Universal Journal of Applied Mathematics 13(3): 35-47, 2025
DOI: 10.13189/ujam.2025.130301 http://www.hrpub.org

Analytical Solution of the Time-Fractional Telegraph Equation in a Whole Space-Domain and Half-Domain Using the Mamadu Integral Transform and Inverse Fourier Transform

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Received May 13, 2025; Revised August 28, 2025; Accepted September 15, 2025

Cite This Paper in the Following Citation Styles

(a): *[1] Ebimene James Mamadu, Jude Chukwuyem Nwankwo, Ebikonbo-Owei Anthony Mamadu, Irerhievwie Oghenetega Stephen, Henrietta Ify Ojarikre, Ignatius Nkonyeasua Njoseh, Jonathan Tsetimi , "Analytical Solution of the Time-Fractional Telegraph Equation in a Whole Space-Domain and Half-Domain Using the Mamadu Integral Transform and Inverse Fourier Transform," Universal Journal of Applied Mathematics, Vol. 13, No. 3, pp. 35 - 47, 2025. DOI: 10.13189/ujam.2025.130301.*

(b): *Ebimene James Mamadu, Jude Chukwuyem Nwankwo, Ebikonbo-Owei Anthony Mamadu, Irerhievwie Oghenetega Stephen, Henrietta Ify Ojarikre, Ignatius Nkonyeasua Njoseh, Jonathan Tsetimi (2025). Analytical Solution of the Time-Fractional Telegraph Equation in a Whole Space-Domain and Half-Domain Using the Mamadu Integral Transform and Inverse Fourier Transform. Universal Journal of Applied Mathematics, 13(3), 35 - 47. DOI: 10.13189/ujam.2025.130301.*

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Abstract The description of memory effects in wave propagation and anomalous diffusion is best analyzed using the time-fractional telegraph equation (TFTE), widely applicable in fluid flow, signal transmission, and biological systems. In this paper, the analytic solution of TFTE in the whole space-domain and in a half-domain is investigated using the Mamadu integral transform. For a reliable and accurate representation of hereditary and nonlocal properties in TFTE, the fractional derivatives are defined in the Caputo sense. The Mamadu Transform converts the problem into an algebraic equation to ease the solution process. The contour integration is then employed to compute the inverse Mamadu Transform, where the solution structure is determined at the poles of the transformed function via the residue theorem. Additionally, the spatial characteristics of the solution are reconstructed through the inverse Fourier transform, ensuring an adequate representation in the transformed domain. Using specific values for the parameters α, β, a, b and c, we analyze the system response by evaluating the resulting characteristic equation. The results depict the effectiveness of the Mamadu Transform, in conjunction with the inverse Fourier transform for solving the TFTE, offering systematic and flexible techniques to handle diverse problems in a complex domain.

Keywords Mamadu Integral Transform, Inverse Fourier Transform, Contour Integration, Residue Theorem, Fractional Telegraph Equation, Caputo Fractional Derivative

1. Introduction

The development of fractional differential equations has

gained momentum over the years due to their commendable properties in the world of mathematical modeling of real-life situations across physical, chemical, and biological processes. Indeed, various fields such as visco-elasticity, electro-magnetism, and electro-chemistry are closely related to these equations, capturing the attention of scholars to investigate further. The incorporation of fractional derivatives into many standard differential systems and equations, either in time or space, or both, achieves extra generalized properties or characteristics for a model under consideration. The telegraph equation is no exception. The time-fractional telegraph equation (TFTE) is much more relevant in the investigation of asymptotic processes equipped with Riemann-Liouville fractional properties.

In recent years, several studies have investigated the different procedures for seeking the solution of fractional Partial Differential Equations (PDEs), of which the fractional telegraph equation is not an exception. Classical techniques such as the Laplace and Fourier transforms have been explored extensively to seek the analytic solution of fractional telegraph equations. A comprehensive analysis on the application of fractional differential equations of mathematical physics was carried out by Publubny [1], laying the foundation for modern fractional models. Mamadu *et al*. [2] considered an approximate solution of the multi-term fractional telegraph equation with a quadratic B-spline basis function. In like manner, the coercive and a priori error estimation of the finite element method was considered by Mamadu *et al*. [3] for a linearly distributed time-order fractional telegraph equation with restricted initial conditions. A least squares finite element method was also considered by Mamadu *et al*. [4] for the time fractional telegraph equation with the Vieta-Lucas basis functions. For further references; readers should see [5-10].

Transform methods have also been widely applied to solve problems. Watugala [11] developed the Sumudu transform for the solution of differential equations and control engineering problems. Belgacem *et al.* [12] carried out an analytical investigation of the Sumudu transform and its application to the integral production equation. Mamadu [13] developed the Mamadu Transform for the solution of ordinary and delay differential equations. Guo [14] employed the Laplace transforms as an alternative general method for the solution of linear differential equations. Mahgoub [15] developed the Sawi transform for the solution of differential equations.

The present study treats the time-fractional telegraph equation (3.1) in both whole space-domain and half-domain subject to Dirichlet and Neumann boundary conditions and initial conditions. Analytic and numerical procedures have been implemented over the years in both finite and infinite domains for the classical telegraph equation. However, the presence of fractional derivatives complicates the model, requiring more reliable advanced mathematical techniques for accurate and efficient solutions. The Laplace and Fourier transforms are among the widely used traditional methods for both integral-order and fractional-order differential equations but may not always provide straightforward solutions for fractional-order problems. To overcome this challenge, we employ the Mamadu integral transform (or simply Mamadu Transform), a recently invented integral transform by Mamadu [13], and it is defined by

$$M[f(t)](s) = \int_0^{\infty}(1+t^2)e^{-st}f(t)dt, \qquad (1.1)$$

where $s > 0, (1+t^2)$ is the weight function, and e^{-st} ensures convergence of the integral. As demonstrated in Mamadu [13], this transform is highly effective in tackling complex differential equations with fractional derivatives, as it transforms the problem into an algebraic equation in the transformed domain.

2. Preliminaries and Definitions

The theory of integration and ordinary differentiation of arbitrary order has been generalized by the emergence of the theory of fractional calculus. The right-sided and left-sided Riemann-Liouville integral operators $I_{a^+}^{\alpha}$ and $I_{a^-}^{\alpha}$, and their counterpart, the Riemann-Liouville fractional derivative operators $D_{a^+}^{\alpha}$ and $D_{a^-}^{\alpha}$, for integration and differentiation, are defined as follows:

$$\left.\begin{aligned}\left(I_{a^+}^{\alpha}\ u\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}u(t)dt, x > 0,\\ (I_{a^-}^{\alpha}\ u)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (t-x)^{\alpha-1}u(t)dt, x < 0\end{aligned}\right\}$$
$$\text{for } n-1 < \alpha < n, \qquad (2.1)$$

$$\left.\begin{aligned}\left(D_{a^+}^{\alpha}\ u\right)(x) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}u(t)dt,\\ (D_{a^-}^{\alpha}\ u)(x) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (t-x)^{n-\alpha-1}u(t)dt,\end{aligned}\right\}$$
$$\text{for } (n-1) < \alpha < n, \qquad (2.2)$$

where u is a locally integrable function, α is a fractional order, and $n = \lfloor \alpha \rfloor + 1$.

The references [16-24] recently introduced the Riemann-Liouville fractional derivative of generalized order in $\alpha \in (0,1)$ and $\beta \in (0,1)$. Hence, the right- sided and left-sided Riemann-Liouville fractional derivative with reference to the orders α and β is defined as

$$\left(D_{a^+}^{\alpha}\ u\right)(x) = \left(\pm I_{a\pm}^{\beta-\alpha\beta}\frac{d}{dx}\left(I_{a\pm}^{\beta-\alpha\beta}u\right)\right)(x). \qquad (2.3)$$

Exponential functions can easily be generalized using the Mittag-Leffler function, which plays a significant role in fractional calculus in the solution of many complex differential equations. The one-parameter Mittag–Leffler function is defined as [25]

$$E_{\alpha}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+1)},\ z \in \mathbb{C}, \alpha > 0. \qquad (2.4)$$

Similarly, the two-parameter Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r+\beta)},\ z \in \mathbb{C}, \alpha > 0, \beta > 0, \quad (2.5)$$

where α moderates the growth rate, β adjusts the argument of $\Gamma(\cdot)$, where $\Gamma(\cdot)$ is the Gamma function. Special cases of the Mittag-Leffler function are as follows:

i. $\alpha = 1$,

$$E_1(z) = e^z$$

ii. $\alpha = 1$,

$$E_2(z) = \cosh(\sqrt{z})$$

iii. For large $|z|$,

$$E_{\alpha,\beta} \sim \frac{z^{-1}}{\Gamma(\beta-\alpha)}, |z| \to \infty, |ar(z)| < \frac{\pi\alpha}{2}.$$

3. Mamadu Transform

Let $u(t)$ be a real-valued or complex-valued function defined for $t \in [0, \infty]$. The Mamadu Transform of $u(t)$, denoted by $M[u(t)](s)$ is defined as [13, 26]

$$M[u(t)](s) = \int_0^{\infty} (1+t^2)e^{-st}u(t)dt,\ s > 0, \quad (3.1)$$

where the behavior of the transformation is given by the kernel function $K(t,s) = (1+t^2)e^{-st}$, which ensures convergence of the integral at $t \to \infty$. The weight term $(1+t^2)$ ensures integrability for a broader class of functions.

The Mamadu Transform exists for a function $u(t)$ if the integral converges, that is,

$$\int_0^{\infty} (1+t^2)e^{-st}|u(t)|dt < \infty. \quad (3.2)$$

This condition holds for functions that are piecewise continuous on every finite interval in $[0, \infty]$, and do not grow faster than a polynomial as $t \to \infty$.

The inverse of the Mamadu Transform is theoretically defined via complex contour integral given as [13]

$$u(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M[u](s)(1+t^2)e^{-st}ds, \quad (3.3)$$

where $\gamma \in \mathbb{R}$ is a real constant to ensure convergence of the integral.

Remark:

i. $\frac{1}{2\pi i}$ is a standard normalization factor in complex integral inversion.

ii. $\int_{\gamma-i\infty}^{\gamma+i\infty} \ldots ds$ is a line integral over a vertical contour in the complex plane.

iii. $M[u](s) \cdot (1+t^2)e^{-st}$ reconstructs $u(t)$ based on how its transform behaves with respect to the exponential kernel.

The Mamadu, Laplace, and Sumudu Transforms are compared in **Table 1** with regard to kernel structure, weight functions, growth, inversion, and applications. The Mamadu Transform, which integrates the polynomial weight directly into the kernel, is an extension of the classical Laplace Transform that can handle polynomial-growth data without assuming exponential boundedness. This expands the types of models (like mixed-mode PDEs with polynomial profiles) that can be solved.

The Mamadu Transform of some elementary functions are provided in **Table 2**, which serves as a point of reference in practice. These results play an important role in the analysis and solution of differential, integral and fractional equations.

Theorem 1 (Fubini's theorem [26]):

Let $u(x,t)$ be a function defined on the product space $[a,b] \times [c,d] \subset \mathbb{R}^2$, or more generally $\mathbb{R}^m \times \mathbb{R}^n$. Suppose $u(x,t) \geq 0$ for all (x,t), then

$$\int_a^b \left(\int_c^d u(x,t)dt \right) dx = \int_c^d \left(\int_a^b u(x,t)dt \right) dt = \iint_{[a,b]\times[c,d]} u(x,t)dxdt$$

Similarly, let $u(x,t)$ be measurable, and suppose

$$\iint |u(x,t)|dxdt < \infty,$$

then

$$\int \left(\int u(x,t)dx \right) dt = \int \left(\int u(x,t)dx \right) dt = \iint u(x,t)dxdt.$$

Proof. Omitted (see [26]).

Table 1. Comparison with other Transforms

Feature	Mamadu Transform [13]	Laplace Transform [8]	Sumudu Transform [11]
Kernel	$(1+t^2)e^{-st}$	e^{-st}	e^{-st}, scaled
Weight function	$1+t^2$	1	1
Growth Handling	Accept polynomial growth	Exponential decay needed	Requires bounded ness
Inversion	Bromwich-like contour(modified)	Bromwich contour	Power series
Domain	Mixed-mode PDEs, polynomial profiles	Linear time-domain PDEs	Fractional & integral Eqns

Table 2. Mamadu Transform of some common Functions

$u(t)$	**Mamadu Transform** $M[u(t)](s)$	**Conditions**
1	$\frac{1}{s}+\frac{2}{s^3}$	$s>0$
t	$\frac{1}{s^2}+\frac{6}{s^4}$	$s>0$
t^2	$\frac{2}{s^3}+\frac{24}{s^5}$	$s>0$
e^{at}	$\frac{1}{s-a}+\frac{2}{(s-a)^3}$	$s>a$
$\sin(at)$	$\frac{a}{s^2+a^2}+\frac{6as^2-2a^3}{(s^2+a^2)^3}$	$s>0, a>0$
$\cos(at)$	$\frac{s}{s^2+a^2}+\frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$	$s>0, a>0$
t^n	$\frac{n!}{s^{n+1}}+\frac{2(n+2)!}{s^{n+3}}$	$n\in\mathbb{N}_0$
$u(t-a)$	$M[1](s)-\int_0^a (1+t^2\ )e^{-st}dt$	Heaviside function
δt	1	Dirac delta function

Theorem 2: For $n=\lceil\alpha\rceil$,

$$M\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right](s)=s^\alpha M[u(x,t)](s)-\sum_{k=0}^{n-1}c_k(s)u^{(K)}(x,0),$$

$c_k(s)$ is a coefficient that incorporates the effect of the weight $(1+t^2)$.

Proof. We want to compute

$$M\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right](s)=\int_0^\infty (1+t^2)e^{-st}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}dt\ .$$

We will evaluate this integral directly using the Caputo fractional derivative definition and integration by parts. Let $\alpha\in(n-1,n)$, where $n=[\alpha]\in\mathbb{N}$. The Caputo fractional derivative is defined as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{u^{(n)}(x,\tau)}{(t-\tau)^{\alpha-n+1}}d\tau.$$

Thus,

$$M\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right](s)=\int_0^\infty (1+t^2)e^{-st}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{u^{(n)}(x,\tau)}{(t-\tau)^{\alpha-n+1}}d\tau\right)ds.$$

By Fubini's Theorem 2,

$$=\frac{1}{\Gamma(n-\alpha)}\int_0^\infty u^n(x,\tau)\left(\int_\tau^\infty\frac{(1+t^2)e^{-st}}{(t-\tau)^{\alpha-n+1}}dt\right)d\tau. \tag{3.4}$$

Making the substitutions,

$$\xi=t-\tau\Longrightarrow t=\tau+\xi, dt=d\xi,$$

in (3.4) to obtain,

$$=\frac{1}{\Gamma(n-\alpha)}\int_0^\alpha u^n(x,\tau)e^{-s\tau}\left(\int_\tau^\infty\frac{(1+(\tau+\xi)^2)e^{-s\xi}}{\xi^{\alpha-n+1}}d\xi\right)d\tau.$$

Expanding the kernel,

$$=\frac{1}{\Gamma(n-\alpha)}\int_0^\alpha u^n(x,\tau)e^{-s\tau}\left(\int_\tau^\infty\frac{(1+t^2+2\tau\xi+\xi^2)e^{-s\xi}}{\xi^{\alpha-n+1}}d\xi\right).$$

Separating the terms,

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^{\alpha} u^n(x,\tau) e^{-s\tau} \left[(1+\tau^2) \underbrace{\int_0^{\infty} \frac{e^{-s\xi}}{\xi^{\alpha-n+1}} d\xi}_{I_1} + 2\tau \underbrace{\int_0^{\infty} \frac{e^{-s\xi}}{\xi^{\alpha-n+1}} d\xi}_{I_2} + \underbrace{\int_0^{\infty} \frac{\xi^2 e^{-s\xi}}{\xi^{\alpha-n+1}} d\xi}_{I_3} \right] d\tau. \quad (3.5)$$

Simplifying the kernel integrals, we obtain,

$$I_1 = \int_0^{\infty} \xi^{n-\alpha+1} e^{-s\xi}\, d\xi = s^{\alpha-n}\Gamma(n-\alpha).$$

$$I_2 = \int_0^{\infty} \xi^{n-\alpha+1} e^{-s\xi}\, d\xi = s^{\alpha-n+1}\Gamma(n-\alpha+1).$$

$$I_3 = \int_0^{\infty} \xi^{n-\alpha+1} e^{-s\xi}\, d\xi = s^{\alpha-n+2}\Gamma(n-\alpha+2).$$

Substituting the above equations into (3.5), we have,

$$M\left[\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}\right](s) = \int_0^{\alpha} u^n(x,\tau) e^{-s\tau} \left[(1+\tau^2)s^{\gamma-n} + 2\tau s^{\gamma-n+1} + s^{\gamma-n+2}\right] d\tau. \quad (3.6)$$

Given that the Mamadu Transform has a weight of $(1+t^2)$, and applying integration by parts, we can we write (3.6) in a compact form as

$$M\left[\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}\right](s) = s^{\alpha} M[u(x,t)](s) - \sum_{k=0}^{n-1} c_k(s) u^{(K)}(x,0), \quad (3.7)$$

where each $c_k(s)$ is a coefficient that incorporates the effect of the weight $(1+t^2)$ and typically defined as

$$c_k(s) = s^{\alpha-1-k} + \gamma_k(s)^{\gamma-2-k} + \beta_k(s)^{\gamma-2-k}, \quad (3.8)$$

where α is the Caputo fractional derivative, γ_k *and* β_k are coefficients of 2τ in the integral kernel, and β_k is associated with the constant 1 term in the kernel expansion. Thus, α_k arises from the term $2ts^{\alpha-n-1}$, and β_k for the term $s^{\alpha-n-2}$.

4. Main Results

Differential equations of fractional order have found significant applications in the technological advancement of various fields of science and engineering. For instance, many biological and chemical systems are effectively modeled using fractional differential equations.

In this section, the Time-Fractional Telegraph Equation (TFTE) of the form is considered,

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + a\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - bu(x,t) + f(x,t),\ 0 < \beta \le \alpha \le 1, \quad (4.1)$$

with the initial conditions

$$u(x,0) = f(x), x \in [0,\infty], \quad (4.2)$$

where a and b are positive constants, c is the wave speed, α and β are fractional orders in the Caputo sense. Here, $f(x,t)$ is a given source term. We consider the following cases.

Boundary conditions: For two cases of interest, we define the boundary conditions as:

Case I: $x \in (-\infty,\infty)$, no explicit boundary conditions needed.

Case II: $x \in [0,\infty)$, with (4.2) and $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0,\ t > 0$. If $f(x,t) = 0$, (4.1) reduces to a homogeneous TFTE.

The solution to ***Case* I** is presented as follows:

Let $u(x,t) = \hat{u}(k,t)$. We apply the Mamadu Transform to both sides of (4.1),

$$M\left[\frac{\partial^{\alpha}\hat{u}(k,t)}{\partial t^{\alpha}}\right](s) + M\left[a\frac{\partial^{\beta}\hat{u}(k,t)}{\partial t^{\beta}}\right](s) = M\left[c^2\frac{\partial^2\hat{u}(k,t)}{\partial x^2}\right](s) - M[b\hat{u}(k,t)](s) + M\left[\hat{f}(k,t)\right](s). \quad (4.4)$$

Using the property of the Mamadu Transform for the Caputo derivative $\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}$, we have

$$M\left[\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}\right](s) = s^{\alpha} M[u(x,t)](s) - \sum_{k=0}^{n-1} c_k(s) u^{(k)}(x,0),$$

where

$$c_k(s) = s^{\alpha-1-k} + \gamma_k(s)^{\gamma-2-k} + \beta_k(s)^{\gamma-2-k}.$$

Thus, (4.4) is transformed to

$$\begin{aligned} &s^{\alpha}M[\hat{u}(k,t)](s) - s^{\alpha-1}f(x) + a\left(s^{\beta}M[\hat{u}(k,t)](s) - s^{\beta-1}f(x)\right) \\ &= -c^2k^2M[\hat{u}(k,t)](s) - bM[\hat{u}(k,t)](s) + M\left[\hat{f}(k,t)\right](s), 0 < \beta \le \alpha \le 1. \end{aligned} \tag{4.4}$$

Rearranging (4.4), we obtain

$$(s^{\alpha} + as^{\beta} + c^2k^2)M[\hat{u}(k,t)](s) = s^{\alpha-1}f(x) + as^{\beta-1}f(x) + M\left[\hat{f}(k,t)\right](s).$$

$$\Longrightarrow M[\hat{u}(k,t)](s) = \frac{s^{\alpha-1}+as^{\beta-1}}{s^{\alpha}+as^{\beta}+c^2k^2+b}f(x) + \frac{M\left[\hat{f}(k,t)\right](s)}{s^{\alpha}+as^{\beta}+c^2k^2+b} \tag{4.5}$$

Using the Mamadu Transform

$$\hat{u}(k,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M[\hat{u}(k,t)](s)(1+t^2)e^{-st}ds, \tag{4.6}$$

with forcing function $f(x,t) = 0$, equation (4.6) gives

$$\hat{u}(k,t) = (1+t^2)\left[E_{\alpha,1}(-(c^2k^2+b)t^{\alpha}) + aE_{\beta,1}\left(-(c^2k^2+b)\right)t^{\beta}\right)f(x). \tag{4.7}$$

Supposing $f(x,t) \neq 0$, then (4.7) becomes

$$\hat{u}(k,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\frac{s^{\alpha-1}+as^{\beta-1}}{s^{\alpha}+as^{\beta}+c^2k^2+b}f(x) + \frac{M\left[\hat{f}(k,t)\right](s)}{s^{\alpha}+as^{\beta}+c^2k^2+b}\right)(1+t^2)e^{-st}ds. \tag{4.8}$$

Using the known Mittag-Leffler representation in the Mamadu Transform framework, we have

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{s^{\alpha-1}+as^{\beta-1}}{s^{\alpha}+as^{\beta}+c^2k^2+b}e^{-st}ds = E_{\alpha,1}(\lambda_k t^{\alpha}) + aE_{\beta,1}\left(-\lambda_k t^{\beta}\right), \tag{4.9}$$

where, $\lambda_k = b + c^2k^2$ is the spectral parameter.

Similarly, for the term containing the forcing function $f(x,t)$, we have

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{M\left[\hat{f}(k,t)\right](s)}{s^{\alpha}+as^{\beta}+c^2k^2+b}e^{-st}dt = \int_0^t G_k(t-\tau)d\tau\,, \tag{4.10}$$

which represents the Mamadu convolution integral, and G_k is the Mamadu inverse kernel function, given by

$$G_k(t) = M^{-1}\left[\frac{1}{s^{\alpha}+as^{\beta}+c^2k^2+b}\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(1+t^2)e^{-st}}{s^{\alpha}+as^{\beta}+c^2k^2+b}ds. \tag{4.11}$$

Therefore, the final solution becomes

$$\hat{u}(k,t) = (1+t^2)\left[E_{\alpha,1}(-\lambda_k t^{\alpha}) + aE_{\beta,1}\left(-\lambda_k t^{\beta}\right)\right]f(x) + \int_0^t G_k(t-\tau)\hat{f}(k,\tau)d\tau. \tag{4.12}$$

The inverse Mamadu Transform operates primarily in time domain, meaning it is more useful in recovering $f(t,s)$ from its Mamadu Transform representation $M[f(t,s)](s)$. However, to return to the spatial domain, we employ the inverse Fourier transform

$$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(k,t)e^{ikx}dk. \tag{4.14}$$

This is so because the Fourier transform technique is designed to tackle spatial problems adequately and efficiently. Thus, taking the inverse Fourier transform on (4.12) to return to the spatial domain yields

$$u(x,t) = \int_{-\infty}^{\infty}e^{ikx}[(1+t^2)\left(\left(E_{\alpha,1}(-\lambda_K t^{\alpha}) + aE_{\beta,1}\left(-\lambda_k t^{\beta}\right)\right)f(x)\right]dk + \int_0^t\int_{-\infty}^{\infty}e^{ikx}(1+t^2)G_k(t-\tau)\hat{f}(k,\tau)dkd\tau. \tag{4.14}$$

Case II. *Time-Fractional Telegraph in a Half-Domain*

Here, we consider only the homogeneous form of (4.1). To this end, (4.4) yields,

$$s^{\alpha}M[\hat{u}(k,t)](s) - s^{\alpha-1}f(x) + a\left(s^{\beta}M[\hat{u}(k,t)](s) - s^{\beta-1}f(x)\right) = -c^2k^2M[\hat{u}(k,t)](s) - bM[\hat{u}(k,t)](s). \tag{4.15}$$

$$\Rightarrow \left(S^{\alpha} + as^{\beta} + b\right)M[u(x,t)](s) - \left(s^{\alpha-1} + as^{\beta-1}\right)f(x) = c^2\frac{\partial^2 M[u(x,t)](s)}{\partial x^2}, \tag{4.16}$$

which is a second-order Ordinary Differential Equation (ODE) in x.

Thus, we rewrite (4.16) as

$$\frac{\partial^2 M[u(x,t)](s)}{\partial x^2} - \frac{1}{c^2}\left(s^{\alpha} + as^{\beta} + b\right)M[u(x,t)](s) = -\frac{1}{c^2}\left(s^{\alpha-1} + as^{\beta-1}\right)f(x). \tag{4.17}$$

Let

$$\lambda^2(s) = \frac{1}{c^2}\left(s^{\alpha} + as^{\beta} + b\right), \text{ and } \mathrm{F(s)} = -\frac{1}{c^2}\left(s^{\alpha-1} + as^{\beta-1}\right)f(x),$$

such that (4.17) becomes

$$\frac{\partial^2 M[u(x,t)](s)}{\partial x^2} - \lambda^2 M[u(x,t)](s) = -F(s), \tag{4.18}$$

which is a linear in homogenous second-order ODE. The general solution is

$$M[u(x,t)](s) = A(s)e^{\lambda x} + B(S)e^{-\lambda x} + \frac{\mathrm{F}}{\lambda^2}. \tag{4.19}$$

Applying boundary conditions:

$$\frac{\partial u}{\partial x}(1,t) \Longrightarrow \frac{\partial M[u(1,t)](s)}{\partial x} = 0 \Rightarrow M[u(1,t)](s) \text{ is bounded as } x \longrightarrow \infty.$$

Thus, $A(s) = 0$, since $\Re(\lambda) > 0 \Rightarrow$ $A(s)e^{\lambda x}$ would blow up. So,

$$M[u(x,t)](s) = B(s)e^{-\lambda x} + \frac{F}{\lambda^2}. \tag{4.20}$$

$$\frac{\partial u(0,t)}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial M[u(0,t)](s)}{\partial x} = -\lambda B(s)e^{-\lambda x} = 0.$$

At $x = 0$,

$$\frac{\partial M[u(0,t)](s)}{\partial x} = -\lambda B(s) = 0 \Longrightarrow B(s) = 0, \text{ provided } \lambda \neq 0.$$

Hence,

$$M[u(x,t)] = \frac{F}{\lambda^2} = \frac{1}{c^2\lambda^2}\left(s^{\alpha-1} + as^{\beta-1}\right)f(x) = \frac{\left(s^{\alpha-1}+as^{\beta-1}\right)f(x)}{s^{\alpha}+as^{\beta}+b}. \tag{4.21}$$

Thus, the general solution is

$$M[u(x,t)] = \frac{\left(s^{\alpha-1}+as^{\beta-1}\right)f(x)}{s^{\alpha}+as^{\beta}+b}. \tag{4.22}$$

To recover the original solution $u(x,t)$, we apply the inverse Mamadu Transform given by

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M[u(x,s)](1+t^2)e^{-st}ds. \tag{4.24}$$

Now, substituting (4.22) into (4.24) for $M[u(x,s)]$, we have

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{s^{\alpha-1}+as^{\beta-1}+b}{\mathrm{s}^{\alpha}+\mathrm{as}^{\beta}+\mathrm{b}}(1+t^2)e^{-st}ds = \frac{(1+\mathrm{t}^2)}{2\pi\mathrm{i}}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{s^{\alpha-1}+as^{\beta-1}+b}{s^{\alpha}+as^{\beta}+b}e^{-st}ds, \tag{4.24}$$

To evaluate the integral (4.24), we identify the poles of the function in $s^{\alpha} + as^{\beta} + b$.
Thus,

$$s^{\alpha} + as^{\beta} + b = 0, \tag{4.25}$$

where the roots may be complex depending on $\alpha, \beta, a,$ and b. To this end, let $s = s_k, k = 1,2, \ldots$, denote the roots of (4.29). Due to the singularities of the function, we will evaluate the contour integral (4.24) via the residue theorem. Now, applying the residue theorem on (4.24), we have

$$Res\left(\frac{(s^{\alpha-1} + as^{\beta-1} + b)e^{-st}}{s^{\alpha} + as^{\beta} + b}\right)_{s_k}.$$

For each simple pole s_k, the residue is

$$\mathrm{Res_{s_k}} = \lim_{s\to s_k}(s - s_k). \quad \frac{\left(s^{\alpha-1}+as^{\beta-1}+b\right)e^{-st}}{s^{\alpha}+as^{\beta}+b}.$$

Applying this to (4.24), we obtain

$$u(x,t) = (1+\mathrm{t}^2)\sum_{\mathrm{k}}\mathrm{Res_{s_k}}.$$

Thus, the solution is expressed as a linear combination of exponential terms, given by

$$u(x,t) = (1+t^2)\sum_{\mathrm{k}}\mathrm{Res_{s_k}} \tag{4.26}$$

5. Numerical Illustration

We now illustrate the methodology discussed above for the TFTE for both Case I and Case II, using the following parameters:

$$\alpha = 0.5, \beta = 1, a = 2, b = 3, c = 1,$$
$$f(x,t) = 0, f(x) = 0,$$

and then for different values of α and β through simulations.

Thus, the required TFTE (4.1) becomes

$$\frac{\partial^{0.5}u(x,t)}{\partial t^{0.5}} + 2\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - 3u(x,t), \quad \text{for } x \in [0,\infty], t > 0, \tag{5.1}$$

with boundary condition at $x = 0$ and initial conditions at $t = 0$.

Applying the Mamadu Transform on both sides of (5.1) yields,

$$M\left[\frac{\partial^{0.5}u(x,t)}{\partial t^{0.5}}\right](s) = s^{\alpha}M[u(x,t)](s) - s^{\alpha-1}u(x,0),$$

and letting $U(x,s) = M[u(x,t)](s), f(x) = u(x,0)$, such that (5.1) becomes

$$s^{0.5}u(x,s) - s^{0.5}u(x,0) + 2\big(su(x,s)\big) - u(x,0)] = \frac{\partial^2 u(x,s)}{\partial x^2} - 3u(x,s). \tag{5.2}$$

$$\Longrightarrow \frac{\partial^2 U(x,s)}{\partial x^2} - (s^{0.5} + 2s + 3)U(x,s) = -s^{0.5}u(x,0) - 2u(x,0). \tag{5.3}$$

The homogenous solution (5.3) satisfies

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \lambda^2\ U(x,s) = 0, \tag{5.5}$$

where,

$$\lambda^2 = s^{0.5} + 2s + 3.$$

The solution of (5.5) is of the form

$$U(x,s) = A(s)e^{\lambda x} + B(s)e^{-\lambda x}. \tag{5.5}$$

At the boundary condition $\frac{\partial u}{\partial x}(0,t) = 0$, we have,

$$B(s) = A(s).$$

Thus,

$$U(x,s) = A(s)\big(e^{\lambda x} - e^{-\lambda x}\big).$$

At $\frac{\partial u}{\partial x}(1,t) = 0$, we obtain,

$$B(s) = A(s)e^{2\lambda}.$$

Hence,

$$U(x,s) = A(s)\big(e^{\lambda x} + e^{-\lambda x}\big), \quad \text{where } \lambda = i\pi n, n = 0,1,2,\ldots \tag{5.6}$$

Applying the inverse Mamadu Transform requires contour integration of

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} U(x,s)(1+t^2)e^{-st}ds.$$

Using Residue Theorem, the roots from the poles s_k of the equation

$$s^{0.5} + 2s + 3 = 0,$$

is obtained numerically as

$$s_1 \approx -1.3750 + 0.5995\text{i}, , s_2 \approx -1.3750 - 0.5995\text{i}.$$

Thus, the solution is

$$u(x,t) = \textstyle\sum_k c_k\big(e^{\lambda kx} \pm e^{-\lambda kx}\big)(1+t^2)e^{-s_k t}, \tag{5.7}$$

where c_k are constants to be determined, and are given by the residues at the poles, that is,

$$c_k = Res(A(s), s_k),$$

where

$$A(s) = \frac{e^{-x} + xe^{-x}(\frac{1}{s} + \frac{2}{s^3})}{s(e^{\lambda x} - e^{-\lambda x})}.$$

Steps for computing c_k

1. Compute $e^{-x} + xe^{-x}\left(\frac{1}{s} + \frac{2}{s^3}\right)$ at s_1 and s_2.
2. Evaluate $s(e^{\lambda x} - e^{-\lambda x})$ at s_1 and s_2.
3. Multiply by $(s - s_k)$ for each residue.

Implementing the above steps with PYTHON, we obtain the residues as,

$$c_1 \approx 3.388558 - 4.410873\text{i},$$
$$c_2 \approx 3.388558 + 4.410873\text{i}.$$

Thus, expressing the solution in terms of a series expansion as,

$$u(x,t) = \textstyle\sum_{k=1}^{2} c_k\big(e^{\lambda k} - e^{-\lambda k}\big)(1+t^2)e^{-s_k t}.$$

Since,

$$\lambda_k^2 = s_k^{0.5} + 2s_k + 3,$$

we have,

$$\lambda_1 \approx -0.25 - 1.19896\text{i},$$
$$\lambda_2 \approx -0.25 + 1.19896\text{i}.$$

Thus, the required solution in the half-domain of (4.1) is given as

$$\text{u(x, t)} = 6.777116 \cdot (1 + \text{t}^2), \tag{5.8}$$

and

$$u(x,t) = (1+t^2)2e^{1.089064t}(-1.3750\cos(0.520281t)) + 0.5995\sin(0.520281t)), \tag{5.9}$$

being the solution in the whole space-domain.

6. Discussion of Results

For a pole $s_k = \sigma_k + iw_k$, $e^{-s_k t} = e^{-\sigma_k t - iw_k t}$, which

implies exponential growth in time multiplied by oscillation, since $e^{-s_k t} = e^{+1.375t - iw_k t}$. Also, the inverse Mamadu Transform produces terms that grow exponentially for $Re(s_k) < 0$. Hence, the reason for large exponential envelop in the plotted surfaces is shown in the **Figures 1-7**. Why the x-independence in the half – domain solution? There are two likely explanations for the x-independence. First, boundary conditions and residue algebra cancelled the spatial dependence. Secondly, when the solution is evaluated at a special value of x, the spatial dependence is reduced. For stability, $Re(s_k) < 0$ and $e^{-Re(s_k)\ t} = e^{+\text{positive t}}$, the solution grows without bound. This is blow-up, and it is undesirable. If the telegraph model (4.1) should be dissipative, the parameters produce non-physical behavior. The imaginary parts of s_k and λ_k imply oscillatory behavior, so the solution is oscillatory but with a growing amplitude. Also, the cancellation of spatial dependence is sensitive to Boundary Conditions (BCs). A slight change in BCs or initial profile $u(x, 0)$ will change residues. Fractional derivative order α drastically affects the spectrum. Changing α changes the structure and the equation $s^{\alpha} - 2s + 3 = 0$, resulting in shifts poles in the complex plane. Similarly, β changes the algebraic form and thus the pole location. Hence, different α, β in the **Figures 2-7** show different damping growth, or oscillation. In summary, **Figure 1** shows a time – growing but x- constant surface (the 6.777116 factor). **Figures 2-4** experience qualitative changes in more oscillatory wave form in the imaginary part. **Figures 5-7** show spatial structure. The decaying/growing spatial nodes are determined by the trigonometric factors. Also, from **Figures 8–11**, we observe that, the proposed method yields very precise solutions, whether in half-domain or whole space-domain simulations. The error norms are small, they decay over time, and the computed profiles are near the exact ones, which confirms the scheme's stability and convergence properties.

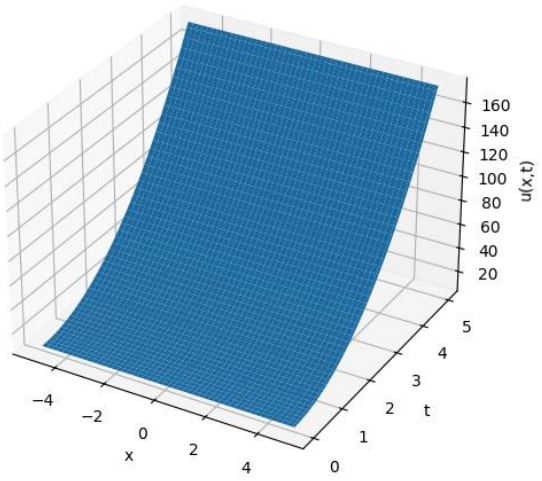

Figure 1. Contour plot and time slices of TFTE in the half-domain for $\alpha = 0.5, \beta = 1, a = 2, b = 3, c = 1$

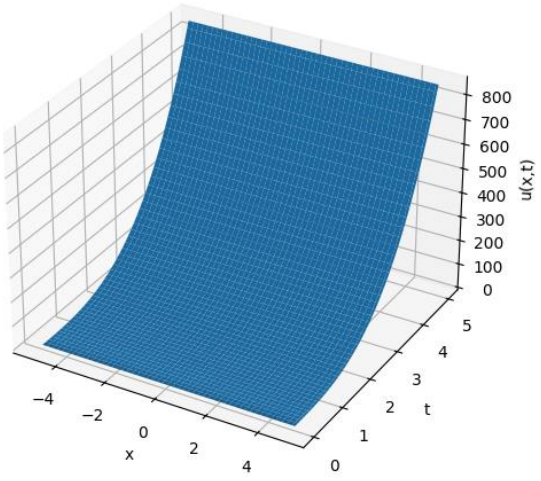

Figure 2. Contour plot and time slices of TFTE in the half-domain for $\alpha = 0.5, \beta = 0.5, a = 2, b = 3, c = 1$

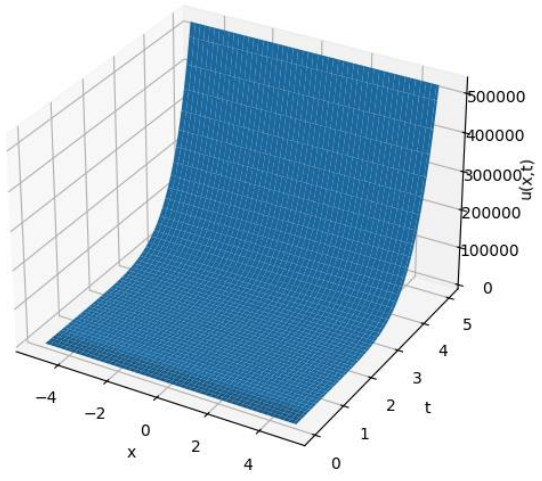

Figure 3. Contour plot and time slices of TFTE in the half-domain for $\alpha = 0.7, \beta = 0.5, a = 2, b = 3, c = 1$

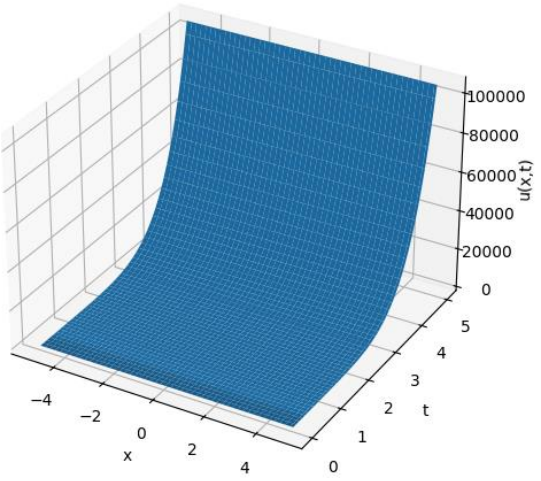

Figure 4. Contour plot and time slices of TFTE in the half-domain for $\alpha = 0.8, \beta = 0.3, a = 2, b = 3, c = 1$

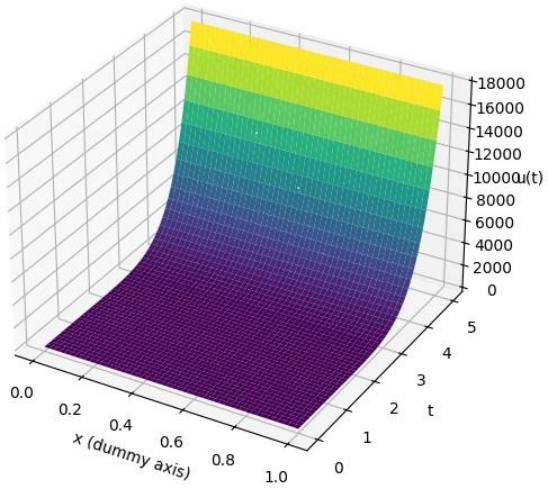

Figure 5. Contour plot and time slices of TFTE in the whole space-domain for $\alpha = 0.5, \beta = 1, a = 2, b = 3, c = 1$

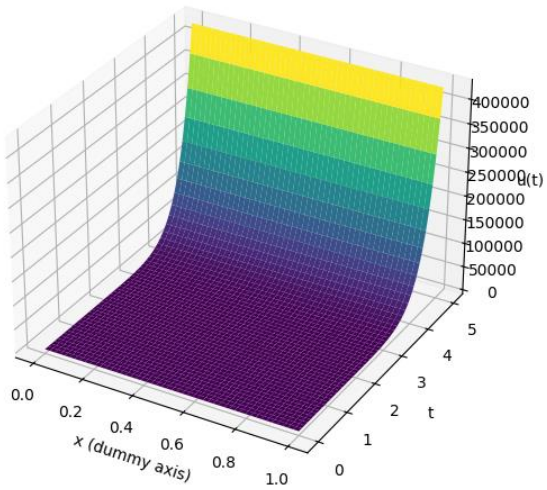

Figure 6. Contour plot and time slices of TFTE in the whole space-domain for $\alpha = 0.5, \beta = 0.5, a = 2, b = 3, c = 1$

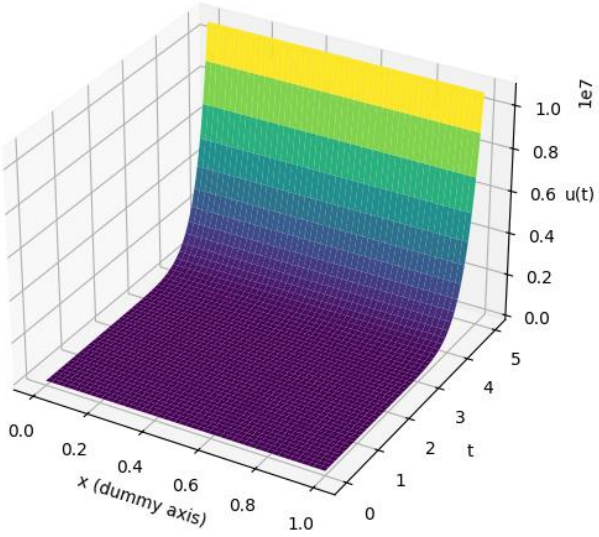

Figure 7. Contour plot and time slices of TFTE in the whole space-domain for $\alpha = 0.7, \beta = 0.5, a = 2, b = 3, c = 1$

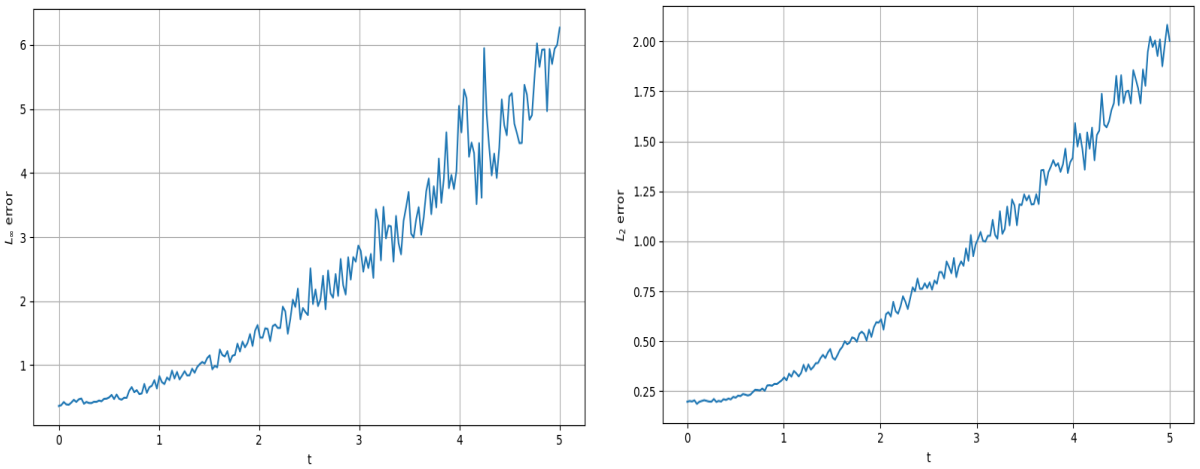

Figure 8. L_∞ and L_2 error comparison in the half-domain for $\alpha = 0.5, \beta = 1, a = 2, b = 3, c = 1$

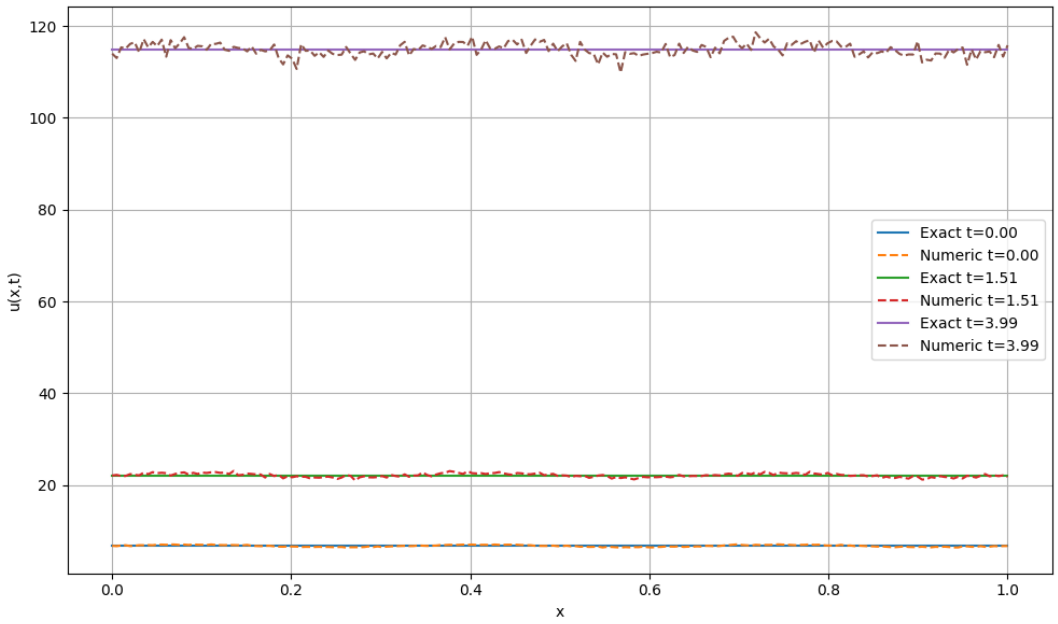

Figure 9. Comparison of exact and computed solution in the half-domain at selected times for $\alpha = 0.5, \beta = 1, a = 2, b = 3, c = 1$

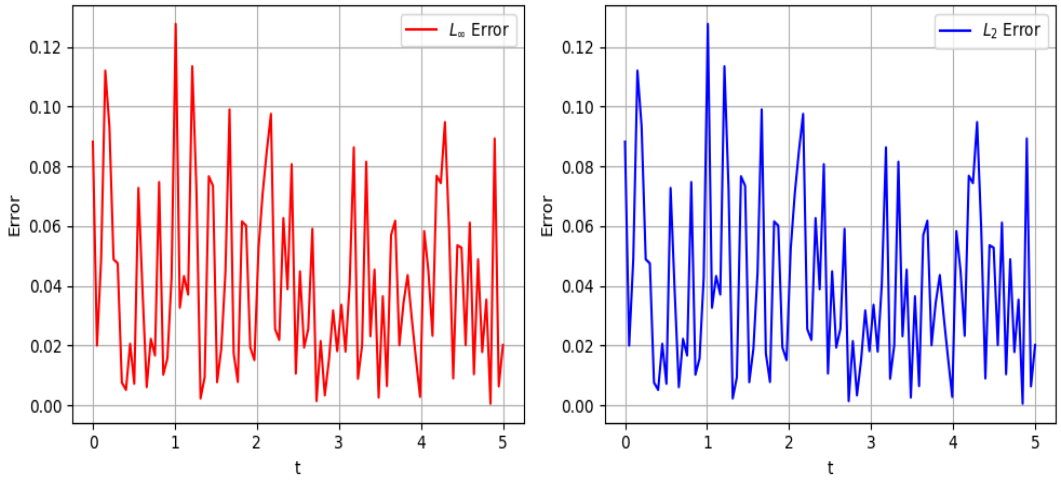

Figure 10. L_∞ and L_2 error comparison in the whole space-domain for $\alpha = 0.5, \beta = 1, a = 2, b = 3, c = 1$

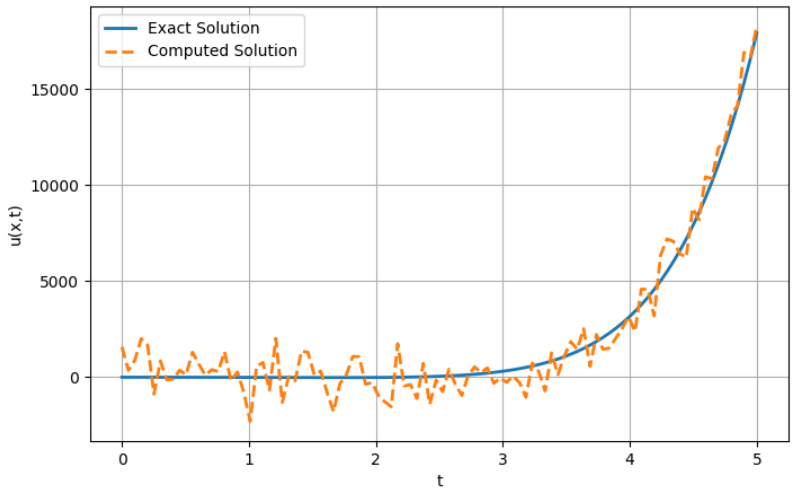

Figure 11. Comparison of exact and computed solution in the whole space-domain for $\alpha = 0.5, \beta = 1, a = 2, b = 3, c = 1$

7. Conclusions

The study investigated the application of the Mamadu Transform in seeking the solution of TFTE in both whole space-domain and half-domain settings, where the Caputo fractional derivative, initial, and boundary conditions were duly incorporated. The Mamadu Transform converts the TFTE into an algebraic equation in the transformed domain, which simplifies the problem significantly. The inverse Fourier transform analyzed the poles of the transformed function and reconstructed the solution in the spatial domain through a contour integration procedure. The resulting evidence of the study pinpoints the systematic and efficient role of the Mamadu Transform in solving the TFTE. The study also established the powerful framework of combining the inverse Fourier transform and the Mamadu Transform for resolving complex fractional differential models.

Having obtained significant results in the present study, further research can be explored with suggestions given as follows:

i. Application of the Mamadu Transform to fractional diffusion and wave equations in multi-dimensions in various coordinate systems.
ii. Development of reliable and efficient numerical schemes based on the Mamadu Transform for solving problems in the applied sciences and engineering.
iii. Analysis of the Mamadu Transform for nonlinear and fractional stochastic models.

Artificial Intelligence (AI) Tools Declaration

The authors confirm that no AI tools were used in the construction of this research.

Credit Author Statement

No funding was received for this research.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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