$\frac{1}{2\pi i}\left(\int_I+\int_{II}\right)\zeta(s)H'(s,1)\frac{x^s}{s}ds\ll$$

$$\ll\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}\zeta(\sigma+iT)G'(\sigma,1)\frac{x^{\sigma}}{T}d\sigma\ll$$

$$\ll\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}T^{\frac{1-\sigma}{4}}\log T\frac{x^{\sigma}}{T}d\sigma\ll\frac{x^{1+\varepsilon}}{T}.$$

Next, we estimate the integral along line III:

$$\frac{1}{2\pi i}\int_{III}\zeta(s)G'(s,1)\frac{x^s}{s}ds\ll$$

$$\ll x^{1/2+\varepsilon}\left(\int_1^T\left|\zeta\left(1/2+\varepsilon+it\right)\right|^2\frac{dt}{t}\right)^{1/2}\left(\int_1^T\left|G'(t,1)\right|^2\frac{dt}{t}\right)^{1/2}\ll$$

$$\ll x^{1/2+\varepsilon}\log T.$$

Note that

$$G'(t,1)=\zeta(2s)\log\zeta(2s)G_3(s)+\zeta(2s)G_4(s),$$

where $G_3(s)$ and $G_4(s)$ are regular to the right of the line $\mathrm{Re}s=\frac{1}{2}$.

Thus, with $T=x^{1+\varepsilon}$, we obtain:

$$J_2=xG'(1,1)+O\left(x^{1/2}\log^2 x\right).$$

Consider the integral J_1:

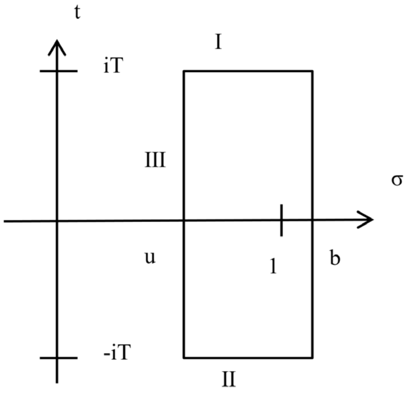
$$\frac{\Omega(g(1))}{2\pi i}\int_{b-iT}^{b+iT}\zeta(s)\log\zeta(s)\frac{x^s}{s}ds=\frac{\Omega(g(1))}{2\pi i}\times$$

$$\times\left(\int_{b-iT}^{b+iT}\frac{1}{s-1}H(s)\log H(s)\frac{x^s}{s}ds-\right.$$

$$\left.-\int_{b-iT}^{b+iT}\frac{\log(s-1)}{s-1}H(s)\frac{x^s}{s}ds\right),$$

where $H(s)=(s-1)\zeta(s)$.

We estimate the first integral. For this, consider the contour:



Then, by the residue theorem:

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{H(s)\log H(s)}{s(s-1)}x^s ds =$$

$$=\frac{1}{2\pi i}\left(\int_I \uparrow + \int_{II} \uparrow + \int_{III} \uparrow\right)\frac{H(s)\log H(s)}{s(s-1)}x^s ds..$$

Further:

$$\int_I \uparrow + \int_{II} \uparrow << \int_u^{1+\varepsilon}\frac{x^\sigma}{T^2}d\sigma << \frac{x}{T^2\log x}$$

$$\int_{III} \uparrow << \int_1^T \frac{x^u}{t^2}dt << \frac{x^u}{T}.$$

Thus, the first integral is $O(1)$ for $T = x^{1+\varepsilon}$.

We estimate the second integral. For this, transform the integrand:

$$\frac{\log(s-1)}{s-1}H(s)\frac{x^s}{s} = x^s(s-1)^{-1}\log(s-1)H_1(s),$$

where $H_1(s)$ is regular in the region $\mathrm{Re}s > 0$, and thus it can be represented by a convergent series $\sum_{m=0}^{\infty} c_m(s-1)^m$.

Since $\lim_{s\to 1} H_1(s) = 0$, we have:

$$\frac{\log(s-1)}{s-1}H(s)\frac{x^s}{s} = x^s(s-1)^{-1}\log(s-1)\sum_{m=1}^{\infty} c_m(s-1)^m.$$

Hence:

$$\int_{b-iT}^{b+iT}\frac{\log(s-1)}{s-1}H(s)\frac{x^s}{s}ds =.$$

$$= \int_{b-iT}^{b+iT} x^s(s-1)^{-1}\log(s-1)\sum_{m=1}^{\infty} c_m(s-1)^m ds <<$$

$$<< \int_{b-iT}^{b+iT}\left|x^s\right|ds << x\log^{-1}x.$$

Here we took into account that the series $\sum_{m=0}^{\infty} c_m(s-1)^m$ converges and $\left|\frac{\log(s-1)}{s-1}\right| < 1$ for $\mathrm{Re}s > 0$. Finally, we obtain:

$$\sum_{n\le x}\Omega(b(n)) = x\frac{\partial}{\partial z}G(1,1) + O\left(x^{1/2+\varepsilon}\right),$$

where

$$G(s,z) = \prod_p\left(1+\frac{z^{\Omega(g(2))} - \frac{z^{2\Omega(g(1))} - z^{\Omega(g(1))}}{2}}{p^{2s}} + \ldots\right),$$

$|z| \le 1$, and $G(s,z)$ is regular in the region $\mathrm{Re}s > \frac{1}{2}$, which completes the proof of the theorem.

Example 2. Exponential-type weights

Let

$$b(p^k) = A^k, \qquad g(k) = A^k, \qquad A > 1.$$

Then $b(p) = A \ne 1$, so $b \in B \setminus B_1$.

Since

$$\Omega(b(p^k)) = k\Omega(A),$$

Theorem 2 yields

$$\sum_{n\le x}\Omega(b(n)) = x\frac{\partial}{\partial z}G(1,1) + O\left(x^{1/2+\varepsilon}\right).$$

This describes the behavior of an important family of "biased" multiplicative weights that arise in analytic models of random multiplicative functions.

Let us consider the third case, when *n* is square-full and $b(n) \in B$.

Theorem 3. Let $b(n) \in B$, and let *n* be square-full, i.e., if $p\,|n$, then $p^2\,|n$. Then

$$\sum_{n\le x}\Omega(b(n)) = 2x^{1/2}\frac{\partial}{\partial z}G\left(1/2, 1\right) + O\left(x^{1/3}\log^2 x\right),$$

where $G(s,z) = \prod_p\left(1+\frac{z^{\Omega(g(3))}}{p^{3s}}+\ldots\right)$, $|z| \le 1$, and $G(s,z)$ is regular in the region $\mathrm{Re}s > 1/3$

Proof. Consider the sum $\sum_{n=1}^{\infty}\frac{z^{\Omega(b(n))}}{n^s}$, where $0 \le |z| \le 1$ and $\mathrm{Re}s > 1$. Due to the multiplicativity of the function $z^{\Omega(b(n))}$ and since $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ with $\alpha_i \ge 2$ for each i, we have:

$$\sum_{n=1}^{\infty}\frac{z^{\Omega(b(n))}}{n^s}=\prod_p\left(1+\frac{z^{\Omega(g(2))}}{p^{2s}}+\ldots+\frac{z^{\Omega(g(\alpha))}}{p^{\alpha s}}+\ldots\right)=$$

$$=\zeta(2s)^{z^{\Omega(g(2))}}\times$$

$$\times\prod_p\left(1+\frac{z^{\Omega(g(3))}}{p^{3s}}+\frac{2z^{\Omega(g(4))}-z^{2\Omega(g(2))}+1}{p^{4s}}+\ldots\right)=$$

$$=\zeta(2s)^{z^{\Omega(g(2))}}G(s,z).$$

The product $G(s,z)$ converges for $0\le z\le 1$, $\mathrm{Re}s>\frac{1}{3}$, since $|G(s,z)|\le\prod\left(1+\frac{3}{p^{3s}}+\ldots\right)$. Hence, the product $G(s,z)$ converges in this region. Then $G(s,z)$ can be represented as a power series in z and differentiated with respect to z in the region of convergence of this series $|z|\le 1$, $\mathrm{Re}s>\frac{1}{3}$.

Thus, differentiating the series $\sum_{n=1}^{\infty}\frac{z^{\Omega(b(n))}}{n^s}$ with respect to *z* and setting $z=1$, we obtain:

$$\sum_{n=1}^{\infty}\frac{\Omega(b(n))}{n^s}=\Omega(g(2))\zeta(2s)\log\zeta(2s)G(s,1)+$$
$$+\zeta(2s)G'(s,1).$$

Hence, as in the previous case, we have:

$$\sum_{n\le x}\Omega(b(n))=$$
$$=\frac{\Omega(g(2))}{2\pi i}\int_{b-iT}^{b+iT}\zeta(2s)\log\zeta(2s)G(s,1)\frac{x^s}{s}ds+$$
$$+\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta(2s)G'(s,1)\frac{x^s}{s}ds+O\left(\frac{x^b}{T(b-1)^{\alpha}}\right).$$

Furthermore, proceeding analogously to the proof of Theorem 2, we estimating the resulting integral along the line $\mathrm{Re}s=\frac{1}{2}+\varepsilon$, we obtain the statement of the theorem.

Example 3. $\tau_k(n)$ on square-full integers

Writing $n=\prod p^{2e_p}$, only prime powers $p^2,p^3,\ldots$ contribute.

From Theorem 3,

$$\sum_{\substack{n\le x\\ n\text{ square-full}}}\Omega(\tau_k(n))=2x^{1/2}\frac{\partial}{\partial z}G(1/2,1)+$$
$$+O\left(x^{1/3}\log^2 x\right).$$

This asymptotic connects with the study of higher moments of divisor functions and with counting problems involving representations by square-full numbers.

4. Conclusions

In this paper, we derived asymptotic formulas for the summatory function $\sum_{n\le x}b(\Omega(n))$, where $\Omega(n)$ counts the prime factors of *n* with multiplicity, and $b(n)$ is a multiplicative function satisfying $b(p^k)=g(k)$, with $g(0)=1$ and $g(k)\in\mathbb{N}$ for $k\ge 1$. We analyzed three distinct cases: when $b(n)\in B_1$, when $b(n)\in B\setminus B_1$, and when *n* is square-full. Using Dirichlet series, contour integration, and estimates for the Riemann zeta function, we obtained precise asymptotic expansions with error terms. These results extend the classical theorems in analytic number theory and can be applied to the study of functions related to divisors, Abelian groups, and other arithmetic structures.

In further research, one may address the question of the asymptotic behavior on special subsequences of the segment of natural numbers — for example, the behavior $\sum_{n\le x}b(\Omega(n))$ on arithmetic progressions with an increasing common difference.

Conflict of Interest

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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REFERENCES

[1] Das S., Kuo W., Y.-R. Liu, “Distribution of ω(n) over h-free and h-full numbers”, International Journal of Number Theory, vol. 21, no. 03, pp. 715-737, 2025. DOI: 10.1142/s1793042125500368.

[2] Jakimczuk R., “Square-full numbers with an even number of prime factors”, Notes on Number Theory and Discrete Mathematics, vol. 26, no 1, pp. 21–30, 2020. DOI: 10.7546/nntdm.2020.26.1.21-30.

[3] Granville A., Soundararajan K., “Large character sums: Pretentious characters and the Pólya-Vinogradov theorem”, Journal of the American Mathematical Society, vol. 20, no 2, pp. 357–384, 2007. DOI: 10.1090/s0894-0347-06-00536-4.

[4] Bhowmik G., Schlage-Puchta J.-C., “Mean representation number of integers as the sum of primes”, Nagoya Mathematical Journal, vol. 200, pp. 27–33, 2010. DOI: 10.1215/00277630-2010-010.

[5] Vaughan R. C., Montgomery H. L., Multiplicative Number Theory I: Classical Theory. Cambridge Univ. Press, 2010.

[6] Mangerel A. P., “Divisor-bounded multiplicative functions in short intervals,” Research in the Mathematical Sciences, vol. 10, article 12, 2023, DOI: 10.1007/s40687-023-00376-0.

[7] Frechette C., Gerbelli-Gauthier M., Hamieh A., and Tanabe N., “A note on large sums of divisor-bounded multiplicative functions,” arXiv preprint, arXiv:2405.00658, 2024.

[8] Murty M. R., A Course in Analytic Number Theory, American Mathematical Society, 2019.

[9] Tenenbaum G., Introduction to Analytic and Probabilistic Number Theory, 3rd ed., American Mathematical Society, 2015.

[10] Bordellès O., Arithmetic Tales: Advanced Topics, Springer, 2020.

[11] Ivić A., The Riemann Zeta-Function: Theory and Applications, Dover Publications, 2003.