Mathematics and Statistics 13(2): 97-104, 2025
DOI: 10.13189/ms.2025.130204
http://www.hrpub.org

Structural Properties and Characteristic Polynomials of Cubic Power Graph of Dihedral Group

Mamika Ujianita Romdhini[1], **Pankaj Rana**[2,*], **Amit Sehgal**[3], **Pooja Bhatia**[2]

[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Mataram, Mataram 83125, Indonesia
[2]Department of Physical Sciences (Mathematics), Faculty of Sciences, Baba Mastnath University, Asthal Bohar (Rohtak), Haryana-124021, India
[3]Department of Mathematics, Pt. NRS Government College, Rohtak, Haryana-124001, India

Received December 15, 2024; Revised March 12, 2025; Accepted April 14, 2025

Cite This Paper in the Following Citation Styles

***(a)**: [1] Mamika Ujianita Romdhini, Pankaj Rana, Amit Sehgal, Pooja Bhatia, "Structural Properties and Characteristic Polynomials of Cubic Power Graph of Dihedral Group," Mathematics and Statistics, Vol.13, No.2, pp. 97-104, 2025. DOI: 10.13189/ms.2025.130204*

***(b)**: Mamika Ujianita Romdhini, Pankaj Rana, Amit Sehgal, Pooja Bhatia (2025). Structural Properties and Characteristic Polynomials of Cubic Power Graph of Dihedral Group, Mathematics and Statistics, 13(2), 97-104. DOI: 10.13189/ms.2025.130204*

Abstract Cubic power graph of dihedral group D_n having order $2n$ and identity element e, $\Gamma_{cpg}(D_n)$ is a finite, simple, undirected graph for which two different vertices $d_1, d_2 \in D_n$ are adjacent if and only if $d_1d_2 = d^3$ or $d_2d_1 = d^3$ for some $d \in D_n$ with $d^3 \neq e$. In this research, conditions on n under which $\Gamma_{cpg}(D_n)$ is planar, are calculated. Also conditions on n under which $\Gamma_{cpg}(D_n)$ is Hamiltonian, are calculated. It is also shown that $\Gamma_{cpg}(D_n)$ is not an Eulerian graph. A polynomial that counts how many different ways there are to color a graph with a specific number of colors is called a chromatic polynomial. We have calculated the chromatic polynomial of $\Gamma_{cpg}(D_n)$ when $gcd(n, 3) = 1$. Also chromatic polynomial of almost complete graph obtained by deleting k number of non-adjacent edges is obtained. Spectral graph theory is a branch of mathematics that examines a graph's characteristics in relation to its characteristic polynomial, eigenvalues, and eigenvectors of related matrices, such as its adjacency matrix, Laplacian matrix or degree based matrices. Characteristic polynomials of degree based matrices such as Laplacian matrix, Signless Laplacian matrix, Maximum matrix, Minimum matrix and Greatest common divisor degree matrix of $\Gamma_{cpg}(D_n)$ when $gcd(n, 3) = 1$.

Keywords Cubic Power Graph, Dihedral Group, Planar, Hamiltonian, Eulerian, Chromatic Polynomial, Degree of A Vertex Based Matrices

1 Introduction

Recently various graphs have been introduced and studied in association with finite groups. Some of them are Power graph, Square power graph, Cubic power graph, Equal-square graph, $k^{th}-$power graph, commuting graph etc. Recently various structural properties, topological indices and characteristics polynomials of Cubic power graph [1, 2], Commuting graph [3], Power graph [4], Square power graph [5, 6] and Equal-square graph [7] are studied.

A connected graph is Eulerian graph if it has an Eulerian circuit i.e a closed walk around the connected graph that visits each edge precisely once before returning to the beginning vertex. Eulerain graph is always connected in which the degree of each vertex must be even. A connected graph is Hamiltonian graph if it contains a Hamiltonian circuit i.e a closed walk around the connected graph that visits each vertex precisely once except the beginning vertex. A planar graph is one that can be drawn on a plane such that no edge crosses any other edge i.e it can be drawn so that its edges only overlap at their endpoints. While the whole bipartite graph $K_{m,n}$ is planar iff either $m < 3$ or $n > 3$, the complete graph K_n is planar iff $n < 5$. Planar graphs have various applications in circuit designing, geographic information systems, transportation networks, VLSI, network designing etc.

We have provided some fundamental definitions and outcomes from previous researchs associated with graph theory in section 2. In section 3, we have given conditions under which $\Gamma_{cpg}(D_n)$ is planar, Hamiltonian and Eulerian. Also chromatic polynomial of $\Gamma_{cpg}(D_n)$ is calculated in section 3. In section 4, we have calculated characteristic polynomials of various matrices based on the degree of vertices associated with $\Gamma_{cpg}(D_n)$.

2 Preliminaries

Let $D_n = \{r^i, r^i s : 1 \leq i \leq n, r^n = e, s^2 = e, r^i s = sr^{n-i}\}$ be dihedral group of order $2n$, $X = \{v : v \in D_n, v = v^{-1}\}$ and $R = \{r^i : 1 \leq i \leq n\}$. When $3 \mid n$, let $R' = \{e, r^3, r^6, \cdots, r^{n-3}\}$.

Theorem 2.1. *[2] Let $gcd(n, 3) = 1$ and $d_1, d_2, d \in V(\Gamma_{cpg}(D_n))$. Then*

$$\textit{(i)}\ \Gamma_{cpg}(D_n) = \begin{cases} \overline{(\frac{n-1}{2})K_2 \cup (n+1)K_1} \textit{ if } n \textit{ is odd,} \\ \overline{(\frac{n-2}{2})K_2 \cup (n+2)K_1} \textit{ if } n \textit{ is even.} \end{cases}$$

$$\textit{(ii)}\ deg_{\Gamma_{cpg}}(d) = \begin{cases} 2n-2 \textit{ when } d \in D_n \setminus X, \\ 2n-1 \textit{ when } d \in X. \end{cases}$$

$$\textit{(iii)}\ distance(d_1, d_2) = \begin{cases} 1, d_1^{-1} \neq d_2, \\ 2, d_1^{-1} = d_2 \neq d_1. \end{cases}$$

Theorem 2.2. *[2] Let $gcd(n, 3) = 3$ and $d \in V(\Gamma_{cpg}(D_n))$. Then*

$$\textit{(i)}\ \Gamma_{cpg}(D_n) = \begin{cases} K_{3,3} \textit{ if } n = 3, \\ [3K_2] + [3K_2] \textit{ if } n = 6, \\ 3K_{\frac{n}{3}} + \left[(\frac{n-3}{3}) - K_{\frac{2n}{3}} \cup \overline{[K_1 \cup (\frac{n-3}{6})K_2]}\right] \\ \textit{if } n \textit{ is odd other than } 3, \\ 3K_{\frac{n}{3}} + \left[(\frac{n-3}{3}) - K_{\frac{2n}{3}} \cup \overline{[2K_1 \cup (\frac{n-6}{6})K_2]}\right] \\ \textit{when } n \textit{ is even other than } 6. \end{cases}$$

$$\textit{(ii)}\ deg_{\Gamma_{cpg}}(d) = \begin{cases} \left(\frac{4n-3}{3}\right) \textit{ when } d \in X \cup R \setminus R', \\ \left(\frac{4n-6}{3}\right) \textit{ when } d \in R' \setminus X. \end{cases}$$

Theorem 2.3 (Ore's Theorem). *[8] Let Γ be a simple and finite graph having vertices $n \geq 3$ and the degree of vertex d is denoted by $deg_\Gamma(d)$. Then, Ore's theorem states that if $deg_\Gamma(d_1) + deg_\Gamma(d_2) \geq n$ for every pair of different non-adjacent vertices d_1 and d_2 of Γ, then Γ is Hamiltonian graph.*

Definition 2.1.[9] The diagonal matrix of vertex degrees of $\Gamma_{cpg}(D_n)$, $D(\Gamma_{cpg}(D_n)) = [dia_{jk}]_{2n \times 2n}$ whose $(j, k)^{th}$ entry is

$$dia_{jk} = \begin{cases} deg_{\Gamma_{cpg}(D_n)}(d_j), \text{ if } j = k \text{ and they are adjacent} \\ 0, \text{ otherwise.} \end{cases}$$

Definition 2.2.[9] The adjacency matrix of $\Gamma_{cpg}(D_n)$, denoted by $A(\Gamma_{cpg}(D_n)) = [adj_{jk}]_{2n \times 2n}$ whose $(j, k)^{th}$ entry is

$$adj_{jk} = \begin{cases} 1, \text{ if } d_k \neq d_j \text{ and they are adjacent} \\ 0, \text{ otherwise.} \end{cases}$$

Definition 2.3.[9] The Laplacian matrix of $\Gamma_{cpg}(D_n)$ is $L(\Gamma_{cpg}(D_n)) = D(\Gamma_{cpg}(D_n)) - A(\Gamma_{cpg}(D_n))$.

Definition 2.4.[9] The Signless Laplacian matrix of $\Gamma_{cpg}(D_n)$ is $SL(\Gamma_{cpg}(D_n)) = D(\Gamma_{cpg}(D_n)) + A(\Gamma_{cpg}(D_n))$.

Definition 2.5.[10] The Maximum matrix of $\Gamma_{cpg}(D_n)$ is $Max(\Gamma_{cpg}(D_n)) = [max_{jk}]_{2n \times 2n}$ whose $(j, k)^{th}$ entry is

$$max_{jk} = \begin{cases} max\{deg_{\Gamma_{cpg}(D_n)}(d_j), deg_{\Gamma_{cpg}(D_n)}(d_k)\}, \\ \text{ if } d_j \neq d_k \text{ and they are adjacent} \\ 0, \text{ otherwise.} \end{cases}$$

Definition 2.6.[11] The Minimum matrix of $\Gamma_{cpg}(D_n)$ is $Min(\Gamma_{cpg}(D_n)) = [min_{jk}]_{2n \times 2n}$ whose $(j, k)^{th}$ entry is

$$min_{jk} = \begin{cases} min\{deg_{\Gamma_{cpg}(D_n)}(d_j), deg_{\Gamma_{cpg}(D_n)}(d_k)\}, \\ \text{ if } d_j \neq d_k \text{ and they are adjacent} \\ 0, \text{ otherwise.} \end{cases}$$

Definition 2.7.[12] The Greatest common divisor degree matrix of $\Gamma_{cpg}(D_n)$ is $GCDD(\Gamma_{cpg}(D_n)) = [g_{ij}]_{2n \times 2n}$ whose $(j, k)^{th}$ entry is

$$g_{jk} = \begin{cases} g.c.d\{deg_{\Gamma_{cpg}(D_n)}(d_j), deg_{\Gamma_{cpg}(D_n)}(d_k)\}, \\ \text{ if } d_j \neq d_k \text{ and they are adjacent} \\ 0, \text{ otherwise.} \end{cases}$$

3 Structural Properties

Theorem 3.1. *$\Gamma_{cpg}(D_n)$ is planar iff $n \in \{1, 2\}$.*

Proof. **Case 1.** $n = 1$
From Theorem 2.1, $\Gamma_{cpg}(D_n) = \overline{2K_1} = K_2$. Thus $\Gamma_{cpg}(D_n)$ is planar if n is one.
Case 2. $n = 2$
From Theorem 2.1, $\Gamma_{cpg}(D_n) = \overline{4K_1} = K_4$. Thus $\Gamma_{cpg}(D_n)$ is planar if n.is two.
Case 3. $gcd(3, n) = 1$ and $n = 2m + 1$ for $m \in \mathbb{N}$
Using Theorem 2.1, we get $\Gamma_{cpg}(D_n) = \overline{(\frac{n-1}{2})K_2 \cup (n+1)K_1}$. In this case, the minimum possible value of n is 5. So K_5 is subgraph of $\Gamma_{cpg}(D_n)$. Thus $\Gamma_{cpg}(D_n)$ is non-planar.
Case 4. $gcd(3, n) = 1$ and $n = 2m$ for $m \in \mathbb{N} \setminus 1$
Fro Theorem 2.1, we get $\Gamma_{cpg}(D_n) = \overline{(\frac{n-2}{2})K_2 \cup (n+2)K_1}$. In this case, the minimum possible value of n is 4. So K_5 is subgraph of $\Gamma_{cpg}(D_n)$. Thus $\Gamma_{cpg}(D_n)$ is non-planar.
Case 5. $n \in \{3, 6\}$
Using Theorem 2.1, we get $\Gamma_{cpg}(D_n) = K_{3,3}$ for $n = 3$ and for $n = 6$, $\Gamma_{cpg}(D_n) = [3K_2] + [3K_2]$ having subgraph $K_{3,3}$. Thus $\Gamma_{cpg}(D_n)$ is non-planar.
Case 6. $n = 3m$ for $m \in \mathbb{N} \setminus \{1, 2\}$
For odd values of n, $\Gamma_{cpg}(D_n) = 3K_{\frac{n}{3}} + \left[(\frac{n-3}{3}) - K_{\frac{2n}{3}} \cup \overline{[K_1 \cup (\frac{n-3}{6})K_2]}\right]$ whereas for even values of n, $\Gamma_{cpg}(D_n) = 3K_{\frac{n}{3}} + \left[(\frac{n-3}{3}) - K_{\frac{2n}{3}} \cup \overline{[2K_1 \cup (\frac{n-6}{6})K_2]}\right]$. In this case, the minimum possible odd value of n is 9 and even value of n is 12. $\Gamma_{cpg}(D_n)$ has K_5 as well as $K_{3,3}$ as its subgraph. Thus $\Gamma_{cpg}(D_n)$ is non-planar.

Hence $\Gamma_{cpg}(D_n)$ is planar iff $n \in \{1, 2\}$. □

Theorem 3.2. *$\Gamma_{cpg}(D_n)$ is Hamiltonian graph for $n \geq 2$.*

Proof. **Case 1.** $n = 1$
From Theorem 2.1, $\Gamma_{cpg}(D_n) = K_2$. As there is no cycle in $\Gamma_{cpg}(D_n)$, so it is not Hamiltonian graph.
Case 2. $gcd(n, 3) = 1$ and $n \neq 1$
From Theorem 2.1, for each pair of non-adjacent distinct vertices d_1, d_2 in $\Gamma_{cpg}(D_n)$, degree of both d_1 and d_2 vertices is $2n - 2$. Thus for each pair of different non-adjacent vertices d_1 and d_2 of $\Gamma_{cpg}(D_n)$, we have

$deg_{\Gamma_{cpg}(D_n)}(d_1) + deg_{\Gamma_{cpg}(D_n)}(d_2) = (2n-2) + (2n-2) = 4n-4 \geq 2n$. Thus using Theorem 2.3, $\Gamma_{cpg}(D_n)$ is Hamiltonian graph.

Case 3. $n = 3$

In this case, $\Gamma_{cpg}(D_n) = K_{3,3}$ by using Theorem 2.2. So for each pair of non-adjacent distinct vertices d_1, d_2 in $\Gamma_{cpg}(D_n)$, degree of both vertices is 3. Thus for each pair of different non-adjacent vertices d_1 and d_2 of $\Gamma_{cpg}(D_n)$, we have $deg_{\Gamma_{cpg}(D_n)}(d_1) + deg_{\Gamma_{cpg}(D_n)}(d_2) = 3 + 3 = 6 =$number of vertices in $\Gamma_{cpg}(D_n)$. Thus using Theorem 2.3, $\Gamma_{cpg}(D_n)$ is Hamiltonian graph.

Case 4. $n = 6$

In this case, $\Gamma_{cpg}(D_n) = [3K_2] + [3K_2]$ by using Theorem 2.2. So for each pair of non-adjacent distinct vertices d_1, d_2 in $\Gamma_{cpg}(D_n)$, degree of both vertices is 7. Thus for each pair of different non-adjacent vertices d_1 and d_2 of $\Gamma_{cpg}(D_n)$, we have $deg_{\Gamma_{cpg}(D_n)}(d_1) + deg_{\Gamma_{cpg}(D_n)}(d_2) = 7 + 7 = 14 \geq 12$. Thus by using Theorem 2.3, $\Gamma_{cpg}(D_n)$ is Hamiltonian graph.

Case 5. $n = 3m$ and $m \in \mathbb{N} \setminus \{1, 2\}$

From Theorem 2.2, there are three types of pairs of non-adjacent distinct vertices. First type of pair of vertices have $(\frac{4n-3}{3})$ degree of vertex on both ends and so $(\frac{4n-3}{3}) + (\frac{4n-3}{3}) = \frac{8n-6}{3} \geq 2n$. Second type of pairs of vertices have $(\frac{4n-6}{3})$ degree of vertex on both ends and so $(\frac{4n-6}{3}) + (\frac{4n-6}{3}) = \frac{8n-12}{3} \geq 2n$. Third type of pairs of vertices have $(\frac{4n-3}{3})$ degree of vertex on one end and $(\frac{4n-6}{3})$ degree of vertex on another end and so $(\frac{4n-3}{3}) + (\frac{4n-6}{3}) = \frac{8n-9}{3} \geq 2n$. Thus by using Theorem 2.3, $\Gamma_{cpg}(D_n)$ is Hamiltonian graph.

Hence $\Gamma_{cpg}(D_n)$ is Hamiltonian graph for $n \geq 2$. □

Theorem 3.3. *$\Gamma_{cpg}(D_n)$ is not an Eulerian graph.*

Proof. **Case 1.** $gcd(n, 3) = 1$

From Theorem 2.1, degree of vertex $x \in \Gamma_{cpg}(D_n)$ is $deg_{\Gamma_{cpg}(D_n)}(d) = 2n-2$ when $d \in D_n \setminus X$ and for $d \in X$, $deg_{\Gamma_{cpg}(D_n)}(d) = 2n-1$. Also $|D_n \setminus X| \neq 0 \neq |X|$. So irrespective of n is odd or even, $\Gamma_{cpg}(D_n)$ have both even and odd degrees of vertices. As $\Gamma_{cpg}(D_n)$ having vertices with odd degree so $\Gamma_{cpg}(D_n)$ is not an Eulerian graph.

Case 2. $n = 3$

$\Gamma_{cpg}(D_n) = K_{3,3}$ by using Theorem 2.2, having degree of every vertex 3. As $\Gamma_{cpg}(D_n)$ having vertices with odd degree so $\Gamma_{cpg}(D_n)$ is not an Eulerian graph.

Case 3. $n = 6$

In this case, $\Gamma_{cpg}(D_n) = [3K_2] + [3K_2]$ by using Theorem 2.2. So every vertex of $\Gamma_{cpg}(D_n)$ have degree 7. As $\Gamma_{cpg}(D_n)$ having vertices with odd degree so $\Gamma_{cpg}(D_n)$ is not an Eulerian graph.

Case 4. $n = 3m$ and $m \in \mathbb{N} \setminus \{1, 2\}$

From Theorem 2.2, $\Gamma_{cpg}(D_n)$ having vertices with degree $\frac{4n-3}{3} = 4m-1$ and $\frac{4n-6}{3} = 4m-2$. Thus $\Gamma_{cpg}(D_n)$ have vertices with odd degree so $\Gamma_{cpg}(D_n)$ is not an Eulerian graph.

Hence $\Gamma_{cpg}(D_n)$ is not an Eulerian graph. □

Theorem 3.4. *Let K_{2n} be a complete graph having $2n$ vertices. Then chromatic polynomial of almost complete graph $K_{2n} - \cup_{j=1}^{r} e_j$, obtained by deleting r non-adjacent edges is $P(K_{2n} - \cup_{j=1}^{r} e_j, \lambda) = \sum_{i=0}^{r} \binom{r}{i} P(K_{2n-i}, \lambda)$, where $r \leq n$.*

Proof. Let K_{2n} be a complete graph with $2n$ vertices and $K_{2n} - \cup_{j=1}^{r} e_j$ be almost complete graph obtained by deleting r non-adjacent edges, where $r \leq n$.

For $r = 1$, $P(K_{2n} - \cup_{j=1}^{1} e_j, \lambda) = P(K_{2n} - e_1, \lambda)$. Using [13, Theorem 8.6], $P(K_{2n} - e_1, \lambda) = P(K_{2n}, \lambda) + P(K_{2n-1}, \lambda)$. Thus, $P(K_{2n} - \cup_{j=1}^{r} e_j, \lambda) = \sum_{i=0}^{r} \binom{r}{i} P(K_{2n-i}, \lambda)$ is true for $r = 1$.

Now let for $r = r'$, $P(K_{2n} - \cup_{j=1}^{r'} e_j, \lambda) = \sum_{i=0}^{r'} \binom{r'}{i} P(K_{2n-i}, \lambda)$ is true.

Now, for $r = r' + 1$, $P(K_{2n} - \cup_{j=1}^{r'+1} e_j, \lambda) = P(K_{2n} - \cup_{j=1}^{r'} e_j - e_{r'+1}, \lambda)$. Using [13, Theorem 8.6], $P(K_{2n} - \cup_{j=1}^{r'} e_j - e_{r'+1}, \lambda) = P(K_{2n} - \cup_{j=1}^{r'} e_j, \lambda) + P(K_{2n-1} - \cup_{j=1}^{r'} e_j, \lambda)$. Now by using above result (at $r = r'$), we get

$P(K_{2n} - \cup_{j=1}^{r'+1} e_j, \lambda) = \sum_{i=0}^{r'} \binom{r'}{i} P(K_{2n-i}, \lambda) + \sum_{i=0}^{r'} \binom{r'}{i} P(K_{2n-1-i}, \lambda)$

$= \sum_{i=0}^{r'} \binom{r'}{i} P(K_{2n-i}, \lambda) + \sum_{i=1}^{r'+1} \binom{r'}{i-1} P(K_{2n-i}, \lambda)$

$= \binom{r'}{0} P(K_{2n}, \lambda) + \sum_{i=1}^{r'} [\binom{r'}{i} + \binom{r'}{i-1}] P(K_{2n-i}, \lambda) + \binom{r'}{r'} P(K_{2n-r'-1}, \lambda)$

$= \binom{r'+1}{0} P(K_{2n}, \lambda) + \sum_{i=1}^{r'} \binom{r'+1}{i} P(K_{2n-i}, \lambda) + \binom{r'+1}{r'+1} P(K_{2n-r'-1}, \lambda)$

$= \sum_{i=0}^{r'+1} \binom{r'+1}{i} P(K_{2n-i}, \lambda)$. Thus, $P(K_{2n} - \cup_{j=1}^{r'+1} e_j, \lambda) = \sum_{i=0}^{r'+1} \binom{r'+1}{i} P(K_{2n-i}, \lambda)$.

Hence, $P(K_{2n} - \cup_{j=1}^{r} e_j, \lambda) = \sum_{i=0}^{r} \binom{r}{i} P(K_{2n-i}, \lambda)$. □

Theorem 3.5. *For $gcd(n, 3) = 1$, chromatic polynomial of $\Gamma_{cpg}(D_n)$ is*

$$P(\Gamma_{cpg}(D_n), \lambda) = \begin{cases} P(K_2, \lambda), n = 1 \\ P(K_4, \lambda), n = 2 \\ \sum_{i=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{i} P(K_{2n-i}, \lambda), \\ n \textit{ is odd number other then } 1 \\ \sum_{i=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{i} P(K_{2n-i}, \lambda), \\ n \textit{ is even number other then } 2 \end{cases}$$

Proof. Let $gcd(n, 3) = 1$.

Case 1. $n = 1$:

From Theorem 2.1, $\Gamma_{cpg}(D_1) = \overline{2K_1} = K_2$. Thus, $P(\Gamma_{cpg}(D_n), \lambda) = P(K_2, \lambda)$ when $n = 2$.

Case 2. $n = 2$:

From Theorem 2.1, $\Gamma_{cpg}(D_2) = \overline{4K_1} = K_4$. Thus, $P(\Gamma_{cpg}(D_n), \lambda) = P(K_4, \lambda)$ when $n = 4$.

Case 3. n is odd other then 1:

Using Theorem 2.1, $\Gamma_{cpg}(D_n) = \overline{(\frac{n-1}{2}) K_2 \cup (n+1) K_1}$ in which all the pairs of vertices which are not adjacent are of type (x, x^{-1}), as $xx^{-1} = e$. So, all such pairs of vertices have no common vertex and so can be obtained by deleting non-adjacent $(\frac{n-1}{2})$ edges from complete graph K_{2n} having vertex set D_n and deleted edges are such that if x

is on one end of edge then $x^{-1}(\neq x)$ is at another end. Thus, $\Gamma_{cpg}(D_n) \cong K_{2n} - \cup_{j=1}^{\frac{n-1}{2}} e_j$. So, $P(\Gamma_{cpg}(D_n), \lambda) = P(K_{2n} - \cup_{j=1}^{\frac{n-1}{2}} e_j, \lambda)$. Now from Theorem 3.4, we have $P(K_{2n} - \cup_{j=1}^{\frac{n-1}{2}} e_j, \lambda) = \sum_{i=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{i} P(K_{2n-i}, \lambda)$. Hence $P(\Gamma_{cpg}(D_n), \lambda) = \sum_{i=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{i} P(K_{2n-i}, \lambda)$.

Case 4. n is even other then 2:
From Theorem 2.1, $\Gamma_{cpg}(D_n) = \overline{(\frac{n-2}{2})K_2 \cup (n+2)K_1}$ in which all the pairs of vertices which are not adjacent are of type (x, x^{-1}), as $xx^{-1} = e$. So, all such pairs of vertices have no common vertex and so can be obtained by deleting non-adjacent $(\frac{n-2}{2})$ edges from complete graph K_{2n} having vertex set D_n and deleted edges are such that if x is on one end of edge then $x^{-1}(\neq x)$ is at another end. Thus, $\Gamma_{cpg}(D_n) \cong K_{2n} - \cup_{j=1}^{\frac{n-2}{2}} e_j$. So, $P(\Gamma_{cpg}(D_n), \lambda) = P(K_{2n} - \cup_{j=1}^{\frac{n-2}{2}} e_j, \lambda)$. Now from Theorem 3.4, we have $P(K_{2n} - \cup_{j=1}^{\frac{n-2}{2}} e_j, \lambda) = \sum_{i=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{i} P(K_{2n-i}, \lambda)$. Hence $P(\Gamma_{cpg}(D_n), \lambda) = \sum_{i=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{i} P(K_{2n-i}, \lambda)$. ☐

4 Characteristic Polynomials of Degree-based Matrices

Theorem 4.1. *Let $\Gamma_{cpg}(D_n)$ be Cubic power graph of dihedral group D_n of order $2n$ and $gcd(n,3) = 1$. Then Laplacian polynomial of $\Gamma_{cpg}(D_n)$ is*

$$\ominus(\Gamma_{cpg}(D_n), \lambda) = \begin{cases} \lambda(\lambda - 2n + 2)^{\frac{n-1}{2}}(\lambda - 2n)^{\frac{3n-1}{2}} \\ \text{when } n \text{ is odd natural number,} \\ \lambda(\lambda - 2n + 2)^{\frac{n-2}{2}}(\lambda - 2n)^{\frac{3n}{2}} \\ \text{when } n \text{ is even natural number.} \end{cases}$$

Proof. **Case (i)** $gcd(n,3) = 1$ and n is odd.
Using Theorem 2.1, we get Laplacian matrix,

$$L = \begin{bmatrix} A_1 & A_2 \\ A_2' & A_3 \end{bmatrix}$$

where A_1 is $(n+1) \times (n+1)$ order matrix having all diagonal elements $2n-1$ and non-diagonal elements -1. A_2 is $(n+1) \times (n-1)$ order matrix having all elements equal to -1.

$$A_3 = \begin{bmatrix} 2n-2 & 0 & \cdots & -1 & -1 \\ 0 & 2n-2 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & 2n-2 & 0 \\ -1 & -1 & \cdots & 0 & 2n-2 \end{bmatrix}_{(n-1)}$$

Thus, $|L - \lambda I| = \begin{vmatrix} A_1 - \lambda I & A_2 \\ A_2' & A_3 - \lambda I \end{vmatrix}$

On applying the following row and column operations:
1. $R_i \Rightarrow R_i - R_1$ for all $2 \leq i \leq n+1$.
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $n+3 \leq i \leq 2n$.
3. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+2 \leq i \leq 2n-1$.
4. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+1}$.
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}$, $a_{(n+3)(n+3)}, a_{(n+5)(n+5)}, \cdots, a_{(2n)(2n)}$.
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n+1}{2}$.
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n+1}{2}}$.
We get,

$$|L - \lambda I| = (2n - \lambda)^n (2n - 2 - \lambda)^{\frac{n-1}{2}} (2n - \lambda)^{\frac{n-3}{2}} \begin{vmatrix} n-1-\lambda & 1-n \\ -(n+1) & n-\lambda+1 \end{vmatrix}$$

$$|L - \lambda I| = \lambda(\lambda - 2n + 2)^{\frac{n-1}{2}}(\lambda - 2n)^{\frac{3n-1}{2}}.$$

Hence $\ominus(\Gamma_{cpg}(D_n), \lambda) = \lambda(\lambda - 2n + 2)^{\frac{n-1}{2}}(\lambda - 2n)^{\frac{3n-1}{2}}$.

Case (ii) $gcd(n,3) = 1$ and n is even.
Using Theorem 2.1, we get Laplacian matrix,

$$L = \begin{bmatrix} A_1 & A_2 \\ A_2' & A_3 \end{bmatrix}$$

where A_1 is $(n+2) \times (n+2)$ order matrix having all diagonal elements $2n-1$ and non-diagonal elements -1. A_2 is $(n+2) \times (n-2)$ order matrix having all elements equal to -1.

$$A_3 = \begin{bmatrix} 2n-2 & 0 & \cdots & -1 & -1 \\ 0 & 2n-2 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & 2n-2 & 0 \\ -1 & -1 & \cdots & 0 & 2n-2 \end{bmatrix}_{(n-2)}$$

Thus, $|L - \lambda I| = \begin{vmatrix} A_1 - \lambda I & A_2 \\ A_2' & A_3 - \lambda I \end{vmatrix}$

On applying the following row and column operations:
1. $R_i \Rightarrow R_i - R_1$ for all $2 \leq i \leq n+2$.
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $i = n+4, n+6, \cdots, 2n$.
3. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+3 \leq i \leq 2n-1$.
4. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+2}$.
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}$, $a_{(n+2)(n+2)}, a_{(n+4)(n+4)}, a_{(n+6)(n+6)}, \cdots, a_{(2n)(2n)}$.
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n}{2}$.
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n}{2}}$.
We get,

$$|L - \lambda I| = (2n - \lambda)^{n+1} (2n - 2 - \lambda)^{\frac{n-2}{2}} (2n - \lambda)^{\frac{n-4}{2}} \begin{vmatrix} n-2-\lambda & 2-n \\ -(n+2) & n-\lambda+2 \end{vmatrix}$$

$$|L - \lambda I| = \lambda(\lambda - 2n + 2)^{\frac{n-2}{2}}(\lambda - 2n)^{\frac{3n}{2}}.$$

Hence $\ominus(\Gamma_{cpg}(D_n), \lambda) = \lambda(\lambda - 2n + 2)^{\frac{n-2}{2}}(\lambda - 2n)^{\frac{3n}{2}}$ when n is even natural number. ☐

Theorem 4.2. *Let $gcd(3,n) = 1$. Then Signless Laplacian polynomial of $\Gamma_{cpg}(D_n)$ is*

$$Q(\Gamma_{cpg}(D_n), \lambda) = \begin{cases} (\lambda - 2n + 2)^{\frac{3n-1}{2}}(\lambda - 2n + 4)^{\frac{n-3}{2}}\{\lambda^2 + \\ 6\lambda(1-n) + 8n^2 - 18n + 6\} \text{ when } n \text{ is odd,} \\ (\lambda - 2n + 2)^{\frac{3n}{2}}(\lambda - 2n + 4)^{\frac{n-4}{2}}\{\lambda^2 + \\ 6\lambda(1-n) + 8n^2 - 18n + 4\} \text{ when } n \text{ is even.} \end{cases}$$

Proof. **Case (i)** $gcd(n,3) = 1$ and n is odd.
From Theorem 2.1, Signless Laplacian matrix,

$$SL = \begin{bmatrix} A_1 & A_2 \\ A_2' & A_3 \end{bmatrix}$$

where A_1 is $(n+1) \times (n+1)$ order matrix having all diagonal elements $2n-1$ and non-diagonal elements 1. A_2 is $(n+1) \times (n-1)$ order matrix having all elements equal to 1.

$$A_3 = \begin{bmatrix} 2n-2 & 0 & \cdots & 1 & 1 \\ 0 & 2n-2 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 2n-2 & 0 \\ 1 & 1 & \cdots & 0 & 2n-2 \end{bmatrix}_{(n-1)}$$

Thus, $|SL - \lambda I| = \begin{vmatrix} A_1 - \lambda I & A_2 \\ A_2^{'} & A_3 - \lambda I \end{vmatrix}$

On applying the following row and column operations:

1. $R_i \Rightarrow R_i - R_1$ for all $2 \leq i \leq n+1$.
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $n+3 \leq i \leq 2n$.
3. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+2 \leq i \leq 2n-1$.
4. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+1}$.
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}, a_{(n+3)(n+3)}, a_{(n+5)(n+5)}, \cdots, a_{(2n)(2n)}$.
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n+1}{2}$.
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n+1}{2}}$.

$|SL - \lambda I| = (-1)^{\frac{3n-1}{2}}(\lambda - 2n + 2)^{\frac{3n-1}{2}}(2n - 4 - \lambda)^{\frac{n-3}{2}} \begin{vmatrix} 3n-1-\lambda & n-1 \\ n+1 & 3n-5-\lambda \end{vmatrix}$

$|SL - \lambda I| = (-1)^{\frac{3n-1}{2}}(\lambda - 2n + 2)^{\frac{3n-1}{2}}(2n - 4 - \lambda)^{\frac{n-3}{2}}\{\lambda^2 + 6\lambda(1-n) + 8n^2 - 18n + 6\}$

Hence $Q(\Gamma_{cpg}(D_n), \lambda) = (\lambda - 2n + 2)^{\frac{3n-1}{2}}(\lambda - 2n + 4)^{\frac{n-3}{2}}\{\lambda^2 + 6\lambda(1-n) + 8n^2 - 18n + 6\}$.

Case (ii) $gcd(n,3) = 1$ and n is even.
From Theorem 2.1, Signless Laplacian matrix,
$SL = \begin{bmatrix} A_1 & A_2 \\ A_2^{'} & A_3 \end{bmatrix}$
where A_1 is $(n+2) \times (n+2)$ order matrix having all diagonal elements $2n-1$ and non-diagonal elements 1. A_2 is $(n+2) \times (n-2)$ order matrix having all elements equal to 1.

$$A_3 = \begin{bmatrix} 2n-2 & 0 & \cdots & 1 & 1 \\ 0 & 2n-2 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 2n-2 & 0 \\ 1 & 1 & \cdots & 0 & 2n-2 \end{bmatrix}_{(n-2)}$$

Thus, $|SL - \lambda I| = \begin{vmatrix} A_1 - \lambda I & A_2 \\ A_2^{'} & A_3 - \lambda I \end{vmatrix}$

On applying the following row and column operations:

1. $R_i \Rightarrow R_i - R_1$ for all $2 \leq i \leq n+2$.
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $i = n+4, n+6, \cdots, 2n$.
3. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+3 \leq i \leq 2n-1$.
4. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+2}$.
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}, a_{(n+2)(n+2)}, a_{(n+4)(n+4)}, a_{(n+6)(n+6)}, \cdots, a_{(2n)(2n)}$.
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n}{2}$.
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n}{2}}$.

$|SL - \lambda I| = (-1)^{\frac{3n}{2}}(\lambda - 2n + 2)^{\frac{3n}{2}}(2n - 4 - \lambda)^{\frac{n-4}{2}} \begin{vmatrix} 3n-\lambda & n-2 \\ n+2 & 3n-6-\lambda \end{vmatrix}$

$|SL - \lambda I| = (-1)^{\frac{3n}{2}}(\lambda - 2n + 2)^{\frac{3n}{2}}(2n - 4 - \lambda)^{\frac{n-4}{2}}\{\lambda^2 + 6\lambda(1-n) + 8n^2 - 18n + 4\}$.

$Q(\Gamma_{cpg}(D_n), \lambda) = (\lambda - 2n + 2)^{\frac{3n}{2}}(\lambda - 2n + 4)^{\frac{n-4}{2}}\{\lambda^2 + 6\lambda(1-n) + 8n^2 - 18n + 4\}$ when n is even natural number. ☐

Theorem 4.3. *Let $gcd(n,3) = 1$. Then characteristic polynomial of the Maximum matrix of $\Gamma_{cpg}(D_n)$ is*

$$P_{Max(\Gamma_{cpg}(D_n))}(\lambda) = \begin{cases} \lambda^{\frac{n-1}{2}}(\lambda + 2n - 1)^n(\lambda + 4n - 4)^{\frac{n-3}{2}} \\ \{\lambda^2 - \lambda(4n^2 - 9n + 6) - 14n^3 + \\ 23n^2 - 10n + 1\} \textit{ when n is odd,} \\ \lambda^{\frac{n-2}{2}}(\lambda + 2n - 1)^{n+1}(\lambda + 4n - \\ 4)^{\frac{n-4}{2}}\{\lambda^2 - \lambda(4n^2 - 9n + 7) - \\ 14n^3 + 19n^2 + 2n - 4\} \textit{ when n} \\ \textit{is even.} \end{cases}$$

Proof. **Case (i)** $gcd(n,3) = 1$ and n is odd.
From Theorem 2.1, Maximum matrix,
$Max(\Gamma_{cpg}(D_n)) = \begin{bmatrix} A_1 & A_2 \\ A_2^{'} & A_3 \end{bmatrix}$
where A_1 is $(n+1) \times (n+1)$ order matrix having all diagonal elements 0 and non-diagonal elements $2n-1$. A_2 is $(n+1) \times (n-1)$ order matrix having all elements equal to $2n-1$.
$A_3 = [a_{jk}]_{(n-1)(n-1)}$ whose $(j,k)^{th}$ entry is

$$a_{jk} = \begin{cases} 0 \text{ if } j = k; \text{ or } j \text{ is odd and } k = j+1; \text{ or } j \\ \text{is even and } k = j-1, \\ 2n-2 \text{ otherwise.} \end{cases}$$

$|Max(\Gamma_{cpg}(D_n)) - \lambda I| = \begin{vmatrix} A_1 - \lambda I & A_2 \\ A_2^{'} & A_3 - \lambda I \end{vmatrix}$

On applying the following row and column operations:

1. $R_i \Rightarrow R_i - R_1$ for all $i = 2, 3, \cdots, n+1$
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $n+3 \leq i \leq 2n$
3. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+1}$
4. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+2 \leq i \leq 2n-1$
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}, a_{(n+3)(n+3)}, a_{(n+5)(n+5)}, \cdots, a_{(2n)(2n)}$
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n+1}{2}$
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n+1}{2}}$

We get,
$|Max(\Gamma_{cpg}(D_n)) - \lambda I| = \lambda^{\frac{n-1}{2}}(\lambda + 4n - 4)^{\frac{n-3}{2}}(\lambda + 2n - 1)^n \begin{vmatrix} -\lambda + n(2n-1) & (4n-2)(\frac{n-1}{2}) \\ (2n-1)(n+1) & -\lambda + (4n-4)(\frac{n-3}{2}) \end{vmatrix}$

$|Max(\Gamma_{cpg}(D_n)) - \lambda I| = \lambda^{\frac{n-1}{2}}(\lambda + 4n - 4)^{\frac{n-3}{2}}(\lambda + 2n - 1)^n\{\lambda^2 - \lambda(4n^2 - 9n + 6) - 14n^3 + 23n^2 - 10n + 1\}$

$P_{Max(\Gamma_{cpg}(D_n))}(\lambda) = \lambda^{\frac{n-1}{2}}(\lambda + 4n - 4)^{\frac{n-3}{2}}(\lambda + 2n - 1)^n\{\lambda^2 - \lambda(4n^2 - 9n + 6) - 14n^3 + 23n^2 - 10n + 1\}$.

Case (ii) $gcd(n,3) = 1$ and n is even.
From Theorem 2.1, Maximum matrix,
$Max(\Gamma_{cpg}(D_n)) = \begin{bmatrix} A_1 & A_2 \\ A_2^{'} & A_3 \end{bmatrix}$
where A_1 is $(n+2) \times (n+2)$ order matrix having all diagonal elements 0 and non-diagonal elements $2n-1$. A_2 is $(n+2) \times (n-2)$ order matrix having all elements equal to $2n-1$.
$A_3 = [a_{jk}]_{(n-2)(n-2)}$ whose $(j,k)^{th}$ entry is

$$a_{jk}=\begin{cases}0 \text{ if } j=k;\ \text{or } j \text{ is odd and } k=j+1;\ \text{or } j\\ \text{ is even and } k=j-1,\\ 2n-2 \text{ otherwise.}\end{cases}$$

Thus, $|Max(\Gamma_{cpg}(D_n))-\lambda I|=\begin{vmatrix}A_1-\lambda I & A_2\\ A_2^{'} & A_3-\lambda I\end{vmatrix}$

On applying the following row and column operations:

1. $R_i \Rightarrow R_i - R_1$ for all $2 \le i \le n+2$
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $i = n+4, n+6, \cdots, 2n$
3. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+3 \le i \le 2n-1$
4. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+2}$
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}, a_{(n+2)(n+2)}, a_{(n+4)(n+4)}, a_{(n+6)(n+6)}, \cdots, a_{(2n)(2n)}$
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n}{2}$
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n}{2}}$

We get,

$$|Max(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}} \times \begin{vmatrix}-\lambda+(2n-1)(n+1) & (4n-2)(\frac{n-2}{2})\\ (2n-1)(n+2) & -\lambda+(4n-4)(\frac{n-4}{2})\end{vmatrix}$$

$$|Max(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}}\{\lambda^2-\lambda(4n^2-9n+7)-14n^3+19n^2+2n-4\}.$$

$$P_{Max(\Gamma_{cpg}(D_n))}(\lambda) = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}}\{\lambda^2-\lambda(4n^2-9n+7)-14n^3+19n^2+2n-4\}.$$

□

Theorem 4.4. *Let* $gcd(n,3)=1$. *Then characteristic polynomial of the Minimum matrix of* $\Gamma_{cpg}(D_n)$ *is*

$$P_{Min(\Gamma_{cpg}(D_n))}(\lambda)=\begin{cases}\lambda^{\frac{n-1}{2}}(\lambda+2n-1)^n(\lambda+4n-4)^{\frac{n-3}{2}}\\ \{\lambda^2-\lambda(4n^2-9n+6)-10n^3+\\ 20n^2-14n+4\} \text{ when } n \text{ is odd,}\\ \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n\\ -4)^{\frac{n-4}{2}}\{\lambda^2-\lambda(4n^2-9n+7)-\\ 10n^3+16n^2-14n+8\} \text{ when } n\\ \text{ is even.}\end{cases}$$

Proof. **Case (i)** $gcd(n,3)=1$ and n is odd.
From Theorem 2.1, Minimum matrix,

$$Min(\Gamma_{cpg}(D_n))=\begin{bmatrix}A_1 & A_2\\ A_2^{'} & A_3\end{bmatrix}$$

where A_1 is $(n+1)\times(n+1)$ order matrix having all diagonal elements 0 and non-diagonal elements $2n-1$. A_2 is $(n+1)\times(n-1)$ order matrix having all elements equal to $2n-2$.
$A_3=[a_{jk}]_{(n-1)(n-1)}$ whose $(j,k)^{th}$ entry is

$$a_{jk}=\begin{cases}0 \text{ if } j=k;\ \text{or } j \text{ is odd and } k=j+1;\ \text{or } j\\ \text{ is even and } k=j-1,\\ 2n-2 \text{ otherwise.}\end{cases}$$

$$|Min(\Gamma_{cpg}(D_n))-\lambda I|=\begin{vmatrix}A_1-\lambda I & A_2\\ A_2^{'} & A_3-\lambda I\end{vmatrix}$$

On applying the following row and column operations:

1. $R_i \Rightarrow R_i - R_1$ for all $2 \le i \le n+1$
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $i = n+3, n+5, \cdots, 2n$
3. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+1}$
4. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+2 \le i \le 2n-1$
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}, a_{(n+3)(n+3)}, a_{(n+5)(n+5)}, \cdots, a_{(2n)(2n)}$
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n+1}{2}$
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n+1}{2}}$

We get,

$$|Min(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-1}{2}}(\lambda+4n-4)^{\frac{n-3}{2}}(\lambda+2n-1)^n\begin{vmatrix}-\lambda+n(2n-1) & (4n-4)(\frac{n-1}{2})\\ (2n-2)(n+1) & -\lambda+(4n-4)(\frac{n-3}{2})\end{vmatrix}$$

$$|Min(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-1}{2}}(\lambda+4n-4)^{\frac{n-3}{2}}(\lambda+2n-1)^n\{\lambda^2-\lambda(4n^2-9n+6)-10n^3+20n^2-14n+4\}.$$

Hence, $P_{Min(\Gamma_{cpg}(D_n))}(\lambda) = \lambda^{\frac{n-1}{2}}(\lambda+4n-4)^{\frac{n-3}{2}}(\lambda+2n-1)^n\{\lambda^2-\lambda(4n^2-9n+6)-10n^3+20n^2-14n+4\}$.

Case (ii) $gcd(3,n)=1$ and n is even.
From Theorem 2.1, Minimum matrix,

$$Min(\Gamma_{cpg}(D_n))=\begin{bmatrix}A_1 & A_2\\ A_2^{'} & A_3\end{bmatrix}$$

where A_1 is $(n+2)\times(n+2)$ order matrix having all diagonal elements 0 and non-diagonal elements $2n-1$. A_2 is $(n+2)\times(n-2)$ order matrix having all elements equal to $2n-2$.
$A_3=[a_{jk}]_{(n-2)(n-2)}$ whose $(j,k)^{th}$ entry is

$$a_{jk}=\begin{cases}0 \text{ if } j=k;\ \text{or } j \text{ is odd and } k=j+1;\ \text{or } j\\ \text{ is even and } k=j-1,\\ 2n-2 \text{ otherwise.}\end{cases}$$

Thus, $|Min(\Gamma_{cpg}(D_n))-\lambda I|=\begin{vmatrix}A_1-\lambda I & A_2\\ A_2^{'} & A_3-\lambda I\end{vmatrix}$

On applying the following row and column operations:

1. $R_i \Rightarrow R_i - R_1$ for all $2 \le i \le n+2$
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $i = n+4, n+6, \cdots, 2n$
3. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+3 \le i \le 2n-1$
4. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+2}$
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}, a_{(n+2)(n+2)}, a_{(n+4)(n+4)}, a_{(n+6)(n+6)}, \cdots, a_{(2n)(2n)}$
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n}{2}$
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n}{2}}$

We get,

$$|Min(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}} \times \begin{vmatrix}-\lambda+(2n-1)(n+1) & (4n-4)(\frac{n-2}{2})\\ (2n-2)(n+2) & -\lambda+(4n-4)(\frac{n-4}{2})\end{vmatrix}$$

$$|Min(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}}\{\lambda^2-\lambda(4n^2-9n+7)-10n^3+16n^2-14n+8\}$$

Hence, $P_{Min(\Gamma_{cpg}(D_n))}(\lambda) = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}}\{\lambda^2-\lambda(4n^2-9n+7)-10n^3+16n^2-14n+8\}$.

□

Theorem 4.5. *Let* $gcd(n,3)=1$. *Then characteristic polynomial of the Greatest common divisor degree matrix of* $\Gamma_{cpg}(D_n)$ *is*

$$P_{GCDD(\Gamma_{cpg}(D_n))}(\lambda) = \begin{cases} \lambda^{\frac{n-1}{2}}(\lambda+4n-4)^{\frac{n-3}{2}}(\lambda+2n-1)^n \times \\ \{\lambda^2-\lambda(4n^2-9n+6)+4n^4-18n^3+ \\ 19n^2-6n+1\} \text{ when } n \text{ is odd,} \\ \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}} \times \\ \{\lambda^2-\lambda(4n^2-9n+7)+4n^4-18n^3+ \\ 3n^2+18n-4\} \text{ when } n \text{ is even.} \end{cases}$$

Proof. **Case (i)** $gcd(n,3)=1$ and n is odd.
From Theorem 2.1, Greatest common divisor degree matrix,
$GCDD(\Gamma_{cpg}(D_n)) = \begin{bmatrix} A_1 & A_2 \\ A_2' & A_3 \end{bmatrix}$
where A_1 is $(n+1)\times(n+1)$ order matrix having all diagonal elements 0 and non-diagonal elements $2n-1$. A_2 is $(n+1)\times(n-1)$ order matrix having all elements equal to 1. $A_3=[a_{jk}]_{(n-1)(n-1)}$ whose $(j,k)^{th}$ entry is

$$a_{jk} = \begin{cases} 0 \text{ if } j=k; \text{ or } j \text{ is odd and } k=j+1; \text{ or } j \\ \text{ is even and } k=j-1, \\ 2n-2 \text{ otherwise.} \end{cases}$$

$$|GCDD(\Gamma_{cpg}(D_n))-\lambda I| = \begin{vmatrix} A_1-\lambda I & A_2 \\ A_2' & A_3-\lambda I \end{vmatrix}$$

On applying the following row and column operations:
1. $R_i \Rightarrow R_i - R_1$ for all $2 \le i \le n+1$
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $n+3 \le i \le 2n$
3. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+1}$
4. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+2 \le i \le 2n-1$
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}, a_{(n+3)(n+3)}, a_{(n+5)(n+5)}, \cdots, a_{(2n)(2n)}$
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n+1}{2}$
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n+1}{2}}$
We get,

$$|GCDD(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-1}{2}}(\lambda+4n-4)^{\frac{n-3}{2}}(\lambda+2n-1)^n \begin{vmatrix} -\lambda+n(2n-1) & 2(\frac{n-1}{2}) \\ n+1 & -\lambda+(4n-4)(\frac{n-3}{2}) \end{vmatrix}$$

$$|GCDD(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-1}{2}}(\lambda+4n-4)^{\frac{n-3}{2}}(\lambda+2n-1)^n\{\lambda^2-\lambda(4n^2-9n+6)+4n^4-18n^3+19n^2-6n+1\}.$$

$$P_{GCDD(\Gamma_{cpg}(D_n))}(\lambda) = \lambda^{\frac{n-1}{2}}(\lambda+4n-4)^{\frac{n-3}{2}}(\lambda+2n-1)^n\{\lambda^2-\lambda(4n^2-9n+6)+4n^4-18n^3+19n^2-6n+1\}.$$

Case (ii) $gcd(3,n)=1$ and n is even.
Using Theorem 2.1, we get Greatest common divisor degree matrix,
$GCDD(\Gamma_{cpg}(D_n)) = \begin{bmatrix} A_1 & A_2 \\ A_2' & A_3 \end{bmatrix}$
where A_1 is $(n+2)\times(n+2)$ order matrix having all diagonal elements 0 and non-diagonal elements $2n-1$. A_2 is $(n+2)\times(n-2)$ order matrix having all elements equal to 1. $A_3=[a_{jk}]_{(n-2)(n-2)}$ whose $(j,k)^{th}$ entry is

$$a_{jk} = \begin{cases} 0 \text{ if } j=k; \text{ or } j \text{ is odd and } k=j+1; \text{ or } j \\ \text{ is even and } k=j-1, \\ 2n-2 \text{ otherwise.} \end{cases}$$

Thus, $|GCDD(\Gamma_{cpg}(D_n))-\lambda I| = \begin{vmatrix} A_1-\lambda I & A_2 \\ A_2' & A_3-\lambda I \end{vmatrix}$
On applying the following row and column operations:
1. $R_i \Rightarrow R_i - R_1$ for all $2 \le i \le n+2$
2. $R_i \Rightarrow R_i - R_{i-1}$ for all $i = n+4, n+6, \cdots, 2n$
3. $C_i \Rightarrow C_i + C_{i+1}$ for all $n+3 \le i \le 2n-1$
4. $C_1 \Rightarrow C_1 + C_2 + \cdots + C_{n+2}$
5. Extending along $a_{22}, a_{33}, a_{44}, \cdots, a_{(n+1)(n+1)}, a_{(n+2)(n+2)}, a_{(n+4)(n+4)}, a_{(n+6)(n+6)}, \cdots, a_{(2n)(2n)}$
6. $R_i \Rightarrow R_i - R_2$ for all $i = 3, 4, \cdots, \frac{n}{2}$
7. $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{n}{2}}$
We get,

$$|GCDD(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}} \begin{vmatrix} -\lambda+(2n-1)(n+1) & 2(\frac{n-2}{2}) \\ n+2 & -\lambda+(4n-4)(\frac{n-4}{2}) \end{vmatrix}$$

$$|GCDD(\Gamma_{cpg}(D_n))-\lambda I| = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}}\{\lambda^2-\lambda(4n^2-9n+7)+4n^4-18n^3+3n^2+18n-4\}$$

$$P_{GCDD(\Gamma_{cpg}(D_n))}(\lambda) = \lambda^{\frac{n-2}{2}}(\lambda+2n-1)^{n+1}(\lambda+4n-4)^{\frac{n-4}{2}}\{\lambda^2-\lambda(4n^2-9n+7)+4n^4-18n^3+3n^2+18n-4\}$$
when n is even natural number.

□

5 Conclusions

Cubic power graph of D_n, $\Gamma_{cpg}(D_n)$ is planar if and only if $n \in \{1,2\}$ and Hamiltonian for $n \in \mathbb{N} \setminus \{1\}$. $\Gamma_{cpg}(D_n)$ is not an Eulerian graph. Chromatic polynomial of almost complete graphs and $\Gamma_{cpg}(D_n)$ is calculated. Characteristic polynomials of Laplacian, signless Laplacian, Minimum, Maximum, Greatest common divisor degree matrices of $\Gamma_{cpg}(D_n)$ are also calculated.

Acknowledgements

The corresponding author acknowledges the Senior Research Fellowship (221610034811) provided by the University Grants Commission (UGC), India.

The first author wishes to express her gratitude to the University of Mataram, Indonesia, for providing partial funding assistance.

REFERENCES

[1] Prathap R. R., Chelvam T. T., “The cubic power graph of finite abelian groups,” AKCE International Journal of Graphs and Combinatorics, vol. 18, no. 1, pp. 16-24, 2021. DOI: 10.1080/09728600.2021.1878868

[2] Rana P., Sehgal A., Bhatia P., Kumar P., “Topological Indices and Structural Properties of Cubic Power Graph of Dihedral Group,” Contemporary Mathematics, vol. 5, no. 1, pp. 61-779, 2024. DOI: 10.37256/cm.5120243029

[3] Romdhini M. U., Nawawi A., Syechah B. N., “Seidel Laplacian and Seidel Signless Laplacian Energies of Commuting Graph for Dihedral Groups,” Malaysian Journal of Fundamental and

Applied Sciences, vol. 20, no. 3, pp. 701-713, 2024. DOI: 10.11113/mjfas.v20n3.3465

[4] Romdhini M. U., Nawawi A., Al-Sharqi F., Al-Quran A., "Characteristic Polynomial of Power Graph for Dihedral Groups Using Degree-Based Matrices," Malaysian Journal of Fundamental and Applied Sciences, vol. 20, no. 2, pp. 328-335, 2024. DOI: 10.11113/mjfas.v20n2.3357

[5] Siwach A., Bhatia V., Sehgal A., Rana P., "Square Power Graph of Dihedral Group of order 2n for odd number n," Palestine Journal of Mathematics, vol. 13, no. SI-III, pp. 168-174, 2024. https://pjm.ppu.edu/paper/1744-square-power-graph-dihedral-group-order-2n-odd-number-n

[6] Rana P., Sehgal A., Bhatia P., Kumar V., "The Square Power Graph of a Finite Abelian Group," Palestine Journal of Mathematics, vol. 13, no. 1, pp. 151-162, 2024. https://pjm.ppu.edu/paper/1618-square-power-graph-finite-abelian-group

[7] Rana P., Aggarwal S., Sehgal A., Bhatia P., "Structural Properties and Laplacian Spectrum of Equal-Square Graph of Finite Groups," Palestine Journal of Mathematics, vol. 13, no. SI-III, pp. 154-161, 2024. https://pjm.ppu.edu/paper/1742-structural-properties-and-laplacian-spectrum-equal-square-graph-finite-groups

[8] Ore O., "Note on hamiltonian circuits," American Mathematical Monthly, vol. 67, no. 1, pp. 55, 1960. DOI: 10.2307/2308928

[9] Cvetkovi ́c D., Doob M., Sachs H., "Spectra of Graphs," Academic press, New York, 1980.

[10] Adiga C., Smitha M., "On maximum degree energy of a graph," International Journal of Contemporary Mathematical Sciences, vol. 4, no. 8, pp. 385-396, 2009. https://www.m-hikari.com/ijcms-password2009/5-8-2009/index.html

[11] Adiga C., Swamy C. S. S., "Bounds on the largest of minimum degree eigenvalues of graphs," International Mathematics Forum, vol. 5, no. 37, pp. 1823-1831, 2010. https://www.m-hikari.com/imf-2010/37-40-2010/index.html

[12] Ramkumar R., Nagarajan K., "Greatest common divisor degree energy of graphs," International Journal of Mathematical Sciences and Engineering Applications, vol. 11, no. 2, pp. 163-171, 2017.

[13] Deo N., "Graph Theory with Applications to Engineering and Computer Science," Dover Publications, INC. Mineola, New York, pp. 176-177, 2016.