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Irreducibility of Polynomials up to Fourth Degree with Some Prescribed Coefficients

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Abstract This paper investigates the irreducibility properties of small-degree polynomials with specific pre-assigned coefficients over the field of rational numbers. Focusing on polynomials of degrees up to four, we analyze families of polynomials where some coefficients are fixed as $\phi(n)$, $\tau(n)$ etc. We establish irreducibility of polynomials using combinatorial techniques and derive irreducibility conditions for others as:

(i) $x^2+p^kx+\phi(p^k)$ is irreducible over $\mathbb{Q}$ if and only if $k>1$,
(ii) $x^2+p^kx+\tau(p^k)$ is irreducible over $\mathbb{Q}$ except for $p=2,\ 3$ and $k=1$,
(iii)$x^3+p^kx^2+p^kx+\phi(p^k)$ is irreducible over $\mathbb{Q}$ iff $k>1$,
(iv) $x^4+p^kx^3+p^kx^2+p^kx+\phi(p^k)$ is irreducible over Q when p is any prime with $k\equiv\ 0,2(mod\,4)$,
(v) $x^3+p^kx+\phi(p^k)$, $x^3+p^kx^2+\phi(p^k)$, $x^3+p^kx+\tau(p^k)$, $x^4+px+\phi(p)$, $x^4+p^2x+\phi(p^2)$ and $x^4+px^2+px+\phi(p)$ are irreducible over $\mathbb{Q}$. In addition to the existing tools, the above mentioned results establish the irreducibility of small degree polynomial at first sight. Moreover, through our results, we extend the applicability of classical irreducibility criteria and provide new classes of irreducible polynomials. The combinatorial and analytical techniques employed offer a foundation for further research into higher-degree polynomials and their irreducibility over other fields.

Keywords Irreducibility, Euler's Totient Function, Number of Divisors Function

1 Context and motivation

The majority of contemporary work on irreducibility stems from classical proofs of irreducibility by C.F. Gauss , Schonemann, Eisenstein, H. Cohen, M. Filesta [1, 2, 3, 4], and others. There are several sufficient conditions available to check irreducibility of polynomials over various fields. Each and every irreducibility criterion has its limitations. No criteria can be applied to all the polynomials. In this sense, there is always scope of generalization of already known criteria and for finding classes of irreducible polynomials. In 2019, Anuj Jakhar [5] established some results on irreducible factors of a polynomial. In 2022, in [6] authors provided some new directions towards irreducibility of polynomials using some already established results in [7, 8, 9]. Motivated by these concepts we have the following results.

Lemma 1.1. *[10]For $f(x)=x^n+a_{n-1}x^{n-1}+\ldots+a_0\in\mathbb{Z}[x]$, if r is rational and $x-r$ divides $f(x)$, then r is an integer. Further, $r|a_0$.*

Lemma 1.2. *[10]Let a be a non-zero element of $\mathbb{Q}$. If $f(x)$ is irreducible over $\mathbb{Q}$, then $f(ax)$ is also irreducible over $\mathbb{Q}$.*

Lemma 1.3. *[10]The quadratic polynomial $x^2+bx+c\in\mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$ iff b^2-4c isn't square of any integer.*

Lemma 1.4. ***Eisenstein's Irreducibility Criteria**[10] "Let p be a prime number and $f(x)=a_0+a_1x+\cdots+a_nx^n$ be an integer polynomial. Suppose that a_n is not divisible by p, that all other coefficients are divisible by p and that a_0 is not divisible by p^2. Then $f(x)$ is irreducible over $\mathbb{Q}$."*

Lemma 1.5. ***Dumas Irreducibility Criteria**[12] In 1906, Swiss mathematician Gustave Dumas generalized the Eisenstien's irreducibility criteria as:*

Let $f(x) = a_nx^n + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. Let p be a fixed prime number such that each $a_i = b_ip^{\alpha_i}$ with $(p, b_i) = 1$ for all $i = 0,\ 1,\ 2,\ 3,\ \cdots,\ n$ with following conditions
(i) $\alpha_n = 0$
(ii)$(\frac{\alpha_i}{n-i}) > \frac{\alpha_0}{n}$ for all $i = 1,\ 2,\ 3,\ \cdots,\ n-1$, and
(iii) $gcd(\alpha_0, n) = 1$.
If we are able to find prime p which satisfies above conditions, then $f(x)$ is irreducible over $\mathbb{Q}$.

Euler's totient function $\phi(n)$ applied to a positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n and Mobius function $\mu(n)$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ is square free and has } r \text{ distinct prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

In the subsequent segments, we identify some classes of irreducible polynomials over $\mathbb{Q}$ with some prescribed coefficients as $\phi(n)$, $\tau(n)$ etc.

2 Irreducibility of Quadratic Equations

Irreducibility of $x^2 + p^kx + \phi(p^k)$

Eisenstein's Irreducibility Criteria 1.4 fail to establish that polynomial $x^2+p^kx+\phi(p^k)$ is irreducible over $\mathbb{Q}$ for all values of $k > 2$. In the next theorem, we establish that polynomial $x^2 + p^kx + \phi(p^k)$ is irreducible over $\mathbb{Q}$ for all $k > 1$.

Theorem 2.1. *$x^2 + p^kx + \phi(p^k)$ is irreducible over $\mathbb{Q}$, for every prime p if and only if $k > 1$.*

Proof. Let, if possible, $x^2 + p^kx + \phi(p^k) \in \mathbb{Z}[x]$ be reducible over $\mathbb{Q}$ for some prime p and $k > 1$. Then by Lemma 1.3, we have $p^{2k} - 4\phi(p^k)$ is square of some integer. Now,

$$\begin{aligned} p^{2k} - 4\phi(p^k) &= p^{2k} - 4p^k + 4p^{k-1} \\ &= (p^k - 2)^2 + 4(p^{k-1} - 1) \\ &> (p^k - 2)^2. \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned} p^{2k} - 4\phi(p^k) - (p^{2k} - 2p^k + 1) &= -2p^k + 4p^{k-1} - 1 \\ &= -2p^{k-1}(p - 2) - 1 \\ &< 0 \\ \implies p^{2k} - 4\phi(p^k) &< (p^k - 1)^2. \end{aligned} \tag{2}$$

From (1) and (2), we obtain

$$(p^k - 2)^2 < p^{2k} - 4\phi(p^k) < (p^k - 1)^2$$

which leads to contradiction because no such integer exists such that square of it lies between $(p^k - 2)^2$ and $(p^k - 1)^2$. Hence $x^2 + p^kx + \phi(p^k)$ is irreducible over $\mathbb{Q}$ where p is any prime except for $k = 1$. ☐

Remark 2.2. For $k = 1$, polynomial $x^2+p^kx+\phi(p^k)$ becomes $x^2 + px + p - 1 = (x + (p - 1))(x + 1)$ which is reducible over $\mathbb{Q}$ because both $-1, -p + 1$ are rationals.

Irreducibility of $x^2 + p^kx + \tau(p^k)$

For polynomial $x^2 + p^kx + \tau(p^k)$ there does not exist any prime p which satisfies all the conditions of Eisenstein's Irreducibility Criteria except for the case when $p|(k + 1)$ and $p^2 \nmid (k + 1)$. In the following result, we establish the irreducibility of $x^2+p^kx+\tau(p^k)$ over $\mathbb{Q}$ for all values of k except when $p = 2$ and $k = 1$.

Lemma 2.3. *$2^{k+1} > 4k + 3$ for all natural numbers $k > 2$.*

Lemma 2.4. *$2.3^k > 4k + 3$ for all natural numbers $k > 1$.*

Lemma 2.5. *$2.p^k > 4k + 3$ for all natural numbers k, whenever $p > 3$.*

Theorem 2.6. *$x^2 + p^kx + \tau(p^k)$ is irreducible over $\mathbb{Q}$ for all prime p and all natural numbers k except when $p = 2$ or 3 and $k = 1$.*

Proof. If possible, suppose that polynomial $x^2+p^kx+\tau(p^k) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ for some prime p except when $p = 3$ and $k > 1$. By Lemma 1.3, we have $p^{2k} - 4\tau(p^k)$ is a square of some integer. Now

$$p^{2k} - 4\tau(p^k) - (p^k - 1)^2 = 2p^k - 4k - 3.$$

If $p \geq 5$, then by Lemma 2.5

$$(p^k - 1)^2 < p^{2k} - 4\tau(p^k) < (p^k)^2 \tag{3}$$

which leads to a contradiction that $p^{2k} - 4\tau(p^k)$ is square of some integer.
If $p = 3$ and $k > 1$ then using Lemma 2.4, we have

$$(3^k - 1)^2 < 3^{2k} - 4\tau(3^k) < (3^k)^2.$$

By Lemma 2.3, for $p = 2$, we have

$$(2^k - 1)^2 < 2^{2k} - 4\tau(2^k) < (2^k)^2 \text{ whenever } k > 2$$

For $p = 2$ and $k = 2$, we have
$p^{2k} - 4\tau(p^k) = 2^4 - 12 = -4$ which leads to contradiction that $2^{2k} - 4\tau(2^k)$ is a square of some integer whenever $k \geq 2$. Hence we conclude that polynomial $x^2 + p^kx + \tau(p^k)$ is irreducible over $\mathbb{Q}$ where p is any prime except when $p = 3$ and $k = 1$. ☐

Remark 2.7. For $k = 1$ and $p = 3$, polynomial $x^2 + 3x + \tau(3)$ reduces to $x^2 + 3x + 2 = (x + 1)(x + 2)$ which is reducible over $\mathbb{Q}$.

3 Irreducibility of Cubic Equations

Irreducibility of $x^3 + p^kx + \phi(p^k)$

Theorem 3.1. *$x^3 + p^kx + \phi(p^k)$ is irreducible over $\mathbb{Q}$ for every prime p.*

Proof. Let, if possible, polynomial $x^3 + p^k x + \phi(p^k) \in \mathbb{Z}[x]$ be reducible over $\mathbb{Q}$ for some prime p which in turn implies $x^3 + p^k x + \phi(p^k)$ has a rational root say r. By Lemma 1.1, we say that r must be an integer. Here constant term of given polynomial is non-zero, so r is a non-zero integer.

Case 1. If $r \in \mathbb{Z}^+$ then,

$$r^3 + p^k r + \phi(p^k) > 0$$

Therefore, r cannot be the root of given polynomial.

Case 2 If $r \in \mathbb{Z}^-$, then we can write $r = -q$ for some $q \in \mathbb{Z}^+$
Now, since

$$\begin{aligned} &\phi(p^k) < p^k \\ &\implies \phi(p^k) < p^k < p^k q < p^k q + q^3 \\ &\implies -q^3 - p^k q + \phi(p^k) < 0 \end{aligned}$$

Therefore, r cannot be root of the given polynomial.

From both the cases, we conclude that $x^3 + p^k x + \phi(p^k) \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$ for every prime p. ☐

Irreducibility of $x^3 + p^k x^2 + \phi(p^k)$

Theorem 3.2. *$x^3 + p^k x^2 + \phi(p^k)$ is irreducible over $\mathbb{Q}$ for every prime p.*

Proof. If possible, suppose polynomial $x^3 + p^k x^2 + \phi(p^k) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ for some prime p, then the given polynomial has a rational root, say, r. By Lemma 1.1, we assert that r must be an integer. Here constant term of given polynomial is non-zero, so r is a non-zero integer.

Case 1. If $r \in \mathbb{Z}^+$, then

$$r^3 + p^k r^2 + \phi(p^k) > 0.$$

Therefore, r cannot be a root of given polynomial.

Case 2. If $r \in \mathbb{Z}^-$, we can write $r = -q$, where $q \in \mathbb{Z}^+$.
By Lemma 1.1 $r|\phi(p^k)$ which implies $q|\phi(p^k)$. Therefore,

$$\begin{aligned} q \le \phi(p^k) &< p^k \\ \implies q^3 < p^k q^2 &< p^k q^2 + \phi(p^k) \\ \implies -q^3 + p^k q^2 &+ \phi(p^k) > 0. \end{aligned}$$

Thus, r cannot be the root of the given polynomial.

From both the cases, we have $x^3 + p^k x + \phi(p^k)$ $\in \mathbb{Z}[x]$ is irreducible over Q for every prime p. ☐

Irreducibility of $x^3 + p^k x + \tau(p^k)$

Theorem 3.3. *$x^3+p^k x+\tau(p^k)$ is irreducible over $\mathbb{Q}$ for every prime p.*

Proof. Let, if possible, $x^3 + p^k x + \tau(p^k) \in \mathbb{Z}[x]$ be reducible over $\mathbb{Q}$ for some prime p, then polynomial $x^3 + p^k x + \tau(p^k)$ has necessarily a rational root say r. By use of Lemma 1.1, we assert that r must be integer. Here, constant term of given polynomial is non-zero, so r is non-zero integer.

Case 1. If $r \in \mathbb{Z}^+$, then

$$r^3 + p^k r + \tau(p^k) > 0$$

Therefore, r cannot be a root of the given polynomial.

Case 2. If $r \in \mathbb{Z}^-$, then we can express r as $r = -q$ for some $q \in \mathbb{Z}^+$
We know that $\tau(p^k) < p^k$.

$$\begin{aligned} &\implies \tau(p^k) < p^k < p^k q < p^k q + q^3 \\ &\implies -q^3 - p^k q + \tau(p^k) < 0. \end{aligned}$$

Therefore, r cannot be root of the polynomial.

From both the cases, we conclude that r cannot be root of mentioned polynomial, hence, $x^3 + p^k x + \tau(p^k) \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$ for every prime p. ☐

Irreducibility of $x^3 + p^k x^2 + p^k x + \phi(p^k)$

Theorem 3.4. *$x^3 + p^k x^2 + p^k x + \phi(p^k)$ is irreducible over $\mathbb{Q}$, where p is any prime if and only if $k > 1$.*

Proof. If possible, assume $x^3 + p^k x^2 + p^k x + \phi(p^k) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ for any prime p, then it has a rational root say r. By Lemma 1.1, we assert that r must be integer. Here, constant term of given polynomial is non-zero, so r is a non-zero integer.

Case 1. If $r \in \mathbb{Z}^+$, then

$$r^3 + p^k r^2 + p^k r + \phi(p^k) > 0. \tag{4}$$

Case 2. If $r \in \mathbb{Z}^-$, we can express r as $r = -q$, where $q \in \mathbb{Z}^+$. Now,

$$-q^3 + p^k q^2 - p^k q + \phi(p^k) = q(-q^2 + p^k(q-1)) + \phi(p^k). \tag{5}$$

Sub-case 1. q is a proper divisor of $\phi(p^k)$ i.e. other than 1 and $\phi(p^k)$, then

$$\begin{aligned} &2q \le \phi(p^k) < (p^k) \\ \implies &2q(q-1) < p^k(q-1) \\ \implies &q^2 \le q^2 + q^2 - 2q < p^k(q-1). \end{aligned} \tag{6}$$

From (5) and (6), we have

$$-q^3 + p^k q^2 - p^k q + \phi(p^k) > 0. \tag{7}$$

Sub-case 2. If $q = \phi(p^k) = p^k - p^{k-1}$, then

$$\begin{aligned} &-q^3 + p^k q^2 - p^k q + \phi(p^k) = \\ &-p^{3k} + 3p^{3k-1} - 3p^{3k-2} + 3p^{3k-3} + p^{3k} \\ &-2p^{3k-1} + p^{3k-2} - p^{2k} + p^{2k-1} + \phi(p^k) \\ &= p^{3k-1} - 2p^{3k-2} + 3p^{3k-3} - p^{2k} + p^{2k-1} + \phi(p^k). \end{aligned}$$

If $k = 2$ and p is an odd prime, then

$$\begin{aligned} p^{3k-1} - 2p^{3k-2} &+ p^{3k-3} - p^{2k} + p^{2k-1} + \phi(p^k) \\ &= p^5 - 2p^4 + p^3 - p^4 + p^3 + p^2 - p \\ &= p^4(p-3) + 2p^3 + p(p-1) > 0. \end{aligned} \tag{8}$$

If $k = 2$ and p is even prime, then

$$\begin{aligned} p^{3k-1} - 2p^{3k-2} + p^{3k-3} - p^{2k} + p^{2k-1} + \phi(p^k) \\ &= p^5 - 2p^4 + p^3 - p^4 + p^3 + p^2 - p \\ &= p^3(p-2)^2 + p(p(p-1)^2 - 1) = 2 > 0. \end{aligned} \tag{9}$$

If $k \geq 3$, then

$$\begin{aligned} p^{3k-1} - 2p^{3k-2} + p^{3k-3} - p^{2k} + p^{2k-1} + \phi(p^k) \\ &= p^{3k-2}(p-2) + p^{2k-1}(p^{k-2} - p + 1) + \phi(p^k) > 0 \\ &\implies -q^3 + p^k q^2 - p^k q + \phi(p^k) > 0. \end{aligned} \tag{10}$$

Sub-case 3. If $q = 1$, then

$$\begin{aligned} & -q^3 + p^k q^2 - p^k q + \phi(p^k) \\ &= -1 + p^k - p^k + \phi(p^k) \\ &= -1 + \phi(p^k) > 0, \qquad p^k \geq 2^k > 2. \end{aligned} \tag{11}$$

From (4) -(11), we conclude that $x^3 + p^k x^2 + p^k x + \phi(p^k) \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$ for every prime p and $k > 1$. ☐

Remark 3.5. For $k = 0$, the above mentioned polynomial reduces to
$x^3 + x^2 + x + 1 = (x+1)(x^2+1)$ in $\mathbb{Z}[x]$.

Remark 3.6. For $k = 1$, the above mentioned polynomial reduces to
$x^3 + px^2 + px + p - 1 = (x - 1 + p)(x^2 + x + 1)$ in $\mathbb{Z}[x]$.

4 Irreducibility of Quartic Equations

Irreducibility of $x^4 + px + \phi(p)$

Theorem 4.1. *Polynomial $x^4 + px + \phi(p)$ is irreducible over $\mathbb{Q}$ for every prime p.*

Proof. If $x^4 + px + \phi(p)$ has a rational root say r, then, by Lemma 1.1, r must be an integer. Also, the constant term of given polynomial is non-zero, so r is non-zero integer.
Case 1. If $r \in \mathbb{Z}^+$, then

$$r^4 + pr + \phi(p) > 0. \tag{12}$$

Thus, r cannot be a root of given polynomial.
Case 2. If $r \in \mathbb{Z}^-$, then we can write $r = -q$, where $q \in \mathbb{Z}^+$
Now, if r is a root of given polynomial, we have

$$\begin{aligned} & q^4 - qp + \phi(p) = 0 \\ \implies & p = 1 + q + q^2 + q^3 = (1+q)(1+q^2) \end{aligned} \tag{13}$$

a contradiction to p being prime, so r cannot be root of given polynomial.
Thus, we conclude that polynomial $x^4 + px + \phi(p)$ does not possess any rational root and consequently has no linear factor in $\mathbb{Q}[x]$.
Now, sum of all four roots of polynomial $x^4 + px + \phi(p)$ is zero. If $\alpha,\ \beta,\ \gamma,\ \delta$ are the four roots, then by putting $a = \alpha + \beta = -(\gamma + \delta)$, $b = \alpha\beta$ and $c = \gamma\delta$, we can rewrite $x^4 + px + \phi(p) = (x^2 - ax + b)(x^2 + ax + c)$, where

$$b + c = a^2,\ b - c = \frac{p}{a},\ bc = p - 1. \tag{14}$$

Here, polynomial $x^4 + px + \phi(p) \in \mathbb{Z}[x]$ is monic and reducible as

$$(x^2 - ax + b)(x^2 + ax + c),$$

then, without loss of generality, let us consider the case $a,\ b,\ c \in \mathbb{Z}$.
Using (14) in $(b+c)^2 - (b-c)^2 = 4bc$, we get

$$(a^2)^3 - 4(p-1)(a^2) - p^2 = 0. \tag{15}$$

Thus polynomial $\phi(y) = y^3 - 4(p-1)y - p^2 \in \mathbb{Z}[x]$ has $a^2 \in \mathbb{Z} - \mathbb{Z}^-$ as its root. Further $a^2 = 1$ or p^2.
Consider zero of polynomial as 1, then

$$1 - 4(p-1) - p^2 = 5 - 4p - p^2 = -(p+5)(p-1)$$

which is a contradiction because p can neither be 1 nor -5. So 1 cannot be zero of polynomial $\phi(y)$.
Consider zero of $\phi(y)$ as p^2, then

$$\begin{aligned} p^6 - 4(p-1)p^2 - p^2 = p^4 - 4p + 3 &= p(p^3 - 4) + 3 \\ &= 0 \end{aligned}$$

which is a contradiction because $p(p^3 - 4) + 3 > 0$. So p^2 cannot be zero of polynomial $\phi(y)$.

Thus, $x^4 + px + \phi(p)$ can not be factorized as $(x^2 - ax + b)(x^2 + ax + c)$ and hence $x^4 + px + \phi(p)$ is irreducible over $\mathbb{Q}$. ☐

Irreducibility of $x^4 + p^2x + \phi(p^2)$

Theorem 4.2. *Polynomial $x^4 + p^2x + \phi(p^2)$ is irreducible over $\mathbb{Q}$ for every prime p.*

Proof. By comparing $x^4 + p^2x + \phi(p^2)$ to the polynomial $f(x) = a_0 + a_1x + \cdots + a_4x^4$, we have $a_0 = \phi(p^2),\ a_1 = p^2,\ a_2 = 0,\ a_3 = 0,\ a_4 = 1$. Using criterion Eisenstein's Irreducibility for prime p, we assert that polynomial $x^4 + p^2x + \phi(p^2)$ is irreducible over Q, where p is any prime. ☐

Irreducibiity of $x^4 + p^kx^3 + p^kx^2 + p^kx + \phi(p^k)$

Theorem 4.3. *$x^4 + p^kx^3 + p^kx^2 + p^kx + \phi(p^k)$ is irreducible over Q, where p is any prime with $k \equiv 0, 2 (mod\ 4)$.*

Proof. By comparing $x^4 + p^kx^3 + p^kx^2 + p^kx + \phi(p^k)$ to the polynomial $f(x) = a_0 + a_1x + \cdots + a_4x^4$, we have $a_0 = \phi(p^k),\ a_1 = a_2 = a_3 = p^k,\ a_4 = 1$.
Suppose p be a fixed prime number which is required to apply Dumas Criterion 1.5 such that each $a_i = b_i p^{\alpha_i}$ with $(p, b_i) = 1$ for all $i = 0,\ 1,\ 2,\ 3,\ 4$.
$\implies \alpha_0 = k - 1,\ \alpha_1 = \alpha_2 = \alpha_3 = k,\ \alpha_4 = 0$, which in turn yields
(i) $\alpha_4 = 0$

(ii)$(\frac{\alpha_i}{4-i}) > \frac{\alpha_0}{4}$ for all $i = 1,\ 2,\ 3$ and
(iii) $gcd(\alpha_0, 4) = 1$ (because α_0 is odd).
So, there exists a fixed prime p which satisfies all the conditions of Dumas Irreducibility Criteria 1.5.
Hence $x^4 + p^k x^3 + p^k x^2 + p^k x + \phi(p^k)$ is irreducible over $\mathbb{Q}$, where p is any prime with $k \equiv 0, 2(mod\, 4)$. ☐

Irreducibility of $x^4 + px^2 + px + \phi(p)$

Theorem 4.4. *$x^4 + px^2 + px + \phi(p)$ is irreducible over $\mathbb{Q}$, for every prime p.*

Proof. Let, if possible, $f(x) = x^4 + px^2 + px + \phi(p)$ have a rational root say r. By Lemma 1.1, r must be an integer. Also, the constant term of $f(x)$ is non-zero, so r is a non-zero integer.

Case 1. If $r \in \mathbb{Z}^+$,then

$$r^4 + pr^2 + pr + \phi(p) > 0$$

Thus r cannot be root of $f(x)$.
Case 2. If $r \in \mathbb{Z}^-$, then we can rewrite $r = -q$, where $q \in \mathbb{Z}^+$. Now, if r is a root of given polynomial, we have

$$\begin{aligned} q^4 + q^2 p - qp + \phi(p) &= 0 \\ \implies p &= \frac{1 - q^4}{q^2 - q + 1} \end{aligned} \tag{16}$$

R.H.S. of (16) is either zero or negative. But p is a prime number, so r cannot be root of $f(x)$.

Thus, we conclude $f(x)$ does not possess any rational root or any linear factor in $\mathbb{Q}[x]$.
Now, sum of all four roots of polynomial $x^4 + px^2 + px + \phi(p)$ is zero. If $\alpha,\ \beta,\ \gamma,\ \delta$ are four roots of polynomial $x^4 + px^2 + px + \phi(p)$, then by putting $a = \alpha + \beta = -(\gamma + \delta)$, $b = \alpha\beta$ and $c = \gamma\delta$, we can rewrite
$f(x) = (x^2 - ax + b)(x^2 + ax + c)$,
where

$$b + c - a^2 = p,\ b - c = \frac{p}{a},\ bc = p - 1. \tag{17}$$

Here, polynomial $f(x) \in \mathbb{Z}[x]$ is monic and reducible as

$$f(x) = (x^2 - ax + b)(x^2 + ax + c),$$

then without loss of generality, consider $a, b, c \in \mathbb{Z}$.
Using (17) in $(b + c)^2 - (b - c)^2 = 4bc$, we get

$$a^2 (p + a^2)^2 - 4(p - 1)a^2 - p^2 = 0. \tag{18}$$

Now, polynomial $\psi(y) = y^3 + 2py^2 + (p - 2)^2 y - p^2 \in \mathbb{Z}[x]$ has $a^2 \in \mathbb{Z} - \mathbb{Z}^-$ as its zero, then that zero can be $1,\ p$ or p^2.
Consider 1 as zero of $\psi(y)$, then

$$\begin{aligned} 1 + 2p + p^2 - 4p + 4 - p^2 &= 0 \\ \implies p &= \frac{5}{2} \end{aligned}$$

a contradiction, so 1 cannot be zero of polynomial $\psi(y)$.
Taking p as zero of $\psi(y)$, we have

$$\begin{aligned} p^3 + 2p^3 + p^3 - 4p^2 + 4p - p^2 &= 0 \\ \implies p(4p^2 - 5p + 4) &= 0 \end{aligned}$$

again, a contradiction, because no prime p satisfies $p(4p^2 - 5p + 4) = 0$. Therefore, p cannot be zero of $\psi(y)$.
Lastly, consider zero of $f(x)$ as p^2, then

$$\begin{aligned} p^6 + 2p^5 + p^6 - 4p^5 + 4p^4 - p^2 = p^2(2p^4 - 2p^3 + 4p^2 - 1) \\ = 0. \end{aligned}$$

To check the irreducibility of the polynomial $2p^4 - 2p^3 + 4p^2 - 1$ over $\mathbb{Z}$, using GAP Code [11] given below, we find that there is no prime p which satisfies $p^2(2p^4 - 2p^3 + 4p^2 - 1) = 0$.
GAP code gap> q:=UnivariatePolynomial(Rationals,[-1,0,4,-2,2],1);
$2 * x_1^4 - 2 * x_1^3 + 4 * x_1^2 - 1$
gap > Factors(q);
$[2 * x_1^4 - 2 * x_1^3 + 4 * x_1^2 - 1]$
Thus p^2 cannot be zero of polynomial $y^3 + 2py^2 + (p - 2)^2 y - p^2$.
Hence $f(x) = (x^2 - ax + b)(x^2 + ax + c)$ is not possible. So $x^4 + px^2 + px + \phi(p)$ is irreducible over $\mathbb{Q}$. ☐

5 Conclusion

In this study we have used some classical tools, existing criteria and combinatorial techniques to check irreducibility of some classes of polynomials up-to degree four with some specified coefficients.

- Quadratic polynomials of the form $x^2 + p^k x + \phi(p^k)$ and $x^2 + p^k x + \tau(p^k)$ were shown to be irreducible over $\mathbb{Q}$ under specific conditions on p and k, with detailed exceptions noted.
- Cubic polynomials such as $x^3 + p^k x + \phi(p^k)$ and $x^3 + p^k x^2 + \phi(p^k)$ were proven irreducible over $\mathbb{Q}$ for all primes p and appropriate conditions on k.
- Biquadratic polynomials, including $x^4 + px + \phi(p)$, $x^4 + px^2 + px + \phi(p)$, and other similar forms, were rigorously analyzed to establish their irreducibility over $\mathbb{Q}$ under various constraints.

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