

A Brief Review on Fuzzy Fractional Differential Equations and Their Applications

Mohammed Rabih

Department of Mathematics, College of Science, Qassim University, Saudi Arabia

Received April 16, 2025; Revised November 7, 2025; Accepted November 20, 2025

Cite This Paper in the Following Citation Styles

(a): [1] Mohammed Rabih , "A Brief Review on Fuzzy Fractional Differential Equations and Their Applications," *Mathematics and Statistics*, Vol. 13, No. 6, pp. 472 - 485, 2025. DOI: 10.13189/ms.2025.130604.

(b): Mohammed Rabih (2025). A Brief Review on Fuzzy Fractional Differential Equations and Their Applications. *Mathematics and Statistics*, 13(6), 472 - 485. DOI: 10.13189/ms.2025.130604.

Copyright©2025 by authors, all rights reserved. Authors agree that this article remains permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

Abstract Differential equation is a very important topic of interest for continuous system modelling as well as theoretical purposes. For real-life situation modelling, several variations are used by scientists and engineers. A fractional differential equation is one of them. Due to several reasons, such as measurement difficulty and environmental noise, the data set for real-life modelling is sometimes imprecise in nature. Fuzzy set theory is one of the suitable ideologies to deal with uncertainty. When a fuzzy set is associated with a fractional differential equation, then fuzzy fractional differential equations (FFDEs) come. Fuzzy fractional differential equations have garnered significant attention due to their ability to model processes with uncertainty and memory dependence in various scientific and theoretical contexts. Also, the concepts are applied to various engineering sciences and technical applications for modelling real-life problems. This review paper provides a comprehensive summary of FFDEs, a key concept in theoretical advancements, solution methodologies, and practical applications. The study examines previously published materials and conducts a comparative analysis of several components. The different crisp and fuzzy fractional derivatives are also compared with respect to different components. The various forms of past FFDEs are described, along with their theoretical and practical applications. Finally, tasks and probable future research directions in this emerging field are outlined.

Keywords Fractional Calculus, Fractional Derivative, Fuzzy Set Theory, Fuzzy Fractional Differential Equation

1. Introduction

1.1. Fractional Derivative and Fractional Differential Equation

Differential equations are very important topics for modelling in realistic situations of real-world problems. Now, a differential equation involves differential operators, such as ordinary or partial differential equations. Fractional order concepts are one of the new ideologies. A fractional derivative is a generalization of the ordinary derivative to non-integer order derivatives, which permits operations like taking a half or α ($0 < \alpha < 1$) derivative of a function. Contrasting standard derivatives, which only depend on the current behaviors of a function, fractional derivatives are imprisonment of memory and hereditary effects, which means they contemplate the entire history of the function's values. This makes them very useful in modelling real-world problems with long-term memory effects, such as diffusion processes and control systems. Fractional differential equations contain these fractional derivatives and define dynamics where the current state depends not only on current rates of variation but also on past states. It offers an accurate framework for different complex natural and concocted systems. For details about the fractional derivative and fractional differential equation, one can refer to the works [1-10].

1.2. Fuzzy Set Theory

Theories of uncertainty and uncertainty quantification play a crucial role in modelling as well as theoretical

research. Fuzzy set theory is one of them. Fuzzy set theory is a mathematical context for dealing with uncertainty and imprecision, where each element in the set has a graded membership value. The elements can partially belong to a set rather than having a strict yes-or-no concept as in classical set theory approaches. The theory was first coined by Lotfi Zadeh in 1965. It assigns each element a membership value between 0 and 1, representing the degree to which the element belongs to and does not belong to the set. For example, in the fuzzy set "tall people," someone who is 7 feet tall might have a membership value of 0.9, while someone who is 5'8" might have 0.3 rather than being entirely "tall" or "not tall." This kind of approach is especially useful in systems where reasoning, perception, or linguistic terms are involved in problems, for example, support systems, control systems, decision-making, pattern recognition, clustering and artificial intelligence. The idea allowed for more flexible and realistic modelling of several complex and uncertain ideas. For application purposes, anyone can follow the papers [11-20].

1.3. Fuzzy Fractional Differential Equation

A fuzzy fractional differential equation (FFDE) is the association of fuzzy set theory and fractional calculus to continuous model systems that involve both uncertainty and memory dependent effects. The variables, parameters, initial conditions and boundary conditions are represented as fuzzy numbers which reflect imprecision. In those cases, the derivatives are taken in a fractional order that is a non-integer order. The derivative captures long-term memory or hereditary behavior. This particular approach is particularly useful for real-world problems where data is uncertain in nature, and also the system dynamics are subject not only to the present but also to historical states.

The concepts of fractional differential equations (FDEs) are very influential mathematical tools and techniques used to model several complex systems with memory and hereditary properties, whereas classical differential equations cannot be effectively modelled in this way. They have initiated wide applications in real-life based problems where the procedures are predisposed by past states. FDEs have applications in several important real-life fields. In solid and fluid mechanics, where stress depends not only on the current strain but also on the history of deformation, FDE plays a crucial role. In the fields of bioengineering and medicine, they help define anomalous diffusion in tissues and drug delivery kinetics in blood flow dynamics. In control system theory applications and signal processing problems, FDEs deliver more correct models of damping

and filtering behaviours. In environmental science applications, FDEs model groundwater flow and pollutant transport with improved pragmatism. In economics and financial markets analysis, it is categorised by long-term dependencies and irregular fluctuations, and also benefits from FDE-inspired models. In electrical sciences, the fractal geometries and memory elements are more accurately designated using FDEs. They are also applied in biomathematical modelling to capture non-instantaneous infection dynamics or in other ways. Robotics, artificial intelligence, and image processing are other emerging areas where FDEs play an important role in developing new theories for better improvement of existing results. Moreover, FDEs offer a multipurpose framework for modelling and analysing various systems with memory effects and complex progressive behaviours.

Suppose we consider the fractional differential equation with initial value:

$$D_t^\alpha y(t) + ay(t) = f(t)$$

with initial condition $y(0) = y_0$, where $D_t^\alpha y(t)$ is a fractional derivative of order α .

The above fractional differential equation is converted to a fuzzy fractional differential equation if one or all of the following criteria hold:

- (i) The initial condition y_0 is fuzzy valued
- (ii) The coefficient a is fuzzy valued
- (iii) The forcing function $f(t)$ is fuzzy valued.

It should be noted that in all cases, the solution comes in a fuzzy nature.

Similarly, if we consider the boundary value problem in the following form

$$D_t^\alpha y(t) = f(t, y(t)), \quad a < t < b, \quad 0 < \alpha \leq 1$$

with subject to boundary conditions like $y(a) = A$ and $y(b) = B$.

Like the former cases, if any one of A, B or any present coefficient or forcing function is fuzzy valued, it is converted to a fuzzy fractional boundary value problem whose solutions come fuzzy in nature.

On the other hand, the system of fractional differential equations is in the form

$$D_t^\alpha U(t) = G(t, U(t))$$

where $U(t) = [u_1(t), u_2(t), \dots, u_n(t)]^T$ is the vector of unknown functions. $G(t, U(t))$ may be a vector fuzzy valued function. Initial or boundary conditions may also be fuzzy in nature.

The relationship between crisp and fuzzy with fractional and integer order differential equations is classified as follows:

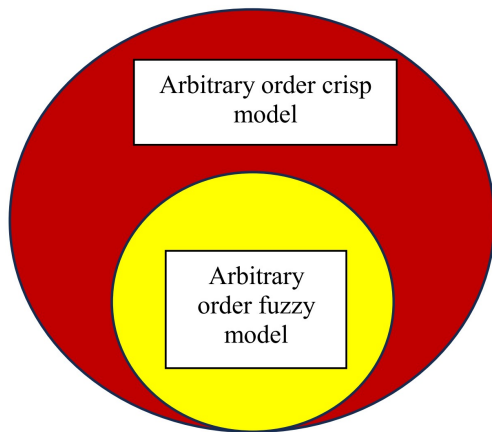


Figure 1. Comparison between an arbitrary order crisp and fuzzy model

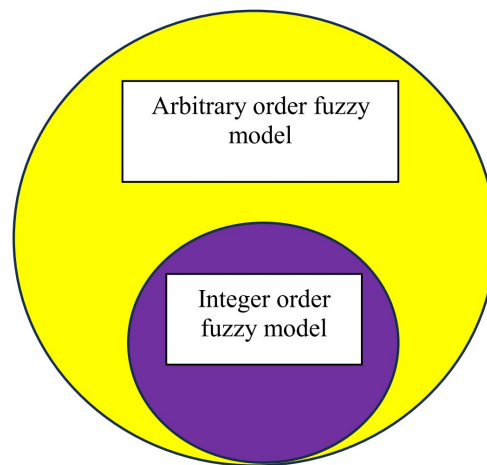


Figure 4. Comparison between an arbitrary order fuzzy and an integer order fuzzy model

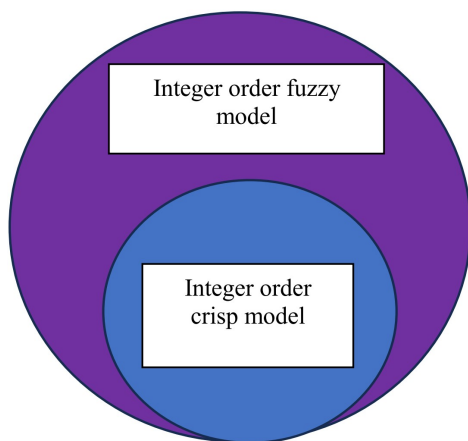


Figure 2. Comparison between integer order fuzzy and integer order crisp models

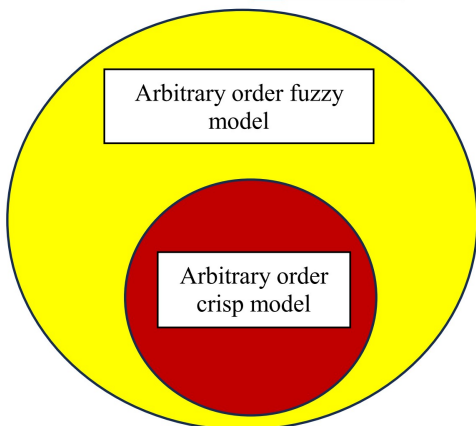


Figure 3. Comparison between an arbitrary order crisp and fuzzy model

The above Figures 1, 2, 3 and 4 present the conceptual set theoretical comparison between different models.

2. Fractional Order Derivative and Fractional Differential Equation

The necessity for numerous different fractional derivatives arises from the various and complex nature of real-world spectacles that cannot be satisfactorily modelled using classical integer-order derivatives. Several ideologies of fractional order derivatives offer unique mathematical properties and advantages that are particularly suitable for precise applications. These various definition ideologies enable further flexible and accurate real-world modelling. Here are some fractional derivatives. With respect to this derivative, the fractional differential type may be classified.

- (1) Riemann–Liouville Fractional Derivative (RLFD) [21,22]

$$D_b^\alpha G(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_b^x \frac{G(\tau)}{(x - \tau)^{\alpha-n+1}} d\tau$$

- (2) Caputo Fractional Derivative (CFD) [23,24]

$${}^c D_b^\alpha G(x) = \frac{1}{\Gamma(n - \alpha)} \int_b^x \frac{G^{(n)}(\tau)}{(x - \tau)^{\alpha-n+1}} d\tau$$

- (3) Grünwald–Letnikov Fractional Derivative (GLFD) [25,26]

$$D_{GL}^\alpha G(x) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lceil \frac{t-a}{h} \rceil} (-1)^k \binom{\alpha}{k} G(x - kh)$$

(4) Hadamard Fractional Derivative (HFD) [27,28]

$$D_H^\alpha G(x) = \frac{1}{\Gamma(n-\alpha)} \left(x \frac{d}{dx}\right)^n \int_b^x \left(\ln \frac{x}{\tau}\right)^{n-\alpha-1} \frac{G(\tau)}{\tau} d\tau$$

(5) Riesz Fractional Derivative (RFD) [29,30]

$$D_R^\alpha G(x) = -\frac{1}{2 \cos\left(\frac{\pi\alpha}{2}\right)} \left(-\infty D_x^\alpha G(x) + \infty D_x^\alpha G(x)\right)$$

(6) Erdélyi–Kober Fractional Derivative (EKFD) [31,32]

$$\left(D_{EK}^{\alpha,\eta} G\right)(x) = x^{-\alpha-\eta} \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \left[t^{n+\eta-1} \int_0^x (x-\tau)^{n-\alpha-1} \tau^{-\eta} G(\tau) d\tau \right]$$

(7) Marchaud Fractional Derivative (MFD) [33,34]

$$D_M^\alpha G(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{G(x) - G(x-\tau)}{\tau^{\alpha+1}} d\tau$$

(8) Chen Fractional Derivative (ChFD) [35,36]

$$D_C^\alpha G(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{G'(\tau)}{(x-\tau)^\alpha} d\tau$$

(9) Jumarie Fractional Derivative (JFD) [37,38]

$$D_J^\alpha G(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{G(\tau) - G(0)}{(x-\tau)^\alpha} d\tau$$

(10) Hilfer Fractional Derivative [39,40]

$$D_{b+}^{\alpha,\beta} G(x) = I_{b+}^{\beta(1-\alpha)} \left(\frac{d}{dx} \left[I_{b+}^{(1-\beta)(1-\alpha)} G(x) \right]\right)$$

Each of these derivatives has its own definitions, properties, and applications in fractional calculus. The comparison table is shown in Table 1.

Table 1. Comparison table between different crisp fractional derivatives

Derivative name	Definition based on	Initial conditions	Singularity situation at origin	Local or Nonlocal	Main application area and features
1. Riemann-Liouville	Integral, then derivatives	Fractional initial conditions	Yes	Nonlocal	Classical formulation, general-purpose
2. Caputo	Derivatives, then integral	Integer-order initial conditions	Yes	Nonlocal	Common in engineering and physical sciences problem
3. Grünwald–Letnikov	Limit of finite difference	Compatible with physical initial conditions	Yes	Nonlocal	Suitable for numerical methods
4. Hadamard	Logarithmic kernel Based integral	Complex	Mild	Nonlocal	Problems with scaling and symmetry
5. Riesz	Symmetric combination of left & right RL derivatives	Not standard	No	Nonlocal	fractional partial differential equations, quantum mechanics
6. Erdélyi-Kober	Generalized integral with Scaling of parameters	Some specialized conditions	Yes	Nonlocal	Problems with self-similarity or scaling based
7. Marchaud	Integral form like RL, defined on real domain	Functions not differentiable at origin	No	Nonlocal	Defined even when functions have no derivative
8. Chen	Extension of Newton-Leibniz ideology	Uses conformable approach	No	Local	General physical systems
9. Jumarie Modified RL	Modified RL by avoiding singularity	Zero-order initial conditions	No	Local	By handlings non-differentiable functions
10. Hilfer	Interpolates between RL and Caputo	Mixed-type ICs	Yes	Nonlocal	For generalization of model

The following Figure 5 shows the word density count for different fractional derivatives.

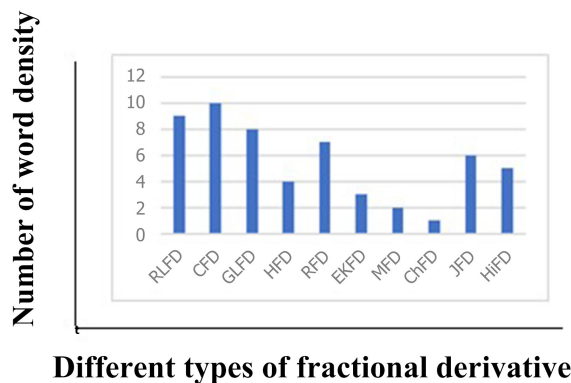


Figure 5. Approximated keyword frequency of different fractional derivatives during 2020 to 2024 in terms of billions of words in web search

Note: Diverse types of fractional derivative ideology are essential because numerous definitions capture several geometrical, physical, and mathematical properties that are appropriate to specific applications and important theories. For instance, the Riemann–Liouville and Caputo type fractional derivatives are well-suited for modelling memory and hereditary effects, but they differ in their handling of initial conditions. The Caputo-type derivative is frequently ideal in engineering applications due to its compatibility with classical initial conditions. Further definitions like the Hadamard, Grünwald–Letnikov or Conformable derivatives propose different formulations that may be more appropriate for numerical based computation or scale-invariant processes. Thus, the multiplicity of fractional derivative concepts permits greater flexibility and accuracy in modelling the dynamics of complex systems across important disciplines, such as physics, biology, economics and social sciences.

3. Fuzzy Fractional Order Derivative

Fuzzy fractional derivative perceptions are desirable for successfully modelling and analysing different types of dynamical systems that exhibit both memory-dependent behaviour with uncertainty or imprecision, which are common phenomena in real-world problems involving modelling. Traditional fractional derivatives handle memory effects only, but they assume precise initial conditions and parameters, which may not be realistic in complex systems modelling, which involves vague or incomplete information. By incorporating fuzzy set concepts or uncertainty into fractional calculus, fuzzy fractional derivatives offer a powerful framework for representing and solving problems where uncertainty and fractional dynamics coexist.

Like the different concepts of crisp fractional derivatives, fuzzy fractional derivatives are in a different form. Property-wise, all the fuzzy fractional derivatives are different and applicable for specific applications. Here are several well-known fuzzy fractional-order derivatives:

- (1) Fuzzy Riemann–Liouville Fractional Derivative [41,42]
- (2) Fuzzy Caputo Fractional Derivative [43,44]
- (3) Fuzzy Grünwald–Letnikov Fractional Derivative [45,46]
- (4) Fuzzy Hadamard Fractional Derivative [47,48]
- (5) Fuzzy Erdélyi–Kober Fractional Derivative [49]
- (6) Fuzzy Hilfer Fractional Derivative [50,51]
- (7) Fuzzy Jumarie Fractional Derivative [52,53]
- (8) Fuzzy Atangana–Baleanu Fractional Derivative [54,55]
- (9) Fuzzy Conformable Fractional Derivative [56,57]

These derivatives extend classical fractional derivatives to the fuzzy setting, enabling the modelling of systems with uncertainty and memory effects. The comparison table is shown in Table 2.

Table 2. Comparison table between different fuzzy fractional derivatives

Fractional Fuzzy Derivative	Used ideology	Approach for handling fuzzy functions	Initial Conditions	Applications area
1. Fuzzy Riemann–Liouville	RL Integral and derivative	Hukuhara or generalized Hukuhara derivative	Non-standard (fuzzy fractional initial conditions)	Useful for modeling fuzzy memory based problems
2. Fuzzy Caputo	Caputo derivative and integral	Strongly generalized derivative	Fuzzy initial values	Ideal in physical fuzzy applied systems
3. Fuzzy Grünwald-Letnikov	Limit of difference operator	Discrete fuzzy arithmetic	Flexible for numerical fuzzy problem simulations	Fuzzy numerical schemes and Simulations based problems
4. Fuzzy Hadamard	Logarithmic kernel based integration	Alpha-cut and gH-differentiability	Involves specific domain scaling	Modeling fuzzy systems with logarithmic based scaling
5. Fuzzy Erd ̇y-Kober	Generalized scaling integral	gH-differentiability or fuzzy transform methods	Involves fuzzy scaled conditions	Fuzzy self-similar problems
6. Fuzzy Hilfer	Interpolation of RL and Caputo idea	Generalized fuzzy Hilfer kernel	Mixed fuzzy initial conditions	Flexibility between fuzzy RL and fuzzy Caputo
7. Fuzzy Jumarie	Modified RL for Non differentiable functions	gH- or strongly gH-derivative for fuzzy continuity	More physical and simple fuzzy initial conditions	Good for fuzzy nondifferentiable function
8. Fuzzy Atangana–Baleanu	AB with Mittag-Leffler kernel	Fuzzy convolution integral using ML function	Non-local fuzzy memory conditions	Fuzzy non-singular kernel based modeling
9. Fuzzy Conformable	Linearized conformable derivative	Local fuzzy derivative via limiting definition	Compatible with fuzzy dynamics	Modeling under fuzzy uncertainty

4. Developments of Fuzzy Fractional Differential Equations

Here are the developments of fuzzy fractional differential equations and their applications. In Table 3, the comparative analysis is done.

Table 3. Comparison table between fuzzy fractional differential equations published works

Article details	Fuzzy Fractional differential equation form/types	Fractional derivative use	Theoretical or Application
[58] Agarwal et al.	$D^\alpha x(t) = f(t, x(t)), t \in (0, T], \alpha \in (0, 1]$	Crisp Riemann–Liouville fractional derivative	Theoretical
[59] Arshad and Lupulescu	$D^q u(t) = f(t, u(t)), \lim_{t \rightarrow 0^+} t^{1-q} u(t) = v_0, 0 < q < 1.$	Hukuhara derivative for a class of fractional differential equations with fuzzy initial values	Theoretical
[60] Allahviranloo et al.	$\begin{cases} ({}^{RL}D^q x)(t) = f(t, x) \\ x(t_0) = x^0 \in E \end{cases}$	Riemann-Liouville generalised H-differentiability	Theoretical
[61] Arshad	$D^q y(t) = f(t, y(t)), \lim_{t \rightarrow 0^+} t^{1-q} y(t) = y_0,$	Fuzzy derivative is considered in the Goetschel Voxman sense and fractional derivative is considered in the Riemann Liouville sense	Theoretical
[62] Ahmad et al.	$D_a^\beta x(t) = f(t, x(t)), 0 < \beta \leq 1, t > a, x(t_0) = x_0$	Crisp Caputo fractional derivative	Theoretical
[63] Takaci et al.	$D_t^{\beta_2} x(t, \lambda) + A(\lambda)D_t^{\beta_1} x(t, \lambda) = f(t, \lambda), 1 < \beta_2 < 2, 0 \leq \beta_1 \leq 1, 0 \leq \lambda \leq 1, t > 0$	The fuzzy fractional derivatives of function x by t are considered in the Caputo sense and Hukuhara difference	Theoretical
[64] Takaci et al.	$D_t^\beta x(t, \lambda) = A(\lambda)x(t, \lambda) + f(t, \lambda), x(0, \lambda) = x_0(\lambda), 0 < \beta < 1, 0 \leq \lambda \leq 1, t > 0$	Caputo Hukuhara-differentiability	Theoretical
[65] Ngo	$\begin{cases} {}_{gH}^C D_{a^+}^\alpha x(t) = f(t, x_t), t \geq 0, 0 < \alpha < 1 \\ x(t) = \varphi(t - a), a \geq t \geq a - \sigma \end{cases}$	Caputo’s generalized Hukuhara derivative	Theoretical

Table 3 continued

[66] Ahmadian et al.	$\begin{cases} ({}^C D_{0^+}^v y)(x) + y(x) = f(x), 0 < v \leq 1 \\ y(0) = y_0 \in E \end{cases}$	fuzzy Caputo fractional-order derivatives ($0 < v \leq 1$) in terms of shifted Chebyshev polynomials	Theoretical
[67] Mansouri et al.	$\begin{cases} {}^C D^\alpha w(t) = f(w(t), t) + g(w(t), t)\hat{C}(t), \alpha \in (0,1) \\ w(t) = w_t \end{cases}$	Caputo fractional derivative	Theoretical
[68] Rahman, and Ahmad	$\begin{cases} {}^C D^\beta \tilde{y}(x) = f[x, \tilde{y}(x)] \\ \tilde{y}(x_0) = [y_\alpha(0), \bar{y}_\alpha(0)] \end{cases}$ where $f \in C^F(a, b) \cap L^F(a, b)$ and $x_0 \in (a, b)$	Caputo fuzzy fractional derivative	Theoretical
[69] Abdollahi et al.	$\begin{cases} {}^C D^\alpha x(t) = \lambda x(t) + b(t), \lambda \in R \text{ and } b(t) \in R_F \\ x(0) = x_0 \in R_F \end{cases}$	Caputo generalized Hukuhara derivative	Theoretical
[70] Allahviranloo et al.	$\begin{cases} {}^F D_t^\alpha y_{gH}(t) = f(t, y(t)), t \in [0, T] \\ y(0) = y_0 \in R_F \end{cases}$ where $f: [0, T] \times R_F \rightarrow R_F$	fuzzy Caputo fractional derivative for fuzzy-valued functions	Theoretical
[71] Chen et al.	$\begin{cases} ({}^\rho D_{a^+}^{\alpha, \beta} x)(t) = f(x, x(t)), t \in [a, b] \\ ({}^\rho I_{a^+}^{1-\gamma} x)(a) = x_0 = \sum_{i=1}^m C_i x(t_i), \gamma = \alpha + \beta(1 - \alpha) \end{cases}$ where, $x \in R, 0 < \alpha < 1, 0 \leq \beta \leq 1, \gamma = \alpha + \beta(1 - \alpha)$ and $\rho > 0$	Hilfer–Katugampola fractional derivative	Theoretical
[42] Rahaman et al.	$\begin{cases} {}^{RL} D_x^\alpha \tilde{y}(x) = A\tilde{y}(x) + B \\ {}^{RL} D_x^{\alpha-1} \tilde{y}(x) = y_0^{RL} \end{cases}$	Riemann Liouville Hukuhara differentiability	Both
[72] Alderremy et al.	System of nonlinear equations	Caputo definition of fuzzy fractional differentiation	Applications
[73] Khakrangin et al.	$\begin{cases} {}^C D_t^\alpha y(t) = f(t, y(t)) \\ y^{(i)}(t_0) = \delta_i \in R_F \end{cases}$ where $y \in C^F[a, b] \cap L^F[a, b], t_0 \in [a, b]$	Caputo derivative	Theoretical
[74] Harir et al.	$\begin{cases} T_q x(t) = F(t, x(t)), q \in (0,1] \\ x(0) = x_0 \end{cases}$ where $F: (0, a) \times R_F \rightarrow R_F$	Fuzzy Conformable Fraction derivative	Theoretical
[75] Haq et al.	System of fractional differential equation	Caputo fractional derivative of a fuzzy function	Theoretical
[76] Akram et al.	Two-dimensional system of homogeneous fuzzy fractional differential equation,	Caputo fractional derivative	Both
[77] Akram et al.	$\begin{cases} {}^C D_{0^+}^{\tau_1} v(m) - \lambda {}^C D_{0^+}^{\tau_2} v(m) - \mu v(m) = g(m) \\ v_i(0) = v_i(0, \alpha), 0 \leq i \leq q - 1 \end{cases}$ where, $q - 1 \leq \tau_2 \leq q - 1, q - 1 \leq \tau_1 \leq q, q \geq 2, \lambda_2 \geq 1$	Caputo-type fractional derivative	Both
[78] Dwivedi et al.	$\begin{cases} {}^C D_t^\alpha x(t) = f(t, x(t)), t \in [t_0, T] \\ x(t_0) = x_0 \in F_R \end{cases}$	Fuzzy Caputo fractional derivative	Theoretical
[79] Dwivedi et al.	$\begin{cases} {}^C D_t^\alpha \tau(t) = f(t, \tau(t)), t \in [t_0, T] \\ \tau(t_0) = \tau_0 \in F_R \end{cases}$	Fuzzy fractional Caputo derivative	Theoretical
[80] Shahryari et al.	$\begin{cases} {}^C D_{t_0}^\alpha (\tilde{v}(t)) = \tilde{F}(t, \tilde{v}(t), \lambda_k(\tilde{v}(t_k))), t \in [t_k, t_{k+1}] \\ \tilde{v}(t_0) = \tilde{v}0 \end{cases}$ where $\tilde{F}: R^+ \times R_\tau \times R_\tau \rightarrow R_\tau$ and $\lambda_k: R_\tau \rightarrow R_\tau$	Fuzzy conformable fractional derivative	Theoretical
[81] Rahaman et al.	${}^C D \bar{q}(t) = -\bar{D}$ with $\bar{q}(0) = \bar{Q}$ and $\bar{q}(T) = \bar{0}$	Caputo gH derivative	Applications
[82] Shiri et al.	$\begin{cases} {}^C D_t^\beta u(t) = \mu u(t)[1 \ominus_{gH} u(t)], t \in [0, T] \\ u(0) = u_0 \end{cases}$ where $0 < \beta \leq 1 \dots$	Caputo Fractional Derivative and generalized Hukuhara derivative	Applications

Table 3 continued

[83] Harir et al.	$\begin{cases} y^{(q)}(t) = F(t, y(t)), q \in (0,1] \\ y(0) = y_0 \end{cases}$ <p>where $F: (0, a) \times R_F \rightarrow R_F$</p>	Fuzzy generalized conformable fractional derivative	Theoretical
[84] Mukherjee et al.	$\frac{\partial^\gamma \tilde{w}(\xi, t)}{\partial t^\gamma} = \tilde{p}(\xi) \frac{\partial^2 \tilde{w}(\xi, t)}{\partial \xi^2} + \tilde{q}(\xi)$ <p>where $\xi \in R, 0 < \xi < 1, t > 0, \gamma \in (0,1]$</p>	Caputo fuzzy fractional derivative	Theoretical
[85] Ghrissi et al.	$\begin{cases} {}_{gH}^C D^\rho w(\xi) = \bar{w}(\xi, w(\xi)), \rho \in [0,1], \xi \in [a, b] \\ w_{gH}^{(k)} = c_k, 0 \leq k < n \end{cases}$ <p>where $c_k (k = 0, \dots, n)$ is constant and \bar{w} is a continuous fuzzy function</p>	Caputo generalized Hukuhara differentiability	Theoretical
[86] El Mfadel et al.	$\begin{cases} {}^C D^q x(t) = f(t, x(t)), t \in J = [t_0, t_0 + \delta] \\ x(t_0) = x_0 \end{cases}$ <p>where $q \in [0,1], \delta > 0$</p>	Fuzzy Caputo fractional differentiability	Theoretical
[87] Subramanian et al.	System of fractional nonlinear equations	Fuzzy fractional Caputo's derivative	Applications
[88] Padmapriya et al.	$\begin{cases} {}^C D^\alpha \tilde{y}(t) + a_1 \tilde{y}'(t) + a_2 \tilde{y}(t) = f(t), 1 \leq \alpha \leq 2 \\ \tilde{y}(0) = \tilde{A} \\ \tilde{y}(T) = \tilde{B} \end{cases}$	Caputo fractional derivative	Theoretical
[89] Arshad et al.	Non-linear fuzzy fractional partial differential equation	Fractional fuzzy derivative in Caputo sense	Theoretical
[90] Ibrahim et al.	$\begin{cases} \Psi^{(a)}(\tau) = A\Psi(\tau) \oplus \Phi(\tau, \Psi(\tau)), \tau > 0 \\ \Psi(0) = \Psi_0 \end{cases}$ <p>where $\Psi^{(a)}(\tau)$ denotes the fuzzy conformable derivative</p>	Fuzzy conformable derivative	Theoretical
[91] Dhanalakshmi et al.	Integro differential equation	q-fractional Caputo generalized Hukuhara derivative	Theoretical
[92] Singh et al.	$\begin{cases} \frac{d^\beta \tilde{\eta}}{d\xi^\beta} \beta = \tilde{A} \otimes \tilde{\eta} \oplus \tilde{B} \\ \tilde{\eta}(\zeta_0) = \tilde{\eta}_0 \end{cases}$	Caputo fractional derivative and Hukuhara difference	Theoretical
[93] Hariharan and Udhayakumar	$\begin{cases} {}^{\omega} D_{0+}^{\tau, \omega} \tilde{x}(t) = A\tilde{x}(t) + F(t, \tilde{x}(t)) \\ {}^{\omega} I_{0+}^{\tau, \omega} \tilde{x}(0) = \tilde{x}_0 \end{cases}$ <p>where ${}^{\omega} D_{0+}^{\tau, \omega} \tilde{x}(t)$ is a Hilfer-Katugampola fractional derivative of order $\tau \in (0,1]$ and $\omega \in [0,1]$</p>	Hilfer-Katugampola fractional derivative	Theoretical
[94] Momena et al.	$\begin{cases} \frac{d^\alpha I_1(t)}{dt^\alpha} = -ap^{-b}, & 0 \leq t \leq t_1 \\ \frac{d^\alpha I_2(t)}{dt^\alpha} = -cp^{-b}, & t_1 \leq t \leq T \end{cases}$ <p>where $I_1(t_1) = I_2(t_1) = 0, 0 < \alpha \leq 1.0$.</p>	Caputo fractional-order derivative	Applications
[95] Rahaman et al.	$D_t^\beta \tilde{x}(t) = k\tilde{x}(t)$ with initial condition $D_t^{\beta-1} \tilde{x}(t_0) = x_0$	Riemann-Liouville fuzzy fractional derivative with Laplace transformation	Theoretical

5. Numerical Examples and Discussion

Example 1: Riemann-Liouville derivative of a fuzzy function: Consider $\tilde{g}(\tau) = (\tau^2 - 1, \tau^2, \tau^2 + 1)$ be a triangular fuzzy-valued function. Then evaluate the Riemann-Liouville fractional derivative of $\tilde{g}(\tau)$ of order $\lambda = \frac{1}{2}$, which is denoted by $D_{RL}^\lambda(\tilde{g}(\tau))$ for $\tau > 0$.

Solution: The Riemann-Liouville derivative for a crisp function $g(\tau)$ of order $\lambda \in (0,1)$ is

$$D_{RL}^\lambda(g(\tau)) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{d\tau} \int_0^\tau \frac{g(t)}{(\tau-t)^\lambda} dt$$

Now, we assume $g(\tau) = \tau^2 - 1$, then the Riemann-Liouville derivative of order $\lambda = \frac{1}{2}$ is

$$D_{RL}^\lambda(g(\tau)) = D_{RL}^{\frac{1}{2}}(\tau^2 - 1) = \frac{8}{3\sqrt{\pi}} \tau^{\frac{3}{2}} - \frac{1}{\sqrt{\pi}} \tau^{-\frac{1}{2}}$$

Similarly, when $g(\tau) = \tau^2$, then the Riemann-Liouville derivative of order $\lambda = \frac{1}{2}$ is

$$D_{RL}^\lambda(g(\tau)) = D_{RL}^{\frac{1}{2}}(\tau^2) = \frac{8}{3\sqrt{\pi}} \tau^{\frac{3}{2}}$$

and $g(\tau) = \tau^2 + 1$, then the Riemann-Liouville derivative of order $\lambda = \frac{1}{2}$ is

$$D_{RL}^\lambda(g(\tau)) = D_{RL}^{\frac{1}{2}}(\tau^2 + 1) = \frac{8}{3\sqrt{\pi}}\tau^{\frac{3}{2}} + \frac{1}{\sqrt{\pi}}\tau^{-\frac{1}{2}}$$

Therefore, then the Riemann-Liouville derivative of a triangular fuzzy-valued function $\tilde{g}(\tau) = (\tau^2 - 1, \tau^2, \tau^2 + 1)$ of order $\lambda = \frac{1}{2}$ is

$$D_{RL}^{\frac{1}{2}}(\tilde{g}(\tau)) = \left(\frac{8}{3\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{1}{\sqrt{\pi}}\tau^{-\frac{1}{2}}, \frac{8}{3\sqrt{\pi}}\tau^{\frac{3}{2}}, \frac{8}{3\sqrt{\pi}}\tau^{\frac{3}{2}} + \frac{1}{\sqrt{\pi}}\tau^{-\frac{1}{2}} \right)$$

for all $\tau > 0$.

Example 2: Caputo Derivative of a fuzzy function:

Assume $\tilde{h}(\tau)$ be a fuzzy-valued function at time τ and define as $\tilde{h}(\tau) = (\tau^2 - t, \tau^2, \tau^2 + t)$. Then compute the Caputo fractional derivative of order $\lambda \in (0,1)$, defined by $D_C^\lambda(\tilde{h}(\tau))$.

Solution: The Caputo derivative for a crisp function $h(\tau)$ of order $\lambda \in (0,1)$ is

$$D_C^\lambda(\tilde{h}(\tau)) = \frac{1}{\Gamma(1-\lambda)} \int_0^\tau \frac{h'(t)}{(\tau-t)^\lambda} dt$$

Then the Caputo derivative for a crisp function $h(\tau) = \tau^n$ of order α is

$$D_C^\lambda(\tau^n) = \frac{\Gamma(n+1)}{\Gamma(n-\lambda+1)} \tau^{n-\lambda}.$$

Now, we assume $h(\tau) = \tau^2 - t$, then the Caputo derivative of order $\lambda = \frac{1}{2}$ is

$$\begin{aligned} D_C^\lambda(h(\tau)) &= D_C^{\frac{1}{2}}(\tau^2 - t) = D_C^{\frac{1}{2}}(\tau^2) - D_C^{\frac{1}{2}}(t) \\ &= \frac{2}{\Gamma(3-\lambda)} \tau^{2-\lambda} - \frac{1}{\Gamma(2-\lambda)} \tau^{1-\lambda} \end{aligned}$$

Similarly, for $h(\tau) = \tau^2$, then the Caputo derivative of order $\lambda = \frac{1}{2}$ is

$$D_C^\lambda(h(\tau)) = D_C^{\frac{1}{2}}(\tau^2) = \frac{2}{\Gamma(3-\lambda)} \tau^{2-\lambda}$$

and $h(\tau) = \tau^2 + t$, then the Caputo derivative of order $\lambda = \frac{1}{2}$ is

$$\begin{aligned} D_C^\lambda(h(\tau)) &= D_C^{\frac{1}{2}}(\tau^2 + t) = D_C^{\frac{1}{2}}(\tau^2) + D_C^{\frac{1}{2}}(t) \\ &= \frac{2}{\Gamma(3-\lambda)} \tau^{2-\lambda} + \frac{1}{\Gamma(2-\lambda)} \tau^{1-\lambda} \end{aligned}$$

Therefore, then the Caputo derivative of a triangular fuzzy-valued function $\tilde{h}(\tau) = (\tau^2 - t, \tau^2, \tau^2 + t)$ of order $\lambda = \frac{1}{2}$ is

$$\begin{aligned} D_C^{\frac{1}{2}}(\tilde{h}(\tau)) &= \left(\frac{2}{\Gamma(3-\lambda)} \tau^{2-\lambda} - \frac{1}{\Gamma(2-\lambda)} \tau^{1-\lambda}, \right. \\ &\quad \left. \frac{2}{\Gamma(3-\lambda)} \tau^{2-\lambda}, \frac{2}{\Gamma(3-\lambda)} \tau^{2-\lambda} \right. \\ &\quad \left. + \frac{1}{\Gamma(2-\lambda)} \tau^{1-\lambda} \right) \end{aligned}$$

for all $\tau > 0$.

Example 3: Solution of fuzzy Riemann–Liouville fractional differential equation: Solve $D_{RL}^\lambda(\tilde{g}(\tau)) = 4\tilde{g}(\tau)$ with $\lambda \in (0,1]$ and satisfy the initial condition $D_{RL}^\lambda(\tilde{g}(\tau_0)) = \tilde{g}_0$ with $\tilde{g}_0 = (8,10,12)$.

Solution: The solution of the fuzzy Riemann Liouville differential equation is

$$\tilde{g}(\tau)^\alpha = \left[(8 + 2\alpha)\tau^{\lambda-1} E_{\lambda,\lambda}(4\tau^\lambda), (12 - 2\alpha)\tau^{\lambda-1} E_{\lambda,\lambda}(4\tau^\lambda) \right]$$

Example 4: Solution of fuzzy Caputo differential equation: Solve fuzzy fractional differential equation $D_C^\lambda(\tilde{g}(\tau)) = \tilde{g}(\tau) + 1$ with the initial condition $\tilde{g}(1) = (0.5,1,1.5)$.

Solution: The solution is written as

$$\tilde{g}(\tau)^\alpha = \left[(0.5 + 0.5\alpha)E_\lambda(\tau^\lambda) + \tau^\lambda E_{\lambda,\lambda+1}(\tau^\lambda), (1.5 - 0.5\alpha)E_\lambda(\tau^\lambda) + \tau^\lambda E_{\lambda,\lambda+1}(\tau^\lambda) \right]$$

6. Future Research Scope

Here are some conceivable future extensions for fuzzy fractional differential equations:

1. Cutting-edge solution techniques: FFDEs can be extended with several cutting-edge solution techniques by integrating unconventional analytical and computational approaches that enhance accuracy, effectiveness, and applicability in complex situations. For example, it combines fuzzy logic with machine learning/ artificial intelligence algorithms to approximate solutions. The development of new numerical schemes, such as adaptive mesh refinement, spectral methods, and high-order finite difference methods, has addressed several complex issues in FFDEs.

2. Higher-dimensional and complex systems modelling: Fuzzy fractional differential equations (FFDEs) can be extended to higher-dimensional and complex systems by participating multidimensional fuzzy sets and matrix-valued fractional operators to model uncertainties and relations among multiple state variables. This extension may enable the illustration of complex, dynamic behavior in systems characterized by both memory effects and uncertain information across several interconnected dimensions.

3. Modified Optimizations techniques applications: Fuzzy fractional differential equations (FFDEs) can be solved through modified optimization methods to address complex and real-world problems that include uncertainty, memory effects, as well as several dynamic constraints. By employing modified optimization algorithms, such as fuzzy-enhanced particle swarm optimization (PSO), genetic algorithms (GA), differential evolution (DE), and swarm intelligence approaches, it is possible to successfully identify optimal parameters and control strategies within FFDE-related models.

4. Considering hybrid models: Fuzzy fractional differential equations (FFDEs) can be prolonged by taking hybrid models which integrate multiple mathematics-based frameworks to better capture the complexity of real-world issues characterized by fuzziness, memory effects, and nonlinear dynamics. For example, these hybrid models may combine FFDEs with stochastic processes, neural networks, cellular automata and other approaches, allowing for the representation of both deterministic and probabilistic behaviors.

5. Increase computational efficiency: Fuzzy fractional differential equations (FFDEs) can be stretched to increase computational efficiency by employing advanced numerical algorithms and parallel computing techniques. Methods of this kind can meaningfully accelerate computations without sacrificing accuracy.

6. Interdisciplinary applications: Fuzzy fractional differential equations (FFDEs) can be applied to several interdisciplinary applications by acclimating their unique ability to real-life model systems with both memory effects and vagueness across diverse scientific and engineering fields. In biomedical applications, FFDEs can designate complex physiological processes involving uncertain measurements and transmissible properties, such as the drug diffusion problem or neural dynamics applications. In the field of financial theory, it offers robust tools for modelling market behaviors with uncertain data. Environmental science models from FFDEs aid in simulating climate modelling dynamics or pollutant diffusion under uncertain conditions. Furthermore, in engineering problems, they augment the design and control of systems subjected to uncertain inputs and time-dependent behaviors, such as robotics, smart grids, and structural engineering. By integrating the domain-specific information with FFDE modelling-based frameworks, interdisciplinary submissions adopt adaptive and innovative solutions to complex as well as real-world problems involving uncertainty, nonlinearity, and memory effects.

These guidelines can offer several profound insights and open new avenues for research in this emerging field.

7. Conclusions

Fuzzy fractional differential equations (FFDEs) provide a powerful structural framework for modelling various complex systems, including those with uncertainty and memory-based effects. This review paper concisely summarizes the theoretical progress, solution practices, and miscellaneous applications of FFDEs in various real-life fields, including economics, sciences, and engineering domains. While noteworthy progress has been made, tasks remain in developing more well-organized scientific computational methods that encompass FFDEs in higher-dimensional systems and exploring their real-world applicability. The future research should

emphasize hybrid models, optimization procedures, and interdisciplinary applications to further improve the usefulness of FFDEs.

Acknowledgments

This publication was supported by College of Science, Qassim University, Buraydah, Saudi Arabia.

REFERENCES

- [1] Podlubny I, "Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications," in Academic Press, Elsevier, 1998, vol. 198, pp. 1-308.
- [2] Oldham K, Spanier J, "The fractional calculus theory and applications of differentiation and integration to arbitrary order," in Academic Press, Elsevier, 1974, vol. 111, pp. 1-218.
- [3] Caputo M, "Linear models of dissipation whose Q is almost frequency independent—II," in Geophysical Journal International, 1967, vol. 13, no. 5, pp. 529-539.
- [4] Hilfer R, "Applications of fractional calculus in physics," in World Scientific, 2000, pp. 1-429.
- [5] Mainardi F, "Fractional relaxation-oscillation and fractional diffusion-wave phenomena," in Chaos, Solitons & Fractals, 1996, vol. 7, no. 9, pp. 1461-1477.
- [6] Herrmann R, "Fractional calculus: an introduction for physicists," in World Scientific, 2nd ed, 2011, pp. 65-90.
- [7] Rahaman M, Chalishajar D, Gazi K H, Alam S, Salahshour S, Mondal S P, "Fractional calculus for type 2 interval-valued functions," Fractal and Fractional, vol. 9, no. 2, pp. 102, 2025. DOI: 10.3390/fractalfract9020102
- [8] Ross B, "Fractional Calculus and Its Applications," Lecture Notes in Mathematics, Springer Berlin, Heidelberg, 1st ed, 1975, vol. VI, pp. 1-386. DOI: 10.1007/BFb0067095
- [9] Bagley R L, Torvik P J, "A theoretical basis for the application of fractional calculus to viscoelasticity," Journal of Rheology, vol. 27, no. 3, pp. 201-210, 1983. DOI: 10.1122/1.549724
- [10] Machado J T, Kiryakova V, Mainardi F, "Recent history of fractional calculus," Communications in nonlinear science and numerical simulation, vol. 16, no. 3, pp. 1140-1153, 2011. DOI: 10.1016/j.cnsns.2010.05.027
- [11] Singh P, Allahviranloo T, Amirteimoori A, Shahriari M, Gazi K H, Mondal S P, "Fuzzy Prey–Predator Model with Holling Type II Response and Its Application in Pest Control," New Mathematics and Natural Computation, 1-26, 2025. DOI: 10.1142/S1793005727500086
- [12] Acharya A, Mahata A, Karak M, Sil N, Mukherjee S, Mondal S P, Roy B, "Stability Analysis of Diabetes

- Mellitus Model in Neutrosophic Fuzzy Environment.” *Franklin Open*, vol. 8, no. 100144, 2024. DOI: 10.1016/j.fraope.2024.100144
- [13] Kahraman C, “Fuzzy applications in industrial engineering,” in Heidelberg: Springer, 2006, vol. 201, pp. 1-613. DOI: 10.1007/3-540-33517-X
- [14] Phuong N H, Kreinovich V, “Fuzzy logic and its applications in medicine,” *International Journal of Medical Informatics*, vol. 62, no. 2-3, pp. 165-173, 2001. DOI: 10.1016/S1386-5056(01)00160-5
- [15] Gazi K H, Raisa N, Biswas A, Azizzadeh F, Mondal S P, “Finding the most important criteria in women's empowerment for sports sector by pentagonal fuzzy DEMATEL methodology,” *Spectrum of Decision Making and Applications*, vol. 2, no. 1, pp. 28-52, 2025. DOI: 10.31181/sdmap21202510
- [16] Shaw A K, Rahaman M, Mondal S P, Chatterjee B, Alam S, “Solution of a mining equipment maintenance system model in imprecise environment,” *International Journal of Operational Research*, vol. 51, no. 4, pp. 562-598, 2024. DOI: 10.1504/IJOR.2024.143342
- [17] Zimmermann H J, “Practical applications of fuzzy technologies,” in Springer Science & Business Media, 2012, vol. 6, pp. 1-646.
- [18] Demirci M, “Fuzzy functions and their applications,” *Journal of Mathematical Analysis and Applications*, vol. 252, no. 1, pp. 495, 2000. DOI: 10.1006%20jmaa.2000.7185
- [19] Mlynek D J, Patyra M J, “Fuzzy logic: implementation and applications,” in Springer Science & Business Media, 1st ed, 2012, pp. 1-314. DOI: 10.1007/978-3-322-88955-3
- [20] Lodwick W A, Kacprzyk J, “Fuzzy optimization: Recent advances and applications,” in Springer Science & Business Media, 2010, vol. 254, pp. 1-530. DOI: 10.1007/978-3-642-13935-2
- [21] Hilfer R, “Fractional diffusion based on Riemann-Liouville fractional derivatives,” *The Journal of Physical Chemistry B*, vol. 104, no. 16, pp. 3914-3917, 2000. DOI: 10.1021/jp9936289
- [22] Nonnenmacher T F, Metzler R, “On the Riemann-Liouville fractional calculus and some recent applications,” *Fractals*, vol. 3, no. 03, pp. 557-566, 1995. DOI: 10.1142/S0218348X95000497
- [23] Almeida R, “A Caputo fractional derivative of a function with respect to another function,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 44, pp. 460-481, 2017. DOI: 10.1016/j.cnsns.2016.09.006
- [24] Gao G H, Sun Z Z, Zhang H W, “A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications,” *Journal of Computational Physics*, vol. 259, pp. 33-50, 2014. DOI: 10.1016/j.jcp.2013.11.017
- [25] Abdelouahab M S, Hamri N E, “The Grünwald-Letnikov fractional-order derivative with fixed memory length,” *Mediterranean Journal of Mathematics*, vol. 13, no. 2, pp. 557-572. DOI: 10.1007/s00009-015-0525-3
- [26] Murio D A, “Stable numerical evaluation of Grünwald-Letnikov fractional derivatives applied to a fractional IHCP,” *Inverse Problems in Science and Engineering*, vol. 17, no. 2, pp. 229-243, 2009. DOI: 10.1080/17415970802082872
- [27] Kilbas A A, “Hadamard-type fractional calculus,” *Journal of the Korean Mathematical Society*, vol. 38, no. 6, pp. 1191-1204, 2001.
- [28] Jarad F, Abdeljawad T, Baleanu D, “Caputo-type modification of the Hadamard fractional derivatives,” *Advances in Difference Equations*, vol. 2012, no. 1, pp. 142, 2012. DOI: 10.1186/1687-1847-2012-142
- [29] Yang Q, Liu F, Turner I, “Numerical methods for fractional partial differential equations with Riesz space fractional derivatives,” *Applied Mathematical Modelling*, vol. 34, no. 1, pp. 200-218, 2010. DOI: 10.1016/j.apm.2009.04.006
- [30] Agrawal O, “Fractional variational calculus in terms of Riesz fractional derivatives,” *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 24, pp. 6287, 2007. DOI: 10.1088/1751-8113/40/24/003
- [31] Odibat Z, Baleanu D, “On a new modification of the Erdélyi-Kober fractional derivative,” *Fractal and Fractional*, vol. 5, no. 3, pp. 121, 2021. DOI: 10.3390/fractalfract5030121
- [32] Pagnini G, “Erdélyi-Kober fractional diffusion,” *Fractional calculus and applied analysis*, vol. 15, no. 1, pp. 117-127, 2012. DOI: 10.2478/s13540-012-0008-1
- [33] Rogosin S, Dubatovskaya M, “Letnikov vs. Marchaud: A survey on two prominent constructions of fractional derivatives,” *Mathematics*, vol. 6, no. 1, pp. 1-15, 2017. DOI: 10.3390/math6010003
- [34] Chandra S, Abbas S, “Analysis of mixed Weyl-Marchaud fractional derivative and box dimensions,” *Fractals*, vol. 29, no. 06, pp. 2150145, 2021. DOI: 10.1142/S0218348X21501450
- [35] Lu J G, Chen G, “A note on the fractional-order Chen system,” *Chaos, Solitons & Fractals*, vol. 27, no. 3, pp. 685-688, 2006. DOI: 10.1016/j.chaos.2005.04.037
- [36] Elbadri M, Abdoon M A, Almutairi D K, Almutairi D M, Berir M, “Numerical Simulation and Solutions for the Fractional Chen System via Newly Proposed Methods,” *Fractal and Fractional*, vol. 8, no. 12, pp. 709, 2024. DOI: 10.3390/fractalfract8120709
- [37] Jumarie G, “Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results,” *Computers & Mathematics with Applications*, vol. 51, no. 9-10, pp. 1367-1376, 2006. DOI: 10.1016/j.camwa.2006.02.001
- [38] Jumarie G, “Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution,” *Journal of Applied Mathematics and Computing*, vol. 24, no. 1, pp. 31-48, 2007. DOI: 10.1007/BF02832299
- [39] Sousa J V D C, De Oliveira E C, “On the ψ -Hilfer fractional derivative,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 60, pp. 72-91, 2018. DOI: 10.1016/j.cnsns.2018.01.005

- [40] Furati K M, Kassim M D, "Existence and uniqueness for a problem involving Hilfer fractional derivative," *Computers & Mathematics with applications*, vol. 64, no. 6, pp. 1616-1626, 2012. DOI: 10.1016/j.camwa.2012.01.009
- [41] Sana G, Khan M B, Noor M A, Mohammed P O, Chu Y M, "Harmonically convex fuzzy-interval-valued functions and fuzzy-interval Riemann-Liouville fractional integral inequalities," *Int. J. Comput. Intell. Syst.*, vol. 14, no. 1, pp. 1809-1822, 2021. DOI: 10.2991/ijcis.d.210620.001
- [42] Rahaman M, Mondal S P, Alam S, Khan N A, Biswas A, "Interpretation of exact solution for fuzzy fractional non-homogeneous differential equation under the Riemann-Liouville sense and its application on the inventory management control problem," *Granular Computing*, vol. 6, no. 4, pp. 953-976, 2021. DOI: 10.1007/s41066-020-00241-3
- [43] El Mfadel A, Melliani S, Elomari M H, "A note on the stability analysis of fuzzy nonlinear fractional differential equations involving the Caputo fractional derivative," *International Journal of Mathematics and Mathematical Sciences*, vol. 2021, no. 1, pp. 7488524, 2021. DOI: 10.1155/2021/7488524
- [44] An T V, Phu N D, Van Hoa N, "A survey on non-instantaneous impulsive fuzzy differential equations involving the generalized Caputo fractional derivative in the short memory case," *Fuzzy Sets and Systems*, vol. 443, pp. 160-197, 2022. DOI: 10.1016/j.fss.2021.10.008
- [45] Khan N A, Razzaq O A, Ara A, Riaz F, "Numerical solution of system of fractional differential equations in imprecise environment," *Num. Simul. Brain Imag. Turb. Flow*, 2016, pp. 167-186. DOI: 10.5772/64150
- [46] Elgezouli D E, Eltayeb H, Abdoon M A, "Novel GPID: Grünwald-Letnikov fractional PID for enhanced adaptive cruise control," *Fractal and Fractional*, vol. 8, no. 12, pp. 751, 2024. DOI: 10.3390/fractalfrac8120751
- [47] Amir F I A, Moussaoui A, Shafqat R, El Omari M H, Melliani S, "The Hadamard ψ -Caputo tempered fractional derivative in various types of fuzzy fractional differential equations," *Soft Computing*, vol. 28, no. 17, pp. 9253-9270, 2024. DOI: 10.1007/s00500-024-09821-w
- [48] Khan M B, Mohammed P O, Noor M A, Hamed Y S, "New Hermite-Hadamard inequalities in fuzzy-interval fractional calculus and related inequalities," *Symmetry*, vol. 13, no. 4, pp. 673, 2021. DOI: 10.3390/sym13040673
- [49] Liang X, Zhu Y, He L, "Solutions and Properties of Caputo-Erdélyi-Kober Uncertain Fractional Differential Equations," *International Journal of Mathematical Analysis*, vol. 16, no. 4, pp. 177-191, 2022. DOI: 10.12988/ijma.2022.912434
- [50] Vanitha K, Saradha M, Antline Nisha B, Udhayakumar R, "An Approach to Solve Gas Dynamic Equation by Fuzzy Hilfer Fractional Differential Equation," *Contemporary Mathematics*, vol. 5, no. 32, pp. 2709-2723, 2024. DOI: 10.37256/cm.5320245035
- [51] Hariharan R, Udhayakumar R, "Approximate Controllability for Sobolev-Type Fuzzy Hilfer Fractional Neutral Integro-Differential Inclusion with Clarke Subdifferential Type," *Qualitative Theory of Dynamical Systems*, vol. 24, no. 1, pp. 53, 2025. DOI: 10.1007/s12346-024-01214-6
- [52] Jumarie G, "Derivation of an amplitude of information in the setting of a new family of fractional entropies," *Information Sciences*, vol. 216, pp. 113-137, 2012. DOI: 10.1016/j.ins.2012.06.008
- [53] Jumarie G, "Path probability of random fractional systems defined by white noises in coarse-grained time applications of fractional entropy," *Frac. Diff. Eq.*, vol. 1, no. 1, pp. 45-87, 2011.
- [54] Allahviranloo T, Ghanbari B, "On the fuzzy fractional differential equation with interval Atangana-Baleanu fractional derivative approach," *Chaos, Solitons & Fractals*, vol. 130, no. 109397, 2020. DOI: 10.1016/j.chaos.2019.109397
- [55] Verma L, Meher R, "Study on generalized fuzzy fractional human liver model with Atangana-Baleanu-Caputo fractional derivative," *The European Physical Journal Plus*, vol. 137, no. 11, pp. 1-20, 2022. DOI: 10.1140/epjp/s13360-022-03396-x
- [56] Harir A, Melliani S, Chadli L S, "Fuzzy generalized conformable fractional derivative," *Advances in fuzzy systems*, vol. 2020, no. 1, pp. 1954975, 2020. DOI: 10.1155/2020/1954975
- [57] Ghaffari M, Allahviranloo T, Abbasbandy S, Azhini M, "Generalized Hukuhara conformable fractional derivative and its application to fuzzy fractional partial differential equations," *Soft Computing*, vol. 26, no. 5, pp. 2135-2146, 2022. DOI: 10.1007/s00500-021-06637-w
- [58] Agarwal R P, Lakshmikantham V, Nieto J J, "On the concept of solution for fractional differential equations with uncertainty," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 6, pp. 2859-2862, 2010. DOI: 10.1016/j.na.2009.11.029
- [59] Arshad S, Lupulescu V, "Fractional differential equation with the fuzzy initial condition," *Electronic Journal of Differential Equations (EJDE)*, vol. 2011, no. 34, pp. 1-8, 2011.
- [60] Allahviranloo T, Abbasbandy S, Salahshour S, "Fuzzy fractional differential equations with Nagumo and Krasnoselskii-Krein condition," in *Proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology*, Atlantis Press, August 2011, pp. 1038-1044. DOI: 10.2991/eusflat.2011.39
- [61] Arshad S, "On existence and uniqueness of solution of fuzzy fractional differential equations," *Iranian Journal of Fuzzy Systems*, vol. 10, no. 6, pp. 137-151, 2013. <https://sid.ir/paper/617989/en>
- [62] Ahmad M Z, Hasan M K, Abbasbandy S, "Solving Fuzzy Fractional Differential Equations Using Zadeh's Extension Principle," *The Scientific World Journal*, vol. 2013, no. 1, pp. 454969, 2013. DOI: 10.1155/2013/454969
- [63] Takači D, Takači A, Takači A, "On the operational solutions of fuzzy fractional differential equations," *Fractional Calculus and Applied Analysis*, vol. 17, no. 4, pp. 1100-1113, 2014. DOI: 10.2478/s13540-014-0216-y
- [64] Takaci D, Takaci A, Takaci A, "On the solutions of fuzzy fractional differential equation," *TWMS Journal of Applied and Engineering Mathematics*, vol. 4, no. 1, pp.

- 98-103, 2014.
- [65] Ngo V H, "Fuzzy fractional functional integral and differential equations," *Fuzzy Sets and Systems*, vol. 280, no. C, pp. 58-90, 2015. DOI: 10.1016/j.fss.2015.01.009
- [66] Ahmadian A, Salahshour S, Chan C S, "Fractional differential systems: a fuzzy solution based on operational matrix of shifted Chebyshev polynomials and its applications," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 1, pp. 218-236, 2016. DOI: 10.1109/TFUZZ.2016.2554156
- [67] Mansouri S S, Gachpazan M, Fard O S, "Existence, uniqueness and stability of fuzzy fractional differential equations with local Lipschitz and linear growth conditions," *Advances in Difference Equations*, vol. 2017, no. 1, pp. 240, 2017. DOI: 10.1186/s13662-017-1293-0
- [68] Rahman N A, Ahmad M Z, "Solving fuzzy fractional differential equations using fuzzy Sumudu transform," *J. Nonlinear Sci. Appl.*, vol. 10, no. 5, pp. 2620-2632, 2017. DOI: 10.22436/jnsa.010.05.28
- [69] Abdollahi R, Farshbaf Moghimi M B, Khastan A, Hooshmandasl M R, "Linear fractional fuzzy differential equations with Caputo derivative," *Computational Methods for Differential Equations*, vol. 7, no. 2, pp. 252-265, 2019. <https://dor.isc.ac/dor/20.1001.1.23453982.2019.7.2.8.2>
- [70] Allahviranloo T, Noeiaghdam Z, Noeiaghdam S, Nieto J J, "A fuzzy method for solving fuzzy fractional differential equations based on the generalized fuzzy Taylor expansion," *Mathematics*, vol. 8, no. 12, pp. 2166, 2020. DOI: 10.3390/math8122166
- [71] Chen X, Gu H, Wang X, "Existence and uniqueness for fuzzy differential equation with Hilfer–Katugampola fractional derivative," *Advances in difference equations*, vol. 2020, no. 1, pp. 241, 2020. DOI: 10.1186/s13662-020-02696-9
- [72] Alderremy A A, Gómez-Aguilar J F, Aly S, Saad K M, "A fuzzy fractional model of coronavirus (COVID-19) and its study with Legendre spectral method," *Results in Physics*, vol. 21, no. 103773, pp. 1-10, 2021. DOI: 10.1016/j.rinp.2020.103773
- [73] Khakrangin S, Allahviranloo T, Mikaeilvand N, Abbasbandy S, "Numerical solution of fuzzy fractional differential equation by Haar wavelet," *Applications and Applied Mathematics: An International Journal (AAM)*, vol. 16, no. 1, pp. 14, 2021.
- [74] Harir A, Melliani S, Chadli L S, "Fuzzy conformable fractional differential equations," *International Journal of Differential Equations*, vol. 2021, no. 1, pp. 6655450, 2021. DOI: 10.1155/2021/6655450
- [75] Haq E U, Hassan Q M U, Ahmad J, Ehsan K, "Fuzzy solution of system of fuzzy fractional problems using a reliable method," *Alexandria engineering journal*, vol. 61, no. 4, pp. 3051-3058, 2022. DOI: 10.1016/j.aej.2021.08.034
- [76] Akram M, Muhammad G, Allahviranloo T, Ali G, "A solving method for two-dimensional homogeneous system of fuzzy fractional differential equations," *AIMS Mathematics*, vol. 8, no. 1, pp. 228–263, 2022. DOI: 10.3934/math.2023011
- [77] Akram M, Muhammad G, Allahviranloo T, Ali G, "New analysis of fuzzy fractional Langevin differential equations in Caputo's derivative sense," *AIMS Mathematics*, vol. 7, no. 10, pp. 18467–18496, 2022. DOI: 10.3934/math.20221016
- [78] Dwivedi A, Rani G, Dabas J, Gautam G R, "On the concept of solutions for fuzzy fractional initial value problem," *J. Comput. Math.*, vol. 6, no. 1, pp. 310-329, 2022. DOI: 10.26524/cm136
- [79] Dwivedi A, Rani G, Gautam G R, "Existence of solutions of fuzzy fractional differential equations," *Palestine Journal of Mathematics*, vol. 11, pp. 125-132, 2022.
- [80] Shahryari N, Allahviranloo T, Abbasbandy S, "Numerical Method to Solve a Hybrid Fuzzy Conformable Fractional Differential Equations," *New Mathematics and Natural Computation*, vol. 18, no. 03, pp. 629-655, 2022. DOI: 10.1142/S1793005722500326
- [81] Rahaman M, Mondal S P, Chatterjee B, Alam S, Shaikh A A, "Generalization of classical fuzzy economic order quantity model based on memory dependency via fuzzy fractional differential equation approach," *Journal of Uncertain Systems*, vol. 15, no. 01, pp. 2250003, 2022. DOI: 10.1142/S1752890922500039
- [82] Shiri B, Alijani Z, Karaca Y, "A power series method for the fuzzy fractional logistic differential equation," *Fractals*, vol. 31, no. 10, pp. 2340086, 2023. DOI: 10.1142/S0218348X23400868
- [83] Harir A, Melliani S, Chadli L S, "Solutions of fuzzy fractional differential equations," *Iranian Journal of Mathematical Sciences and Informatics*, vol. 61, no. 4, pp. 3051-3058, 2022. DOI: 10.18576/pfda/100206
- [84] Mukherjee S, Kumar A, Kumbhakar S, "Numerical simulation of fuzzy fractional differential equations using a reliable technique," *Journal of fuzzy extension and applications*, vol. 5, no. 2, pp. 159-172, 2024. DOI: 10.22105/jfea.2024.449034.1418
- [85] Ghrissi B, Amara A, Henka Y, "Higher-order fuzzy fractional differential equations: on the existence, uniqueness and Hyers–Ulam–Rassias stability of solutions," *Journal of Fuzzy Extension and Applications*, vol. 5, no. 3, pp. 353-364, 2024. DOI: 10.22105/jfea.2024.453123.1448
- [86] El Mfadel A, Melliani S A I D, Elomari M, "On the initial value problem for fuzzy nonlinear fractional differential equations," *Kragujevac J Math*, vol. 48, no. 4, pp. 547-554, 2024. DOI: 10.46793/KgJMat2404.547M
- [87] Subramanian S, Kumaran A, Ravichandran S, Venugopal P, Dhahri S, Ramasamy K, "Fuzzy fractional Caputo derivative of susceptible-infectious-removed epidemic model for childhood diseases," *Mathematics*, vol. 12, no. 3, pp. 466, 2024. DOI: 10.3390/math12030466
- [88] Padmapriya V, Kaliyappan M, "Solutions of Fuzzy Fractional Boundary Value Problems: A Novel Approach," *Mathematics and Statistics*, vol. 12, no. 5, pp. 494-500, 2024. DOI: 10.13189/ms.2024.120511
- [89] Arshad M, Khan S, Khan H, Ali H, Ali I, "Fractional analysis of non-linear fuzzy partial differential equations by using a direct procedure," *Scientific Reports*, vol. 14, no. 1, pp. 9627, 2024. DOI: 10.1038/s41598-024-60123-5

- [90] Ibrahim Abdou Amir F, El Ghazouani A, El Omari M H, Melliani S, "Fuzzy Fractional Differential Equation Involving the Fuzzy Conformable Derivative and the α -Semigroups," *The Scientific World Journal*, vol. 2024, no. 1, pp. 9993669, 2024. DOI: 10.1155/2024/9993669
- [91] Dhanalakshmi P, Padmavathi S, "Outcomes for The Analysis of The Existence of Fuzzy Fractional Differential Equations," *Communications on Applied Nonlinear Analysis*, vol. 31, no. 6s, pp. 35-48, 2024. DOI: 10.52783/cana.v31.1117
- [92] Singh P, Gazi K H, Rahaman M, Salahshour S, Mondal S P, "A fuzzy fractional power series approximation and Taylor expansion for solving fuzzy fractional differential equation," *Decision Analytics Journal*, vol. 10, pp. 100402, 2024. DOI: 10.1016/j.dajour.2024.100402
- [93] Hariharan R, Udhayakumar R, "Existence of mild solution for fuzzy fractional differential equation utilizing the Hilfer-Katugampola fractional derivative," *An International Journal of Optimization and Control: Theories & Applications*, vol. 15, no. 1, pp. 82-91, 2025. DOI: 10.36922/ijocta.1653
- [94] Momena A F, Pakhira R, Haque R, Sobczak A, Mondal S P, "A memory-dependent inventory model with fuzzy price-dependent demand under backlogged shortages," *Journal of Uncertain Systems*, vol. 18, no. 3, pp. 2550003, 2025. DOI: 10.1142/S1752890925500035
- [95] Rahaman M, Gazi K H, Rabih M, Alamin A, Razzaq O A, Khan N A, ... Alam S, "Solutions of Linear Homogeneous Fuzzy Fractional Differential Equations Using the Mittag-Leffler Function," *Journal of Uncertain Systems*, vol. 18, no. 04, pp. 2550013, 2025. DOI: 10.1142/S1752890925500138