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Statistical Framework for Bivariate Point Processes: Conditional Intensity and Parameter Estimation Techniques

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Abstract The point process is a model that is suitable to describe the number of events that occur randomly in a given interval through its intensity function. The conditional intensity of the bivariate point process can be seen more specifically in two separate groups. The purpose of this study is to construct the conditional intensity for the homogeneous bivariate point process and to estimate the parameters using the maximum likelihood method. The conditional intensity construction is obtained from the ratio of the event-time probability density function to the event time survival function. Estimation of the conditional intensity parameter is carried out by constructing a likelihood function of the probability of one event in a small interval multiplied by the probability of no event in the remaining observation time. The results show that the conditional intensity of a bivariate point process depends on the number of events n and the observation time interval T. The pattern application was performed on two datasets, namely dataset the number of active cases of Covid-19 in Indonesia and dataset the number of earthquake in Sulawesi Island. The application to Dataset Covid-19 or Dataset earthquake reveals that the conditional intensity of the type-1 and type-2 in both datasets exhibits a directly proportional to the average frequency of events and inversely proportional to the observation time. Application to Dataset A or Dataset B shows that the conditional intensity of type A and type B events for the two Datasets is inversely proportional to the observation time.

Keywords Point Process, Bivariate, Conditional Intensity, Mean Function, Homogeneous Poisson Process

1 Introduction

Stochastic processes play an important role in the analysis of random phenomena, especially in modelling events scattered across space and time. In many practical applications, such as seismic data analysis [1], epidemiology [2], as well as spatial and spatial-temporal data [3], [4], point processing has become a key analytical tool. In the univariate point process model, the main concern is a single event, in which the spatial or temporal pattern of one type of phenomenon is analyzed. Although this model has provided many insights, there are significant limitations in capturing the complexity of real-world phenomena that often involve interactions between events or interrelationships between the dimensions of variables [5].

In many natural phenomena, the connection between events becomes inevitable. For example, in seismological studies, aftershocks are often associated with the main shocks [1], [6]. Meanwhile, in epidemiology, the spread of disease in one location can affect the pattern of spread in another location [2]. The univariate model is inadequate to describe this relationship because of its limited nature to one dimension of events. Therefore, the bivariate point process approach is relevant to capture these relationships. By incorporating two event variables, the bivariate point process allows for a richer analysis of complex patterns that appear in the data [7].

The bivariate point process is one of the significant developments of the univariate point process. In this approach, two types of events or variables are analyzed simultaneously, al-

lowing the model to capture the interaction between the two events. This approach becomes relevant especially in the context of random events that involve structural dependence between events. For example, in ecological research, the relationship between predator and prey species can be analyzed using a bivariate point process, where the occurrence of predators and prey is considered as two interrelated dimensions [7].
One of the important aspects of point processes, both univariate and bivariate, is the construction of a conditional intensity function. This function plays a key role in describing the probability of the next event based on the previous event. In the bivariate model, the conditional intensity becomes more complex because it must consider the relationship between the two dimensions of events. Proper conditional intensity construction is not only important for understanding event patterns, but also serves as the basis for inferential analysis, including the estimation of model parameters [5], [8].
In addition to conditional intensity, the likelihood function is also an essential element in bivariate point process analysis. The likelihood function allows estimation of model parameters based on observed empirical data. In the context of bivariate point processes, the likelihood construct must include both dimensions of the event simultaneously, making it more complex than the univariate [3] model. However, with the right likelihood construction, the conditional intensity parameters can be estimated more accurately, providing a deeper insight into the phenomenon being studied.
Although there have been significant developments in the literature related to point processes, research on bivariate point processes is still limited, especially in the context of random phenomena in the real world. Many previous studies have tended to focus on developing methods for univariate models or simple applications of bivariate models without in-depth exploration of fundamental aspects such as conditional intensity and likelihood [9]. This leaves an important gap in the literature, especially in the application of bivariate point processes to more complex phenomena.
This research aims to fill this gap by developing an integrated framework for the construction of conditional intensity and likelihood in bivariate point processes. This framework not only provides a solid theoretical foundation, but is also designed to be applied to a variety of random phenomena in the real world. With this approach, it is hoped that it can provide a more comprehensive solution for the analysis of random phenomena that involve interactions between events. If the study of the bivariate point process conducted by [10], [11], [12], [13], and [14], the bivariate point process model is obtained from two or more univariate point processes, then in this study the bivariate point process is obtained from the univariate point process which is partitioned based on the grouping of events in the process into two different types of events.
The main contribution of this study is to the development of a method that integrates the construction of conditional intensity and likelihood in the analysis of bivariate point processes. This approach is expected to provide new insights into the modelling of complex random phenomena and open up opportunities for wider applications in various fields of science. Thus, this research not only contributes to the development of point process theory but also has significant practical implications in various disciplines.

2 Material and Method

If the time can be expressed as a real line. On this real line, randomly scattered event points are then placed. The realization of the distribution pattern of the location of these points gives us different process points. A commonly used and simple process point is the homogeneous Poisson process, which assumes that the points are evenly distributed randomly over time intervals [15].

2.1 Point processes

Observation of the number of events in the time space for $t \geq 0$ is a process. The observation is expressed as a counting process. This event appears randomly, so the number of event points in time $(0, t)$ for $t \in [0, \infty)$ is also random. A random set of event points scattered within a set of time or space is referred to as a point process.

Definition 2.1. *Let $N(t)$ be the number of events in a time t. $\{N(t), t \geq 0\}$ is called a counting process that must satisfy $N(0) = 0$ and $N(t+s) \geq N(s)$ for all $s \geq 0$.*

If the numbers of events that occur in two separate intervals of time are mutually independent, then the process is called mutually independent increment. Let the number of events between time t to $t+s$, $N(t+s) - N(t)$ must be independent of the number of events that occur by time t, $N(t)$. The counting process is a process stationary increment if the number of events that occur in a time interval of the same length has the same probability distribution. If a counting process has independent increment and stationary increment, then the process is called the homogeneous Poisson process.

Definition 2.2. *If $N(t)$ is the number of events in any interval of length, t is a counting process that has mutually independent and stationary increment, then $N(t)$ has Poisson distribution with a rate of λt, and $\lambda > 0$.*

The random variable $N(t)$ is Poisson distribution and suppose the observation is made over the time interval $(a, b]$, then the mean of the distribution $\mu(a, b]$ is equal to the variance $V(a, b]$, of the number of events or points occurring within the observation interval $(a, b]$, and can be expressed as

$$\mu(a, b] = V(a, b] = \lambda(b - a).$$

The constant λ can be interpreted as the average rate or intensity of the points in the process and $b - a$ is the length of time interval. Another result that can be derived is the probability of no events occurring within the interval $(0, t]$, which is given by

$$P(N(t) = 0) = e^{-\lambda t}.$$

The equation above is equivalent to stating that the time between events follows an exponential distribution, $P(X_n < t) = e^{-\lambda t}$. According to [5], suppose that in the observation time interval $(0,T]$, there are N observations. Let the observations times be $t_1, t_2, \cdots, t_N$. For each $i, i = 1, 2, \cdots, N$, the probability of exactly one event occuring in the interval $(t_{i-1} - \Delta_i, t_i]$, and no other events occuring in other time intervals within the observation time $(0,T]$, is given by the likelihood realization:

$$L_{(0,t]}(\lambda; N, t_1, t_2, \cdots, t_N) = \lambda^N e^{-\lambda T}. \qquad (1)$$

2.2 Conditional Intensity

A counting process whose intensity is not stationery increment means that its rate of intensity is no longer constant and depends on time. Below this is the definition for a Poisson process whose intensity varies over time.

Definition 2.3. *$N(t)$ is called a non-homogeneous Poisson process if it is a counting process with an intensity rate $\lambda(t)$, which is a process with independent increment over different time intervals and follows a Poisson distribution with function parameter $\Lambda(t) = \int_0^t \lambda(s)ds$.*

The function $\Lambda(t)$ is called the mean function of the non-homogeneous Poisson process, $E(N(t)) = \Lambda(t)$ while $\lambda(t)$ is called intensity function. In many cases, the counting process $\{N(t), t \geq 0\}$ has an intensity function that depends not only on time t but also on the history of the process itself. Let $\mathcal{H}_t$ denote the history of the process up to time t, then its intensity function is expressed in the general form $\lambda(t|\mathcal{H}_t)$ and called conditional intensity function.
The likelihood function for a Poisson process in definition 2.3 for the realization of observation over the time interval $(0,T]$ and as a generalization of equation 1, becomes:

$$L_{(0,T]}(\cdot) = \exp\left(-\int_0^T \lambda(t)dt + \int_0^T \ln\lambda(t)N(dt)\right). \qquad (2)$$

Suppose that $\{N(t), t \geq 0\}$ represents the number of events or occurrences following a non-homogeneous Poisson process. Specific events that may occur from this process will be considered. For the observation interval $(t, t+h)$ as $h \to 0$, the probability that exactly one event occurs within this interval is given by

$$\begin{aligned} P(N(t,t+h) = 1) &= \int_t^{t+h} \lambda(s)ds \\ &\quad + \exp\left(-\int_t^{t+h} \lambda(s)ds\right) \\ &= \lambda(t)h + o(h), \end{aligned}$$

where $f(\cdot)$ is said to be $o(h)$ if

$$\lim_{h\to 0} \frac{f(h)}{h} = 0.$$

The probability of this specific events is denoted by $p(t)$, which affects the intensity function for that group of specific events.

Theorem 2.1. *Suppose that $\{N(t), t \geq 0\}$ is a Poisson process with a conditional intensity function $\lambda(t)$, and $N_1(t)$ represents the number of type-1 events that occur within the observation time t. Then $\{N_1(t), t \geq 0\}$ is a non-homogeneous Poisson process with a conditional intensity function $\lambda_1(t) = p(t)\lambda(t)$, where $p(t)$ is the probability of type-1 event occurring within time t.*

Proof. Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with a conditional intensity function $\lambda(t)$ and within this Poisson process, there are events categorized as type-1.

- The number of type-1 events at time $t = 0$ is 0, i.e. $N_1(0) \leq N(0) = 0$, then $N_1(0) = 0$.
- The number of type-1 events in the interval time $(0, s)$ and $(s, s+t)$ depends on the number of events from the original process $N(t)$ in these intervals and since the events $N(t)$ are independent, so $N_1(t)$ also has independent increments.
- Let $N_1(t, t+h) = N_1(t+h) - N_1(t)$, as it holds for $N(t, t+h) = N(t+h) - N(t)$ and since $P(N(t, t+h) \geq 2) = o(h)$, we have

$$\begin{aligned} P(N_1(t,t+h) \geq 2) &\leq P(N(t,t+h) \geq 2) = o(h) \\ P(N_1(t,t+h) \geq 2) &= o(h). \end{aligned}$$

- The probability that one event type-1 occurs within time interval $(t, t+h)$ as $h \to 0$ is $P(N_1(t,t+h) = 1) = P(N_1(t,t+h) = 1|N(t,t+h) = 1)P(N(t,t+h) = 1) + P(N_1(t,t+h) = 1|N(t,t+h) \geq 2)P(N(t,t+h) \geq 2)$. Then

$$\begin{aligned} P(N_1(t,t+h) = 1) &= p(t)(\lambda(t)h + o(h)) + o(h) \\ &= p(t)\lambda(t)h + o(h) \end{aligned}$$

Therefore $\{N_1(t), t \geq 0\}$ is a non-homogeneous Poisson process. □

3 Result and Discussion

3.1 Construction of conditional intensity of bivariate point process

A bivariate point process can be described as a process within the product space $T \times \{1, 2\}$, where T represents the time domain, and $\{1, 2\}$ denotes the set of indeces corresponding two types of events within the process. What is unique about the Poisson process is that partitioning the number of events in the process can form sub-processes that also follow a Poisson process. On the other hand, combining independent Poisson processes will also result in a Poisson process.
Let $N(t)$, $N_1(t)$, dan $N_2(t)$ denote the total number of events, the total number of type-1 events, and the total number of type-2 events respectively.

Corollary 3.1. *Suppose that $\{N(t), t \geq 0\}$ is non-homogeneous Poisson process with a conditional intensity* *function $\lambda(t)$, and $N_1(t)$ dan $N_2(t)$ represent the number of type-1 events and the number of type-2 events respectively, that occur within the observation time. Then $\{N_A(t), t \geq 0\}$ and $\{N_B(t), t \geq 0\}$ are non-homogeneous Poisson processes with conditional intensity functions $\lambda_1(t) = p\lambda(t)$ and $\lambda_2(t) = q\lambda(t)$, where p is the probability of type-1 event occurring and q is the probability of type-2 event occurring within the time t.*

Proof. Suppose $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson process with a conditional intensity function $\lambda(t)$ and the events of the process are grouped into two or more event categories.

The probability of the first group of events is $p(t)$ and the probability of the second group of events is $q(t)$. Then according to theorem 2.1, we get

$$P(N_1(s, s+t) = 1) = p(t)\lambda(t)$$

and

$$P(N_2(s, s+t) = 1) = q(t)\lambda(t).$$

So $\{N_1(t), t \geq 0\}$ is a non-homogeneous Poisson process with a conditional intensity function $\lambda_1(t) = p(t)\lambda(t)$ and $\{N_2(t), t \geq 0\}$ is a non-homogeneous Poisson process with a conditional intensity function $\lambda_2(t) = q(t)\lambda(t)$. □

Suppose that within the observation interval time $(0, t)$, the joint probability function of two random variables $N_1(t)$ and $N_2(t)$, which follow Poisson processes as a result corollary 3.1, takes the form of a bivariate Poisson distribution. For $p(t) = p$ and $q(t) = q$ constant, the bivariate Poisson process follows a bivariate Poisson distribution, with its probability function $f(N_1(t) = n_1, N_2(t) = n_2)$ given by:

$$\frac{(\int_0^t \lambda(s)ds)^{n_1+n_2} p^{n_1} q^{n_2} \exp\left(-\int_0^t \lambda^*(s)ds\right)}{n_1! n_2!}, \tag{3}$$

where $\lambda^*(t) = (p+q)\lambda(t)$ untuk $t \in [0, \infty)$.

For the bivariate Poisson process in equation 3, the conditional intensity function will be constructed in several cases. The construction of intensity function includes the constant rate form (λ), the intensity function $\lambda(t)$, and the conditional intensity function $\lambda(t|\mathcal{H}_t)$.

If the intensity function is constant, then the Poisson process is stationary. A constant rate is obtained when the inter-arrival times of the events follow an exponential distribution.

The form of the probability equation for this case is

$$f(N_1(t) = n_1, N_2(t) = n_2) = \frac{(\lambda t)^{n_1+n_2} p^{n_1} q^{n_2} \exp(-\lambda^* t)}{n_1! n_2!},$$

where $\lambda^* = \lambda^*(t|\mathcal{H}_t) = (p+q)\lambda$.

Note that for the type-1 event, the intensity function process is

$$\lambda_1 = p\lambda.$$

While $\lambda(t|\mathcal{H}_t)$ is obtained from

$$\lambda(t|\mathcal{H}_t) = \frac{f(t)}{1 - F(t)} = \lambda,$$

where $f(t) = \lambda e^{-\lambda t}$. It be shown that $\lambda_1(t|\mathcal{H}_t)$ also arises from the exponential probability function with parameter $p\lambda$ where the inter-arrival times of type-1 events follow an exponential distribution with parameter $p\lambda$.

$$\begin{aligned}
\lambda_1(t|\mathcal{H}_t) &= p\frac{f(t)}{1 - F(t)} \\
&= p\frac{\lambda e^{-\lambda t}}{1 - \int_0^t \lambda e^{-\lambda u} du} \\
&= \frac{p\lambda e^{-\lambda t}}{1 - \int_0^t \lambda e^{-\lambda u} du}
\end{aligned}$$

Let $t = p \times s$ for $s \in \mathbb{R}$, then

$$\begin{aligned}
\lambda_1(t|\mathcal{H}_t) &= \frac{p\lambda e^{-p\lambda s}}{1 - \int_0^{p.s} \lambda e^{-\lambda u} du} \\
&= \frac{\lambda_1 e^{-\lambda_1 s}}{1 - \int_0^{p.s} \frac{\lambda_1}{p} e^{-\lambda_1 u/p} du} \\
&= \frac{\lambda_1 e^{\lambda_1 s}}{1 - \int_0^s \lambda_1 e^{-\lambda_1 w} dw} \\
&= \frac{f_1(s)}{1 - F_1(s)}.
\end{aligned}$$

This form confirms that $\lambda_1(t|\mathcal{H}_t) = p\lambda$ is obtained from the ratio of the probability function $f_1(s) = p\lambda e^{-p\lambda s}$ to its survival function $1 - \int_0^s f_1(u)du$. In the same way $\lambda_2(t|\mathcal{H}_t) = q\lambda$ can be expressed as:

$$\lambda_2(t|\mathcal{H}_t) = \frac{f_2(s)}{1 - F_2(s)} = q\lambda,$$

where $f_2(s) = q\lambda e^{-q\lambda s}$, for $s \in \mathbb{R}$. Such that, conditional intensity for bivariate Poisson process is

$$\lambda^*(t|\mathcal{H}_t) = (p+q)\lambda.$$

3.2 Construction of the likelihood function for a bivariate point process

The likelihood function for a bivariate Poisson process describes the probability of a set of events occurring in two distinct event types within a certain time interval. In this case, two Poisson processes with rate λ_1 and λ_2 are independent. Two Poisson processes are obtained from two distinct and independent event groups from a Poisson process with rate λ.

For a bivariate Poisson process obtained from a Poisson process that can be partitioned into two event types, which also follow a Poisson process. Suppose that within the observation interval time $(0, T]$, there are observation points $t_1, t_2, \cdots, t_n$ with n_1 being the number of type-1 events and n_2 being the number of type-2 events, such that $n_1 + n_2 = n$. The likelihood function is:

$$L(\lambda, p, q; n_1, n_2) = \frac{\lambda^n p^{n_1} q^{n_2} \exp(-(p+q)\lambda T)}{n_1! n_2!}, \tag{4}$$

where $T \in [0, \infty)$.

If this bivariate Poisson process is viewed as two independent point processes, assume that for each sub interval (t_{i-1}, t_i), $i = 1, 2, \cdots, n$, only one event of type-1 or type-2 occurs, such that there are n_1 observation for type-1 and n_2 observations for type-2 events. For n_1 observations for type-1 event, probability of exactly one type-1 event occurring in the interval $(t_{i_1-1} - \Delta_{i_1}, t_{i_1})$ and no type-1 events occurring in the other time intervals during the total observation time $(0, T]$ is

$$P(n_1; (0, T]) = (p\lambda)^{n_1} e^{-p\lambda T}. \tag{5}$$

Similarly, for n_2 observations for type-2 event, probability of exactly one type-2 event occurring in the interval $(t_{i_2-1} - \Delta_{i_2}, t_{i_2})$ and no type-2 events occurring in the other time intervals during the total observation time $(0, T]$ is

$$P(n_2; (0, T]) = (q\lambda)^{n_2} e^{-q\lambda T}. \tag{6}$$

Thus, the likelihood function for these two independent Poisson processes is obtained by multiplying the probabilities in equation 5 and 6, which is

$$L(\cdot) = \lambda^{n_1+n_2} p^{n_1} q^{n_2} e^{-(p+q)\lambda T}. \tag{7}$$

The exponential form of equation 7 similar to the form of the likelihood from equation 2 for the bivariate case can be expressed as

$$L(\cdot) = \exp\left((n_1 + n_2)\ln\lambda + n_1 \ln p + n_2 \ln q - (p+q)\lambda T\right). \tag{8}$$

The likelihood function in equation 8 can also be obtained by multiplying the likelihood form in equation 4 by $n_1! n_2!$. The log-likelihoodnya form is:

$$\ln L(\lambda, p, q; n_1, n_2) = n \ln\lambda + n_1 \ln p + n_2 \ln q - (p+q)\lambda T. \tag{9}$$

The resulting likelihood function or log-likelihood function is used to accurately estimate the model parameters based on the observed data.

3.3 Parameters estimation

For observation in the time interval $(0, T]$ and $p + q = 1$, the maximum likelihood estimation for λ, p dan q are as follows:

$$\begin{aligned} \hat{\lambda} &= \frac{n}{T} & (10)\\ \hat{p} &= \frac{n_1}{\hat{\lambda} T} & (11)\\ \hat{q} &= \frac{n_2}{\hat{\lambda} T}. & (12) \end{aligned}$$

Suppose that the observation data in $[0, T)$ only provides the number of type-1 and type-2 events without knowing the exact event times for type-1 and type-2, with m samples. Therefore, the samples obtained are pairs of values $\{(n_{11}, n_{21}), (n_{12}, n_{22}), \cdots, (n_{1m}, n_{2m})\}$. The likelihood function is obtained from equation 8 as:

$$\begin{aligned} L^*(\lambda, p, q) &= \exp\sum_{i=1}^{m}\left((n_{1i} + n_{2i})\ln\lambda\right)\\ &\quad + \exp\left(n_{1i}\ln p + n_{2i}\ln q - (p+q)\lambda T\right). \end{aligned}$$

The log-likelihood form is:

$$\begin{aligned} \ln L^* &= \sum_{i=1}^{m}(n_{1i} + n_{2i})\ln\lambda + \sum_{i=1}^{m} n_{1i}\ln p\\ &\quad + \sum_{i=1}^{m} n_{2i}\ln q - m(p+q)\lambda T. \end{aligned}$$

The first derivatives with respect to the parameters λ, p and q are:

$$\begin{aligned} \frac{\partial \ln L^*}{\partial \lambda} &= -m(p+q)T + \frac{1}{\lambda}\sum_{i=1}^{m}(n_{1i} + n_{2i})\\ \frac{\partial \ln L^*}{\partial p} &= -m\lambda T + \frac{1}{p}\sum_{i=1}^{m} n_{1i}\\ \frac{\partial \ln L^*}{\partial q} &= -m\lambda T + \frac{1}{q}\sum_{i=1}^{m} n_{2i}. \end{aligned}$$

If $p + q = 1$, then the maximum likelihood estimation for λ, p and q are as follows:

$$\begin{aligned} \hat{\lambda} &= \frac{\sum_{i=1}^{m}(n_{1i} + n_{2i})}{mT} & (13)\\ \hat{p} &= \frac{\sum_{i=1}^{m} n_{1i}}{m\lambda T} & (14)\\ \hat{q} &= \frac{\sum_{i=1}^{m} n_{2i}}{m\lambda T}. & (15) \end{aligned}$$

These results obtained are consistent with equations 10, 11, and 12.

4 Simulation two type Poisson experiment

In this section, simulation is involved for parameter estimation using equations 13, 14 and 15. The simulation is for an experiment involving 2 types of Poisson processes derived from a homogeneous Poisson process that can be splitted. The simulation is taken from 2 type experiment Poisson through Applet on the website www.randomservice.org.

In this simulation, data is generated for a homogeneous Poisson process with $\lambda = 3$, $T = 5$ and the probability of type-1 event $p = 0.17$. Observations are made for $m = 10, 100, 500$ and 1000. The simulation results, shown in Figure 1, display two realizations of event points for two types of events, marked in red and green. The realization shows the number of events marked with green and red in the observation time $T = 5$. The green points are the event points of Poisson process with rate $(1 - p)\lambda = 2.49$, and the red points are the event points of Poisson process with rate $p\lambda = 0.51$.

The graph in Figure 2 shows two homogeneous Poisson processes in one plot. The blue and red dots represent the number of events $N(t)$ for each time interval $(0, 5]$, with 10 observations taken. The blue and red dashed lines represent the mean for the observation time interval $(0, 5]$. For 10 observations, the data for the first process, represented by the

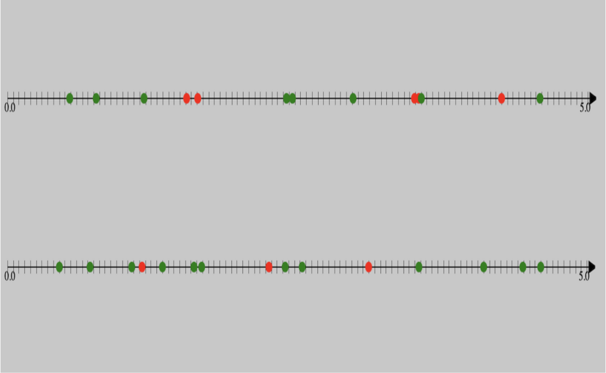

Figure 1. The simulation result for 2 type of Poisson processes from Poisson process with $\lambda = 3, T = 5$ and $p = 0.17$.

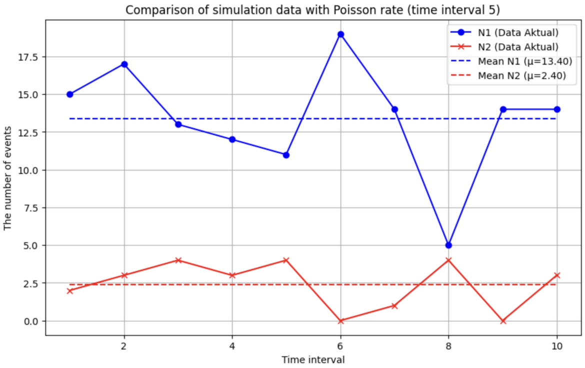

Figure 2. Graph of 10 random observations of 2 types of Poisson Process with time interval $(0.5]$

blue points, is $(15, 17, 13, 12, 11, 19, 14, 5, 14, 14)$ and the data for the second process, represented by the red points, is $(2, 3, 4, 3, 4, 0, 1, 4, 0, 3)$.

The data obtained from the simulation is then used to calculate the estimated parameter values using equations 13 , 14 and 15. As an example, for 10 observations, the calculated values are $\hat{\lambda} = 3.16$, $\hat{p} = 0.848$, and $\hat{q} = 1 - \hat{p} = 0.152$. The calculation results can be seen in Table 1.

Table 1 shows the estimated parameter values for each size $m = 10, 100, 500$ and 1000, which consistently approach the true values. In general, as the sample size increases, the estimated parameters become closer to the true value. For example, the estimate for λ at $m = 10$, $\hat{\lambda} = 3.16$ approaches the true value of 3 as the sample size increases, reaching $\hat{\lambda} = 3.0086$ at $m = 1000$.

Table 2 shows the Mean Squared Error (MSE) for each parameter estimate based on different sample sizes. MSE measures the squared deviation between the estimate and the true value, with smaller values indicating more accurate estimates. For the sample size $m = 10$, the MSE is relatively larger, indicating that the parameter estimates at smaller sample size are less accurate. For instance, the MSE for λ_1 at $m = 10$ is 0.0361, while for larger sample sizes, such as $m = 1000$, the MSE significantly decreases to 0.000004 for λ_1. The same trend is observed for other parameters like p and λ_2, where MSE decreases substantially as the sample size increases, signalling that the estimates become more accurate with more data.

Figure 2 shows the graph of 10 observations within the time interval $(0, 5]$ for two Poisson processes representing two types of events. Figure 2 provides a comparison between the actual data generated from two processes with the expected values based on the bivariate Poisson process, measured using the MLE. The mean value $\mu_1 = 13.40$ is obtained from the estimator $\hat{\lambda}_1 = 2.68$ multiplied by the time interval $T = 5$. Similarly, the mean value $\mu_2 = 2.40$ is obtained from the product of the estimator $\hat{\lambda}_2 = 0.48$ with $T = 5$.

One realization of the two Poisson for $\hat{\lambda}_1 = 2.68$ and $\hat{\lambda}_2 = 0.48$ within the observation time interval $(0.5]$ can be seen in Figure 3.

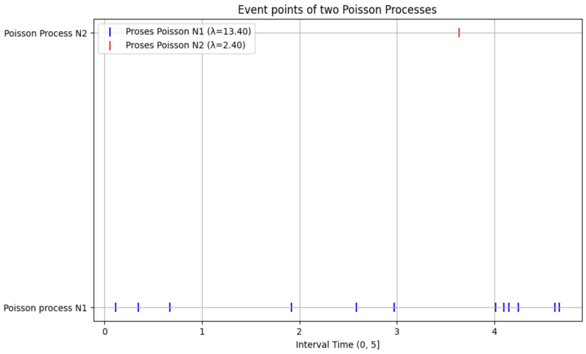

Figure 3. Realization of two Poisson processes in the time interval $(0, 5]$

5 Case study

This section demonstrates the application of the bivariate Poisson process model to two types of data. The first dataset is from the field of epidemiology, while the second dataset is sourced from seismology.

5.1 Bivariate Poisson process for active Covid-19

In this section, a bivariate Poisson process model will be used, as described in equation 8. The data used is https://covid19.go.id/id. The exploration of the reported daily active Covid-19 cases represents the cumulative total of new patients, recovered patients, and deceased patients. For example, in one day, the data on active case count for recovered and deceased patients can be expressed as events that follow a bivariate Poisson process.

Suppose the data observed is from the last month before the official end of the pandemic in Indonesia on June 26, 2023. The data in Table 3 shows the number of patients who died

Table 1. Simulation results of the parameter estimation of the Homogeneous Bivariate Poisson Process for m = 10, 100, 500 and 1000

m	λ	p	λ_1	$q = 1 - p$	λ_2
10	3.16	0.8481	2.68	0.15189873	0.48
100	3.108	0.8301	2.58	0.1699	0.528
500	2.9744	0.8232	2.4484	0.1768	0.5260
1000	3.0086	0.8269	2.4880	0.1730	0.5206
True value	3	0.83	2.49	0.17	0.51

Table 2. MSE for each estimated parameter with its sample size

m	MSE_λ	MSE_p	MSE_{λ_1}	MSE_q	MSE_{λ_B}
10	0.0256	0.00032766	0.0361	0.00032766	0.0009
100	0.011664	0.00000001	0.0081	0.00000001	0.000324
500	0.000655	0.0000468	0.0017306	0.0000468	0.000256
1000	0.000074	0.00000923	0.000004	0.00000923	0.000112

and recovered from Covid-19 over a 30-day period.

Let N_1 be the number of patients who died in a day and N_2 be the number of patients who recovered from Covid-19 in a day. The joint probability function for The bivariate Poisson process is

$$f(N_1(t) = n_1, N_2(t) = n_2) = \frac{(p\lambda t)^{n_1}(q\lambda t)^{n_2}}{n_1! n_2!} e^{-(p+q)\lambda t}.$$

Table 3 shows that for a sample of 30 days of observation ($T = 1$ day), the average number of recovered patients is 351 people and the average number of deceased patients is 5 people. Thus, the log-likelihood function for this sample is

$$\begin{aligned}\ln L(\lambda, p, q) &= -30(p+q)\lambda + \sum_{i=1}^{30}(n_{1i} + n_{2i}) \ln \lambda \\ &\quad + \ln p \sum_{i=1}^{30} n_{1i} + \ln q \sum_{i=1}^{30} n_{2i}.\end{aligned}$$

The estimator for $\lambda, p,$ and q are

$$\begin{aligned}\hat{\lambda} &= \frac{\sum_{i=1}^{30}(n_{1i} + n_{2i})}{30} \\ \hat{p} &= \frac{\sum_{i=1}^{30} n_{1i}}{30\hat{\lambda}} \\ \hat{q} &= \frac{\sum_{i=1}^{30} n_{2i}}{30\hat{\lambda}}.\end{aligned}$$

Thus, the intensity rate of Covid-19 (recovered and deceased) patients, ($\hat{\lambda}$) using the bivariate point process for the 30-day observations is the average number of patients who recovered and died. The probability of death, ($\hat{p}$) is the ratio of deceased patients to the intensity rate over 30 days. The probability of recovery, $\hat{q}$ is the ratio of recovered patients to the intensity rate over 30 days.

Based on Table 3, for the observation from May 28 to June 26, 2023, the number of deceased patients is $\sum_{i=1}^{30} n_{1i} = 134$ and the number of recovered patients is $\sum_{i=1}^{30} n_{2i} = 10,522$.

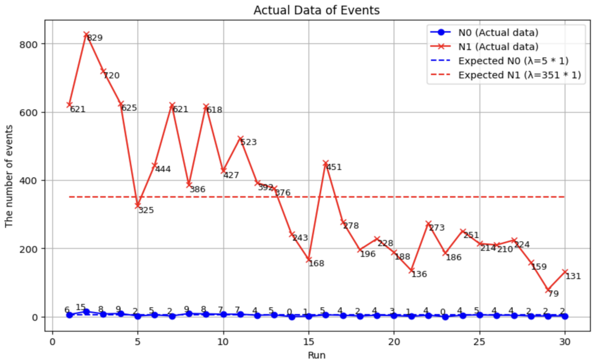

Figure 4. The distribution data Poisson processes with $\lambda_1 = 5$ and $\lambda_2 = 351$

Therefore, the conditional intensity function for the bivariate Poisson process is $\lambda(t|\mathcal{H}_t) = 355$. The probability of death is $p = 0.0125$ and the probability of recovery is $q = 0.9874$. Figure 4 shows two types of events in a Poisson process. For type-1 events, the actual data exhibits greater variation compared to the mean value $\mu_1(T = 1) = \lambda_2 = 351$. For type-2 events, the actual data closely follows the mean value $\mu_1(T = 1) = \lambda_1 = 5$, although some variation is still observed.

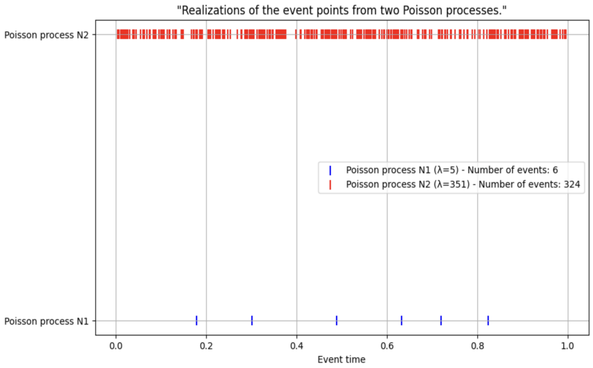

Figure 5. Realization two Poisson processes

Graph in Figure 5 shows a comparison between two Poisson processes for Data Covid-19 with very different event conditional intensity functions or rates. The process Poisson for

Table 3. Data on the number of patients who died and recovered from Covid-19 in Indonesia from May 28 - June 26, 2023

Death	Recovery	Death	Recovery	Death	Recovery
6	621	7	523	1	136
15	829	4	392	4	273
8	720	5	376	0	186
9	625	0	243	4	251
2	325	1	168	5	214
5	444	5	451	4	210
2	621	4	278	4	224
9	386	2	196	2	159
8	618	4	228	2	79
7	427	3	188	2	131

events of deceased patients Covid-19 generates fewer events, as seen from the lower number of blue dots along the x-axis. The Poisson process for events of recovery patients Covid-19 generates more events, as seen from the higher number of red dots along the x-axis. The process for recovery patients has a much higher event intensity, resulting in more events within the same time period compared to the process for deceased patients.

5.2 Bivariate Poisson process for earthquake

In this study, the bivariate Poisson process will be applied to estimate the probability of an earthquake of magnitude 5 and above from any possible earthquake event that occurs on the island of Sulawesi. The dataset spans from January 2018 to December 2020 and is sourced from the Meteorology, Climatology, and Geophysics agency (BMKG), region IV Makassar. The application of the model involves categorising earthquake events into two groups based on earthquake magnitude: one group for earthquake events associated with earthquakes of magnitude less than 5, and another for those linked to earthquakes of magnitude 5 or greater.

The data in Table 4 shows the number of earthquakes in two groups per month for the years 2018 to 2020. It can be seen that there are fluctuations in the number of earthquake in both categories each month.

Let N_1 be the number of earthquakes of magnitude less 5 in a month and N_2 be the number of earthquakes of magnitude 5 or greater in a month. The number of earthquakes of magnitude less 5, $N_1(t)$ is Poisson process with rate $\lambda_1 = p\lambda$, and the number of earthquakes of magnitude 5 or greater, $N_2(t)$ is Poisson process with rate $\lambda_2 = q\lambda$. Such that, the distribution function for the bivariate Poisson process is

$$f(N_1(t) = n_1, N_2(t) = n_2) = \frac{p^{n_1} q^{n_2} (\lambda t)^{n_1+n_2}}{n_1! n_2!} e^{-(p+q)\lambda t}.$$

Based on Table 4, for 36 months of observation ($T = 1$ month), the total number of earthquakes of magnitude less 5 is 8733 and the total number of earthquakes of magnitude 5 or greater only is 126. Thus, the log likelihood function is

$$\begin{aligned} \ln L(\lambda, p, q) &= -36(p+q)\lambda + \sum_{i=1}^{36}(n_{1i} + n_{2i}) \ln \lambda \\ &\quad + \ln p \sum_{i=1}^{36} n_{1i} + \ln q \sum_{i=1}^{36} n_{2i}. \end{aligned}$$

The estimator for parameters λ and probability that an earthquake occurirng at given time is of magnitude 5 or higher q are

$$\begin{aligned} \hat{\lambda} &= \frac{\sum_{i=1}^{36}(n_{1i} + n_{2i})}{36} \\ \hat{q} &= \frac{\sum_{i=1}^{36} n_{2i}}{36\hat{\lambda}}. \end{aligned}$$

Since the total number of earthquakes of magnitude less 5 is $\sum_{i=1}^{36} n_{1i} = 8733$ and the total number of earthquakes of magnitude 5 or greater only is $\sum_{i=1}^{36} n_{2i} = 126$, the estimated values are obtained as

$$\begin{aligned} \hat{\lambda} &= \frac{\sum_{i=1}^{36}(n_{1i} + n_{2i})}{36} = \frac{8733 + 126}{36} = 246.083 \\ \hat{q} &= \frac{\sum_{i=1}^{36} n_{2i}}{36\hat{\lambda}} = \frac{126}{36 \times 246.08} = 0.014223. \end{aligned}$$

The number of earthquake of magnitude less 5 is Poisson process with intensity rate

$$\hat{\lambda_1} = (1 - \hat{q})\hat{\lambda} = 242.5833,$$

and the number of earthquake of magnitude 5 or greater is Poisson process with intensity rate

$$\hat{\lambda_2} = \hat{q}\hat{\lambda} = 3.5.$$

6 Conclusions

The probability model for a homogeneous bivariate Poisson process is

$$f(n_1, n_2 : \lambda, p, q) = e^{-\lambda(p+q)t} \frac{\lambda^{n_1+n_2} p^{n_1} q^{n_2}}{n_1! n_2!},$$

where λ is the overall intensity function, $0 < \alpha < 1$, $0 < \beta < 1$, n_1 is the number of type-1 events, and n_2 is the number of

Table 4. Data on the number of earthquakes that occurred on Sulawesi Island fro January 2018 to December 2020

Month	Mag 1-5	Mag 5 - up	Month	Mag 1-5	Mag 5 - up	Month	Mag 1-5	Mag 5 - up
Jan-18	193	3	Jan-19	195	0	Jan-20	136	3
Feb-18	161	3	Feb-19	146	4	Feb-20	159	4
Mar-18	261	8	Mar-19	132	1	Mar-20	266	2
Apr-18	168	1	Apr-19	72	2	Apr-20	247	1
May-18	138	2	May-19	94	4	May-20	226	0
Jun-18	148	0	Jun-19	160	2	Jun-20	239	7
Jul-18	183	2	Jul-19	108	0	Jul-20	204	0
Aug-18	163	0	Aug-19	72	1	Aug-20	191	6
Sep-18	518	32	Sep-19	99	0	Sep-20	194	1
Oct-18	940	8	Oct-19	74	2	Oct-20	318	7
Nov-18	1393	8	Nov-19	109	2	Nov-20	253	1
Dec-18	431	3	Dec-19	128	0	Dec-20	214	6

type-2 events.

This bivariate Poisson process can be viewed as two independent point processes within the observation time period $(0, T]$, which is divided into n sub-intervals. In each sub-interval (t_{i-1}, t_i), for $i = 1, 2, \cdots, n$, only one event of type-1 or type-2 occurs, with n_1 events of type-1 and n_2 events of type-2, where $n_1 + n_2 = n$. The likelihood for bivariate Poisson process is given by the following expression:

$$L(\cdot) = \exp\left((n_1 + n_2)\ln\lambda + n_1 \ln p + n_2 \ln q - (p+q)\lambda T\right).$$

For observations over the time interval $[0, T)$ and $p + q = 1$ conducted m samples, the maximum likelihood estimators for the parameters λ, p and q are

$$\begin{aligned}\hat{\lambda} &= \frac{1}{mT}\sum_{i=1}^{m}(n_{1i} + n_{2i}) \\ \hat{p} &= \frac{1}{m\hat{\lambda}T}\sum_{i=1}^{m} n_{1i} \\ \hat{q} &= \frac{1}{m\hat{\lambda}T}\sum_{i=1}^{m} n_{2i}.\end{aligned}$$

The intensity rate for the events of type-1 and type-2 are

$$\begin{aligned}\hat{\lambda}_1 &= \hat{p}\hat{\lambda} = \frac{\sum_{i=1}^{m} n_{1i}}{mT} \\ \hat{\lambda}_2 &= \hat{q}\hat{\lambda} = \frac{\sum_{i=1}^{m} n_{2i}}{mT}.\end{aligned}$$

The results show that the conditional intensity of a bivariate point process depends on the number of events n and the time interval of observation T. The application on the number of active cases Covid in Indonesia show that the conditional intensity of the incidence of recovery and death patients is inversely proportional to the time of observation. Analog to the earthquake data on Sulawesi, events with magnitude less than 5 and events with magnitude of 5 or greater show that the intensity of the earthquakes is inversely proportional to the observation time interval.

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