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Antimagic Labeling of Digraphs

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Abstract A directed graph D can be labeled antimagically by assigning different integers to its arcs, ensuring that the computed vertex weights are distinct. Antimagic labeling exists in a digraph D with h arcs and g vertices if it is possible to uniquely match each arc to an integer from 1 to h. For every vertex, the difference between the totals of the labels from incoming arcs and outgoing arcs is unique for each vertex. A directed graph that allows such antimagic labeling is referred to as an antimagic digraph. There are countless methods to create an antimagic digraph. These constructions are essential in fields such as network theory, coding theory, and combinatorial optimization. The subset sum problem is a well-recognized issue in both computer science and combinatorics. These problems play a decisive role in graph labeling, especially when it comes to the creation and evaluation of specific types of labels. In this paper, we connect the idea of subset sum problems with wheel digraphs $\overrightarrow{W_n}$ to represent it as antimagic. The Cartesian product of directed graphs is a fundamental concept in graph theory that holds both theoretical and practical importance. The applications of Cartesian products include network design and analysis, parallel computing, graph decomposition and construction, Game Theory, and Decision-Making. Additionally, in this paper, we have developed antimagic digraphs from the Cartesian product of directed path P_n and $\overrightarrow{K_2}$ by traversing the directed path $\overrightarrow{P_n}$ in alternating and unidirectional ways.

Keywords Cartesian Product, Wheel Digraph, Subset-sum, Antimagic

1 Introduction

Many researchers nowadays prefer working in graph theory, especially in graph labeling owing to its various applications. Potential uses of graph labeling include addressing challenges in Mobile Adhoc Networks. A graph-based model can be utilized to explore issues related to connectivity, scalability, routing, network simulation, and modeling. Graphs can be expressed as matrices, and problems can be examined using algorithms. Concepts from random graphs can illustrate node density, mobility, link creation, and routing. Various methods can be employed to analyze congestion in Mobile Adhoc Networks, and the principles of graph theory can be applied to model these networks. Additionally, the idea of 2-odd labeling in graphs has been applied to design the restricted frequency spectrum for the global mobile communication system as the number of subscribers continues to rise [1]. Gallian's [2] dynamic study of graph labeling provides a comprehensive analysis of many of the different labeling types. Hartsfield and Ringel [3] conceived the concept of antimagic labeling for a graph, and they defined it as follows.

For a graph G consisting of i vertices and j edges, antimagic labeling is a one-to-one correspondence from the edges to the integers $\{1, 2, \ldots, j\}$ in such a way that the sums of the incident edge labels at each vertex are distinct. The total of all the edge labels on a certain vertex is used to calculate its weight. D. Hefetz, T. Mutze, and J. Schwartz [4] expanded the idea of antimagic labeling of graphs to include labeling of digraphs, defined as follows.

A digraph D containing l vertices and m arcs can be labeled in an antimagic way by establishing a one-to-one correspondence between its arcs and the integers $\{1, 2, \ldots, m\}$, ensuring that the l-directed vertex weights are all mutually exclusive. A directed vertex weight represents the sum of the labels of all incoming arcs to that vertex minus the sum of the labels of all outgoing arcs from that vertex.

The following definitions prove Theorem 2.1, Proposition 2.2, and Theorem 2.5.

Definition 1.1 [5] Finding a subset of numbers whose sum matches the target sum supplied from a set of integers and a goal sum is the task at hand in the subset sum problem in computer science. Since there is currently no effective solution to tackle this problem for every potential input, it is NP-complete. However, it is possible to design approximate algorithms that use heuristics to find approximate solutions.

Definition 1.2 [6] For digraphs A and B with vertex sets $V(A) = \{u_1, u_2 \ldots u_n\}$ and $V(B) = \{v_1, v_2 \ldots v_m\}$ are taken into consideration. $V(A \times B) = \{(x, y) \mid x \in V(A), y \in V(B)\}$ is the vertex set for the Cartesian product of digraphs A and B, denoted by $A \times B.\{(x, y), (x_1, y_1) \mid xx_1 \in E(A), y = y_1, or x = x_1, yy_1 \in E(B)\}$ is the arc set.

Definition 1.3 [7] A network with $n + 1$ vertices and $2n$ edges that has a universal vertex (the "rim") attached to every vertex in the cycle is called a wheel, denoted by W_n.

Definition 1.4 [7] If all of a directed wheel's spokes extend from the universal vertex to the rim, the directed wheel is said to be outspoken. A directed wheel is referred to as unicyclic if its rim is unidirectional.

2 Antimagic labeling of $\overrightarrow{W_n}$ and $\overrightarrow{P_n} \times \overrightarrow{K_2}$

Research has demonstrated that lattice grid graphs (the Cartesian product of paths $P_m \times P_n$) and prism grid graphs (the Cartesian product of paths and cycles $P_m \times C_n$) are antimagic, as shown in [8]. In this study, we have broadened the concept of labeling from graphs to digraphs.

This section deals with partitioning a specific set and operating it on the edges of $\overrightarrow{W_n}$ to render it antimagic. Additionally, by moving the path in different directions, a digraph created by taking the Cartesian product of $\overrightarrow{P_n}$ and $\overrightarrow{K_2}$ has been labeled in an antimagic way.

Theorem 2.1 An outspoken wheel $\overrightarrow{W_n}$ is antimagic if its arc set $S = \{1, 2, 3 \ldots (2n-2)\}$ where $n \geq 4$ can be divided into $n - 1$ portions with
i)$\mid s_i \mid = 3$ for $n - 1 \geq i \geq 1$
ii)$\mid s_i \cap s_{i+1} \mid$=1 for $n - 1 \geq i \geq 1$ (where the modulo$(n-1)$ subscripts are used)
iii)For every subset, the sum of its components equals 0 mod 2.

Proof. Consider the set $S = \{1, 2, 3 \ldots (2n-2)\}$ where $n \geq 4$.
This set can be divided into $n - 1$ subsets as $\{(1, 2, 3), (3, 4, 5) \ldots (2n-5, 2n-4, 2n-3), (2n-3, 2n-2, 1)\} \ldots (1)$
The intersection of any two consecutive subsets in (1) has one element in common, and the sum of each subset's elements is congruent to zero mod 2.
Let us describe the digraph $\overrightarrow{W_n}$ as follows:
$\overrightarrow{W_n}$ represents a directed graph that contains n vertices and $2n - 2$ directed edges. Imagine that the universal vertex is v_0 and that the vertices of the rim are $\{v_i\}$ for i ranging from 1 to $n-1$. Now for $1 \leq i \leq n-1$, mark the spoke arcs as $\{e_{2i-1}\}$, and denote the rim arcs oriented unidirectionally (clockwise) as $\{e_{2i}\}$.
Currently, each of the $(n - 1)$ subsets specified in (1) is being allocated to the $(n - 1)$ cycles represented by $\overrightarrow{v_0v_i} \cup \overrightarrow{v_iv_{i+1}} \cup \overrightarrow{v_0v_{i+1}}$ for values of i ranging from 1 to $n - 1$, where the subscripts are taken over modulo $(n - 1)$ as $(e_{2i-1}, e_{2i}, e_{2i+1}) = (-1 + 2i, 2i, 1 + 2i)$ for $n - 1 \geq i \geq 1$ where the subscripts are taken over modulo $(2n - 2)$.
To show that $\overrightarrow{W_n}$ is antimagic we shall first find the weights of the vertices $v_0, v_1 \ldots v_{n-1}$ as follows:
$W(v_o) = -(n-1)^2$
$W(v_1) = (e_1 + e_{2n-2}) - e_2$
$W(v_{i-1}) = (e_{2i-4} + e_{2i-3}) - e_{2i-2}$ for $3 \leq i \leq n$.
We now demonstrate that each of the vertex weights specified above is unique.
Suppose $W(v_o) = W(v_1) = a$.
This implies $-(n-1)^2 - (e_1 + e_{2n-2}) + e_2 = 0$
$\implies -(n-1)^2 - 1 - 2n + 2 + 2 = 0$
$\implies -(2n-3) - (n^2 + 1 - 2n) = 0$
$\implies -2 + n^2 = 0$, which is a contradiction since $n \geq 4$.
Similarly, this contradiction holds good for any pair of distinct vertices $v_0, v_1 \ldots v_{n-1}$.
Hence the proof.

Example 1: The following Figure 1 shows the antimagic labeling of an outspoken wheel $\overrightarrow{W_9}$

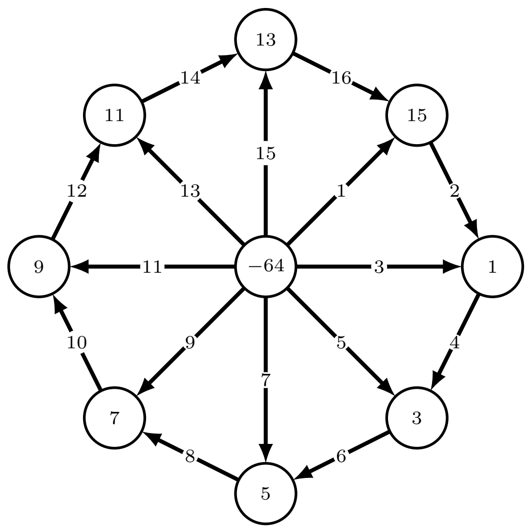

Figure 1: Antimagic labeling of an outspoken wheel $\overrightarrow{W_9}$

Proposition 2.2 Outspoken wheels $\overrightarrow{W_n}$ that are antimagic have rim labels whose sum is equivalent to zero modulo $(-1+n)^2$.

Proof. Marking the arcs of the wheel $\overrightarrow{W_n}$, which is outspoken and deriving the vertex labels as outlined in Theorem 2.1, the total of the rim vertices weights is given as follows:
$W(v_1) + W(v_2) + W(v_3) + \ldots + W(v_{n-1}) = [(e_1 + e_{2n-2}) - e_2] + [(e_2 + e_3) - e_4] + \ldots + [(e_{2n-6} + e_{2n-5}) - e_{2n-4}] + [(e_{2n-4} + e_{2n-3}) - e_{2n-2}]$
$=[1+(2n-2)-2]+[(2+3)-4]+\ldots+[(-6+2n)+(-5+2n)-(-4+2n)]+[(-4+2n)+(-3+2n)-(-2+2n)]$
$=1+3+5+\ldots+(-5+2n)+(-3+2n)+(-1+2n)-(-1+2n)$
$=(n-1)^2 \equiv 0 mod (n-1)^2$
Hence the proof.

Note 2.3. Suppose we consider an inspoken directed wheel $\overrightarrow{W_n}$ and describe the wheel the same as in Theorem 2.1. Also maintaining the same arc labels one can obtain the vertex weights as follows:
$W(v_0) = (n-1)^2$,
$W(v_{n+1}) = -(2n+3)$ for $i = 1, 2, 3...n-2$
$W(v_{n-1}) = -(2n-1)$

Example 2. The following Figure 2 shows the antimagic labeling of an in-spoken wheel $\overrightarrow{W_9}$

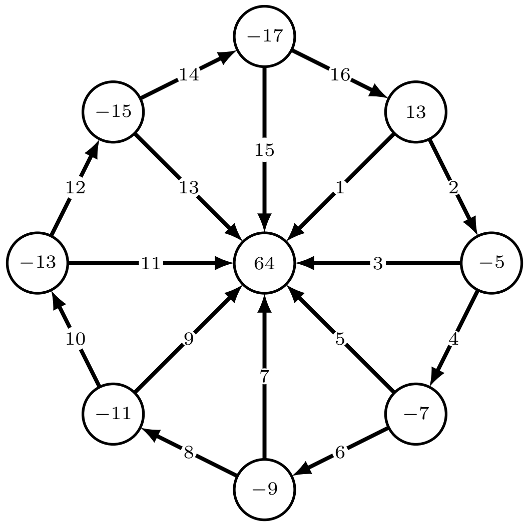

Figure 2: Antimagic labeling of an in-spoken wheel $\overrightarrow{W_9}$

Observation 2.4

1. Inspoken wheels and outspoken wheels that are antimagic have rim labels that add up to zero modulo $(-1+n)^2$.
2. The labeling technique used to label both inspoken and outspoken wheels has no limitations for n. One can use the same labeling techniques discussed above to label W_n for both odd and even n.

Theorem 2.5. The Cartesian product $G = \overrightarrow{P_n} \times \overrightarrow{K_2}$ is antimagic when the path $\overrightarrow{P_n}$ is traversed unidirectionally and alternatively.

Proof: Let $V(\overrightarrow{P_n}) = \{u_1, u_2 \ldots u_n\}$ and $V(\overrightarrow{K_2}) = \{v_1, v_2\}$. The digraph $G = \overrightarrow{P_n} \times \overrightarrow{K_2}$ is a ladder digraph with $2n$ vertices and $3n-2$ arcs.
Case 1: When the path $\overrightarrow{P_n}$ is unidirectional and traversed in the right direction.
Denote the vertices of $G = \overrightarrow{P_n} \times \overrightarrow{K_2}$ as $\{(u_i, v_j) = h_{ij} for 1 \leq i \leq n, 1 \leq j \leq 2\}$ and the arcs as $\overrightarrow{h_{i1}h_{(1+i)1}} \cup \overrightarrow{h_{i2}h_{(1+i)2}} \cup \overrightarrow{h_{i1}h_{i2}}$ for $1 \leq i \leq n$
Define a labeling function g from the arcs of $\overrightarrow{P_n} \times \overrightarrow{K_2}$ to the integers $\{1, 2 \ldots -2+3n\}$ as follows:
The function $g(\overrightarrow{h_{i1}h_{(i+1)1}})$ evaluates to $-1+2i$ for $1 \leq i \leq n-1$.
The function $g(\overrightarrow{h_{i2}h_{(1+i)2}})$ results in $2i$ for $1 \leq i \leq n-1$.
The function $g(\overrightarrow{h_{i1}h_{i2}})$ gives $i-2+2n$ for $1 \leq i \leq n$. The weights of the vertices can be calculated using the below formulas
$W(h_{11}) = -2n$
$W(h_{(-2-i+n)1}) = 2+i-3n$ for $1 \leq i \leq n-4$
$W(h_{(-2+n)1}) = 2-3n$
$W(h_{n1}) = -(1+n)$
$W(h_{i1}) = -3i-2$ for $i = n-1$
$W(h_{12}) = 2n-3$
$W(h_{(i+1)2}) = 2n-3+i$ for $1 \leq i \leq n-2$
$W(h_{n2}) = 5n-4$.
Each vertex weight is unique and the arc labels comprise the set $\{1, 2 \ldots 3n-2\}$. Hence the digraph $\overrightarrow{P_n} \times \overrightarrow{K_2}$ antimagic when the path $\overrightarrow{P_n}$ is unidirectional and traversed in right direction.

Case 2: When the path $\overrightarrow{P_n}$ is unidirectional and traversed in the left direction.
Denote the vertices of $G = \overrightarrow{P_n} \times \overrightarrow{K_2}$ as $\{(u_i, v_j) = h_{ij}$ for $1 \leq i \leq n, 1 \leq j \leq 2\}$ and the arcs as $\overrightarrow{h_{i1}h_{(-1+i)1}} \cup \overrightarrow{h_{2i}h_{2(-1+i)}} \cup \overrightarrow{h_{j1}h_{j2}}$ for $n \leq i \leq 2, 1 \leq j \leq n$.
Define a labeling function $f : E(\overrightarrow{P_n} \times \overrightarrow{K_2}) \rightarrow \{1, 2 \ldots 3n-2\}$ as
$f(\overrightarrow{h_{i1}h_{(-1+i)1}}) = -1+2j$ for $n \leq i \leq 2, 1 \leq j \leq n-1$.
$f(\overrightarrow{h_{2i}h_{2(-1+i)}}) = 2j$ for $n \leq i \leq 2, 1 \leq j \leq n-1$.
$f(\overrightarrow{h_{i1}h_{j2}}) = j+2n-2$ for $n \leq i \leq 1, 1 \leq j \leq n$.
The weights of the vertices can be calculated in the following manner
$W(h_{n1}) = -2n$
$W(h_{11}) = -(n+1)$
$W(h_{21}) = -3n+1$
$W(h_{31}) = -3n+2$
$W(h_{(i+3)1}) = 2+i-3n$ for $1 \leq i \leq n-4$
$W(h_{n2}) = 2n-3$
$W(h_{(-i+n)2}) = -3+i+2n$ for $1 \leq i \leq n-2$
$W(h_{12}) = 5n-4$
Each vertex weight is unique and the arc labels comprise the set $\{1, 2 \ldots 3n-2\}$. Hence the digraph $\overrightarrow{P_n} \times \overrightarrow{K_2}$ antimagic when the path $\overrightarrow{P_n}$ is unidirectional and traversed in the left direction.

Case 3: When the path $\overrightarrow{P_n}$ is orientated in the alternate direction.

When the path $\overrightarrow{P_n}$ has even number of vertices:
Adjust the direction of the arcs along the path in this way.
$\overrightarrow{h_{(-1+2i)1}h_{(2i)1}} \cup \overleftarrow{h_{(1+2i)1}h_{(2i)1}} \cup \overrightarrow{h_{(-1+n)1}h_{n1}} \cup \overrightarrow{h_{(1+2i)2}h_{(2i)2}} \cup \overleftarrow{h_{(1+2i)2}h_{(2i)2}} \cup \overrightarrow{h_{(-1+n)2}h_{n2}} \cup \overrightarrow{h_{j1}h_{j2}}$
for $1 \le i \le (n/2) - 1$ and $1 \le j \le n$.
Define a labeling function $f : E(\overrightarrow{P_n} \times \overrightarrow{K_2}) \rightarrow \{1, 2, \ldots, 3n-2\}$ such that:
$f(\overrightarrow{h_{i1}h_{(1+i)1}}) = i$ for $i = 1, 3, 5 \ldots, n-1$
$f(\overleftarrow{h_{i1}h_{(-1+i)1}}) = -1 + i$ for $i = 3, 4, 5 \ldots, n-1$
$f(\overrightarrow{h_{(-1+2i)2}h_{(2i)2}}) = n - 2 + 2i$ for $i = 1, 2, \ldots, \frac{n}{2}$
$f(\overleftarrow{h_{(1+2i)2}h_{(2i)2}}) = n + 2i - 1$ for $i = 1, 2, \ldots, \frac{n}{2} - 1$
$f(\overrightarrow{h_{(-i+1+n)1}h_{(-i+1+n)2}}) = -i - 1 + 3n$ for $i = 1, 2, 3, 4 \ldots, n$ The following formula is used to determine the vertex weights:
$W(h_{11}) = -2n$
$W(h_{12}) = n - 1$
$W(h_{n1}) = -(2n - 1)$
$W(h_{n2}) = 5n - 4$
For $1 \le i \le -1 + (n/2)$
$W(h_{(-2i+n)1}) = -(2i - 1 + n)$
$W(h_{(1+2i)1}) = -(2n + 6i)$
$W(h_{(2i+1)2}) = -2i$
$W(h_{(2i)2}) = -(4n + 6i - 5)$
Each vertex weight is unique and the arc labels comprise the set $\{1, 2 \ldots 3n-2\}$. Hence the digraph $\overrightarrow{P_n} \times \overrightarrow{K_2}$ antimagic when the path $\overrightarrow{P_n}$ is traversed in alternate direction.

When n is odd
Orient the arcs of the path in the following manner:
$\overrightarrow{h_{(-1+2i)1}h_{(2i)1}} \cup \overleftarrow{h_{(1+2i)1}h_{(2i)1}} \cup \overrightarrow{h_{(-1+2i)2}h_{(2i)2}} \cup \overleftarrow{h_{(1+2i)2}h_{(2i)2}} \cup \overrightarrow{h_{j1}h_{j2}}$ for $1 \le i \le (n-1)/2$, $1 \le j \le n$
Make a labeling function as $q : E(\overrightarrow{P_n} \times \overrightarrow{K_2}) \rightarrow \{1, 2 \ldots -2 + 3n\}$ as
$q(\overrightarrow{h_{(-1+2i)1}h_{(2i)1}}) = 2i - 1$ for i ranging from $1 to (n-1)/2$
$q(\overleftarrow{h_{(2i)1}h_{(1+2i)1}}) = 2i$ for i ranging from $1 to (n-1)/2$
$q(\overrightarrow{h_{(-1+2i)2}h_{(2i)2}}) = -2 + 2i + n$ for $i = 1, 2 \ldots (n-1)/2$
$q(\overleftarrow{h_{(2i)2}h_{(1+2i)2}}) = -1 + n + 2i$ for $i = 1, 2 \ldots (n-1)/2$
$q(\overrightarrow{h_{(-i+1+n)1}h_{(-i+1+n)2}}) = -i - 1 + 3n$ for $i = 1, 2 \ldots n$
The vertex weights are calculated in the following matter:
$W(h_{11}) = -2n$
$W(h_{12}) = n - 1$
$W(h_{n1}) = -(4n - 3)$
$W(h_{n2}) = n$
$W(h_{(1+n-2i)1}) = -(2i - 2 + n)$ for $1 \le i \le (n-1)/2$
$W(h_{(-2i+n)2}) = -(-1 - 2i + n)$ for $1 \le i \le (n-3)/2$
$W(h_{(2i+1)1}) = -(2n + 6i)$ for $1 \le i \le (n-3)/2$
$W(h_{(2i)2}) = 4n + 6i - 5$ for $1 \le i \le (-1 + n)/2$
It is evident that each vertex weight is unique and the arc labels comprise the set $\{1, 2 \ldots 3n-2\}$. Hence the digraph $\overrightarrow{P_n} \times \overrightarrow{K_2}$ antimagic when the path $\overrightarrow{P_n}$ is traversed in alternate direction.

Example 3: We give an illustration for case 3 as follows:
When $n = 7$, the arcs of the digraph $\overrightarrow{P_7} \times \overrightarrow{K_2}$ is labeled in the following manner:
$\overrightarrow{h_{11}h_{21}} = 1$, $\overrightarrow{h_{31}h_{41}} = 3$, $\overrightarrow{h_{51}h_{61}} = 5$, $\overleftarrow{h_{21}h_{31}} = 2$, $\overleftarrow{h_{41}h_{51}} = 4$, $\overleftarrow{h_{61}h_{71}} = 6$, $\overrightarrow{h_{12}h_{22}} = 7$, $\overrightarrow{h_{32}h_{42}} = 9$, $\overrightarrow{h_{52}h_{62}} = 11$, $\overleftarrow{h_{22}h_{32}} = 8$, $\overleftarrow{h_{42}h_{52}} = 10$, $\overleftarrow{h_{62}h_{72}} = 12$, $\overrightarrow{h_{71}h_{72}} = 19$, $\overrightarrow{h_{61}h_{62}} = 18$, $\overrightarrow{h_{51}h_{52}} = 17$, $\overrightarrow{h_{41}h_{42}} = 16$, $\overrightarrow{h_{31}h_{32}} = 15$, $\overrightarrow{h_{21}h_{22}} = 14$, $\overrightarrow{h_{11}h_{12}} = 13$.
The vertices receive the following labels:
$W(h_{11}) = -14$, $W(h_{12}) = 6$, $W(h_{71}) = -25$, $W(h_{72}) = 7$, $W(h_{61}) = -7$, $W(h_{41}) = -9$, $W(h_{21}) = -11$, $W(h_{52}) = -4$, $W(h_{32}) = -2$, $W(h_{31}) = -20$, $W(h_{51}) = -26$, $W(h_{22}) = 29$, $W(h_{42}) = 35$, $W(h_{62}) = 41$
This is represented in Figure 3.

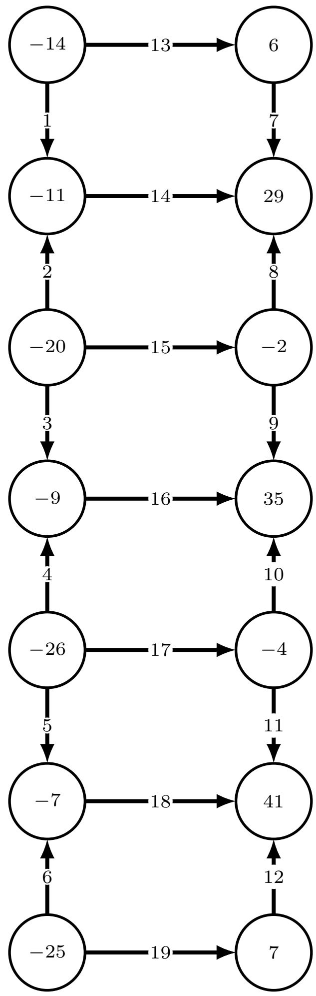

Figure 3: Antimagic labeling of $\overrightarrow{P_7} \times \overrightarrow{K_2}$

Observation 2.6. In the demonstration of Theorem 2.5, in case 1, let us keep the same arc labels for $\overrightarrow{P_n} \times \overrightarrow{K_2}$, but reverse the orientation of K_2 once. The resulting digraph $\overrightarrow{P_n} \times \overleftarrow{K_2}$ does not satisfy antimagic conditions because $W(h_{11}) = W(h_{21}) = 2n - 2$ for every value of n.

In case 2, let's assume we keep the same arc labels for $\overrightarrow{P_n} \times \overrightarrow{K_2}$, but we reverse the orientation of K_2. The digraph $\overrightarrow{P_n} \times \overrightarrow{K_2}$ becomes non-antimagic because $W(h_{n1}) = W(h_{(n-1)1}) = 2n - 2$ for any value of n.

3 Conclusions

This article demonstrated the process of creating antimagic digraphs for specific orientations of $\overrightarrow{W_n}$. One can utilize the idea of subset sum problems to construct similar antimagic digraphs. Likewise, antimagic digraphs can be formed for both inspoken and outspoken wheels by orienting their rims in an anticlockwise direction. Additionally, we established that $\overrightarrow{P_n} \times \overrightarrow{K_2}$ is antimagic for various orientations of the path $\overrightarrow{P_n}$. Furthermore, one can explore the Cartesian product of different families of digraphs to create antimagic digraphs from these combinations.

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