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Advanced Theorems on Double Integral Transforms and Their Applications

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Abstract This article delves into the realm of double integral transforms (DIT), focusing on the pivotal role played by Fox's H-Function in their theoretical framework. The DIT denoted as $\phi(t)$ is intricately defined through Fox's H-Function and expressed as a Mellin-Barnes-type contour integral with various parameters and conditions. The study emphasizes chain properties connecting the DIT, presenting a concise representation as $\phi(t) = DT\left[f(x, y)\right]$. Three theorems are established, leveraging the power series expansions of special functions such as Laplace transforms, Hankel transforms, and specific transforms by Pathak and Narain. These theorems are proven analytically and their results are verified with the help of examples. The Fox H-Function is a powerful mathematical tool, which is explored for its significance in the analysis of DIT. Being a generalization of the hypergeometric series, the H-function finds widespread applications across mathematics, physics, and engineering. Detailed proofs substantiate the three theorems, illustrating the manipulation of the double integral transform under specific conditions. The application of these theorems extends to the evaluation of known and novel integrals involving the product of H-Function and other math-ematical functions. This comprehensive exploration unveils the indispensable role of Fox's H-Function in the theoret-ical landscape of double integral transform and its applications.

Keywords Double Integral Transform, Laplace Transform, Hankel Transform, Fox's H-Function

Highlights of research

1. Use of Fox's H-Function in Double Integral Transforms
2. Establishment of Three Theorems
3. Applications to Engineering Problems
4. Exploration of Multiple Transforms
5. Analytical Proofs and Practical Examples

1 Introduction

Integral equations and transforms like (Laplace transform, Hankel Transform, etc.) are very crucial and useful tools to solve many engineering problems. We use integral equations and transforms to tackle the crack problems in fracture mechanics. These transforms and special functions help in converting complicated problems into simpler forms that can be solved further in a simpler way. This article provides a theoretical overview in the form of theorems which helps researchers to tackle the crack problems by using special Fox's H-function. Fig.1 below shows different types of cracks in a material, as illustrated in fracture mechanics. These crack models are crucial in understanding the behavior of cracks under stress, and are analyzed using integral equations and transforms such as the Laplace and Hankel transforms, alongside special functions like Fox's H-function, to simplify and solve complex fracture problems.

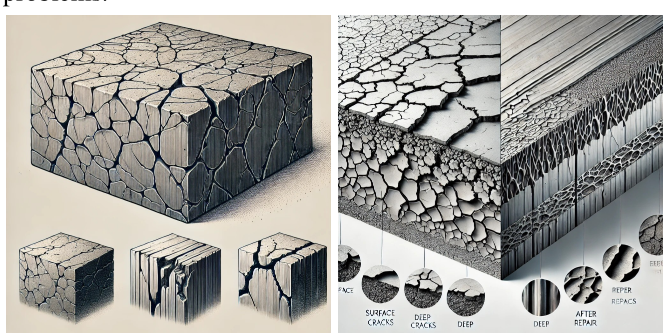

Figure 1. Illustration of different crack patterns in materials

The double integral transform constitutes a sophisticated mathematical methodology employed within fracture mechanics to address intricate challenges associated with dynamic crack phenomena. This approach entails the utilization of integral transformations, including but not limited to Laplace and Fourier transforms, which are applied to the differential equations that dictate the behavior of cracks. Such transformations streamline the equations, making them more amenable to rigorous analysis and resolution. The double integral transform proves to be exceptionally advantageous in the realm of dynamic fracture mechanics, as it helps to determine stress intensity factors and crack opening displacements. This mathematical technique represents an advance of integral transforms into two-dimensional contexts, thereby enhancing the capacity to resolve complex equations, particularly partial differential equations (PDEs). The method has garnered significant interest across various disciplines, encompassing physics, engineering, and applied mathematics, owing to its efficacy in simplifying and resolving intricate issues. Although double integral transforms present considerable benefits in the resolution of complex equations, they necessitate a comprehensive understanding of their inherent properties and associated theorems for effective application. Al-Safi et al. [1] introduce an innovative DIT referred to as the Double Sumudu-Elzaki transform (DSET), which is integrated with the variational iteration method to address nonlinear partial differential equations (PDE) characterized by fractional order derivatives. Elzaki et al. [2] introduce a combined approach centered on the double transformation of Laplace and Sumudu (DLST) and develop results related to this proposed method. Ahmed et al. [3] present an efficient new double transform called the double Laplace-Sumudu transform (DLST) to solve partial differential equations. Theorems are established that address key properties of the double Laplace-Sumudu transform, including the convolution theorem and its proof. Kashuri et al. [4] established a relationship between the double new integral transform and the double Laplace transform. Awasthi et al. [5] explore the mechanics of bone cracks, particularly focusing on Griffith crack models influenced by forces at the crack faces. This research is significant in understanding fracture mechanics in biological materials, which exhibit unique properties compared to traditional materials. The following sections elaborate on key aspects of this study. Ahmed [6] introduces an effective alternative double transform, known as the double Laplace-Sumudu transform (DLST), and establishes several related theorems.

The Laplace transform is a powerful method used to study linear time-invariant systems in mathematics, engineering, and physics. As engineering challenges become more complex, Laplace transforms to offer a simple and efficient way to tackle intricate problems, similar to how transfer functions are employed to solve ordinary differential equations (ODE). Cao et al. [7] emphasize the innovative modeling techniques and their implications for understanding dynamic fracture behaviors in transient flow scenarios, which are vital for effective resource management in fractured reservoirs. Hosseini-Tehrani et al. [8] highlight the integration of established methods and concepts in fracture mechanics while introducing new techniques and considerations that enhance the analysis of dynamic cracks under coupled thermoelastic conditions. The comparisons with existing literature further solidify the relevance and accuracy of the proposed approach. Savruk [9] introduces a new approach to the solution of dynamic problems of the theory of elasticity and fracture mechanics based on the application of the finite-difference method only concerning time. The equation of motion is divided into homogeneous and inhomogeneous systems to compute displacements at time nodes. Davies and Martin [10] evaluated various methods for numerically inverting the Laplace transform, analyzing their effectiveness for practical inversion tasks. They assessed these techniques based on their compatibility with different types of functions, numerical accuracy, computational efficiency, and ease of implementation and programming. Cost [11] explores different approaches for inverting the Laplace transform in the context of viscoelastic stress analysis, focusing on their applications and effectiveness in addressing specific engineering challenges.

The Hankel transform is a powerful mathematical tool in fracture mechanics, particularly for solving problems involving axisymmetric geometries. It simplifies the analysis of stress and displacement fields in materials with cracks or defects by transforming complex differential equations into more manageable forms. This method is particularly useful in polar coordinates, where the Hankel transform can effectively address the governing equations of fracture mechanics. Ueda [12] establishes a context for the research by discussing the significance of piezoelectric materials, previous studies on fracture mechanics, and the methodologies employed to analyze complex problems in this field. Yonglin et al. [13] introduced a novel numerical method for computing the Hankel transform, which is crucial for resolving several physics and engineering issues. This approach is especially helpful for high oscillation integrals using Bessel functions, which are frequently found in domains such as wave propagation and electromagnetic physics. Cinelli [14] developed finite Hankel transforms that include kernels paired with their associated infinite series. These innovations enable the use of integral transform theory to address Bessel's equation with asymmetric boundary conditions. Erdogan [15] proposes that simultaneous dual integral equations with trigonometric and Bessel kernels can be resolved by converting them into simultaneous singular integral equations. However, the study highlights that a direct approach to solving these dual integral equations may fail to capture the oscillatory nature of the solution accurately. The Fox H-function plays a significant role in crack analysis, particularly in the context of integral equations and transformations. This function's versatility allows it to model various physical phenomena, including stress distribution in materials with cracks. Kalia [16] examined the application of dual integral equations with Fox's H-function kernels to address a class of mixed boundary value problems related to the potential of an electrified disc. Awasthi et al. [17] studied dual integral equations involving Fox's H-function and provided formal solutions for certain dual integral equations that incorporate Fox's H-function. Fox [18] defined the H-function as a contour integral of the Mellin-Barnes type, symbolically represented as:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\ \alpha_p)\}\\ \{(b_q,\ \beta_q)\}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j s\right)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\beta_j s\right)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\,x^s ds \tag{1}$$

Where $\{(f_r,\gamma_r)\}$ stands for a set of parameters $(f_r,\gamma_1),\ldots,(f_r,\gamma_r)$; $x\neq 0$ and all other conditions are the same as given in [21].

In this article, we focus on the chain properties of the double integral transform:

$$\phi\left(t\right)=\int_{0}^{\infty}\int_{0}^{\infty}x^{\alpha-1}\,y^{\beta-1}(x+y)^{\sigma}$$
$$\times H_{u,v}^{f,g}\left[\lambda\left(x+y\right)\left|\begin{matrix}\{(A_u,\,\eta_u)\}\\\{(B_v,\,\xi_v)\}\end{matrix}\right.\right]$$
$$\times H_{p,q}^{m,n}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\left|\begin{matrix}\{(c_p,\,\gamma_p)\}\\\{(d_q,\,\delta_q)\}\end{matrix}\right.\right]f\left(x,y\right)dx\,dy \tag{2}$$

Provided that
$0\leq m\leq q,\;\; 0\leq n\leq p,\;\; 0\leq f\leq v,\;\; 0\leq g\leq u,$
$R(\alpha)>0,\;\; R(\beta)>0,\; \alpha_1,\; \beta_1,\; \sigma_1\geq 0,$
$-\min_{1\leq j\leq f}R\left(B_j|\xi_j\right)<R\left(\alpha+\beta+\sigma\right)<-\max_{1\leq i\leq g}R\left\{\frac{A_i-1}{\eta_i}\right\},$
m,n,p,q,f,g,u,v are integers with various known transforms e.g. the Laplace transform, the Hankel transform, the $J_{\nu,\lambda}^{\mu}$ transfrom due to Pathak [19] and Ψ_{ν_1,k_1,m_1} transform due to Narain [20]. For the sake of brevity, we denote the double integral transform (2) as

$$\phi\left(t\right)=DT\;\left[f\left(x,y\right)\right]$$

The motivation behind this article comes from recent findings (4) by Srivastava and Panda [21]. We prove three key theorems, often using power series expansions of special functions present in the integrals. Several examples are provided, which help evaluate both known and new integrals that involve the product of H-functions and other functions. The below-given Fig.2 presents the step-by-step process described in the article. Visually, it outlines each stage in the sequence.

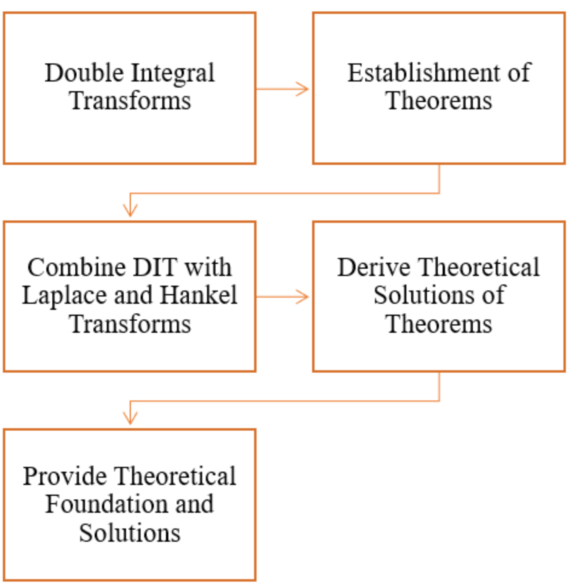

Figure 2. The Process Flow Chart of the article

2 Theorem 1

If

$$\phi\left(t\right)=DT\left[f\left(x\right)\right] \tag{3}$$

and $f(x)$ is self-reciprocal in the ψ_{v_1},k_1,m_1 transform then

$$\phi(t)=\frac{\lambda^{-\alpha-\beta-\sigma-\nu_1-\frac{1}{2}}}{2^{v_1}}$$
$$\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{\Gamma(2m_1-r)\Gamma\left(\frac{1}{2}-k_1+m_1+\nu_1+r\right)}{\Gamma\left(1+2m_1+\nu_1+r\right)}$$
$$\times\,2^{-2r}\frac{1}{\Gamma(1+\nu_1+r)\Gamma\left(-k_1+m_1+\frac{1}{2}-r\right)}H_{p+v+2,\,q+u+1}^{g+m,\,2+f+n}$$
$$\times\left[t\lambda^{-\theta}\left|\begin{matrix}\left(\frac{1}{2}-\alpha-\nu_1-2r,\alpha_1\right),\left(1-\beta,\beta_1\right),\\\left\{\left(\epsilon_f,\theta\xi_f\right)\right\},\left\{\left(c_p,\gamma_p\right)\right\},\\\left(\epsilon_{f+1},\theta\xi_{f+1}\right),\ldots,\left(\epsilon_v,\theta\xi_v\right)\\\\\left\{\left(\Psi_g,\theta\eta_g\right)\right\},\left\{\left(d_q,\delta_q\right)\right\},\\\left(\frac{1}{2}-\alpha-\beta-\nu_1-2r,\alpha_1+\beta_1\right),\\\left(\Psi_{g+1},\theta\eta_{g+1}\right),\ldots,\left(\Psi_u,\theta\eta_u\right)\end{matrix}\right.\right]$$
$$\times\int_0^{\infty}z^{\nu_1+2r+\frac{1}{2}}f(z)\,dz \tag{4}$$

Provided that the integrals

$$\int_0^{\infty}z^{\nu_1+2r+\frac{1}{2}}f(z)\,dz\quad\text{and}\quad\int_0^{\infty}z^{-\nu_1+\frac{1}{2}}f(z)\,dz\text{ exist.}$$

Where $\alpha_1,\beta_1,\sigma_1\geq 0, R\left(\nu_1-k_1+m_1+\frac{1}{2}\right)>0,$ $R(m_1)>0,\nu_1<0,\nu_1<-2m_1,2m_1$ is not an integer;$-\delta'<R\left(\alpha+\beta+\sigma+\nu_1+\frac{1}{2}\right)<-\beta'$,and $\theta,\epsilon_j\left(j=1,\ldots,v\right),\psi_j\left(j=1,\ldots,u\right)$

Proof: We have

$$f(x)=2^{\nu_1}\int_0^{\infty}(xz)^{-\nu_1+\frac{1}{2}}$$
$$\times\quad H_{2,4}^{2,1}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(k_1-m_1-\frac{1}{2},1\right),\left(\nu_1-k_1+2m_1+\frac{1}{2},1\right)\\\left(\nu_1,1\right),\left(\nu_1+2m_1,1\right),\left(-2m_1,1\right),\left(0,1\right)\end{matrix}\right.\right]$$
$$\times\,f(z)\,dz \tag{5}$$

On substituting for $f(x)$ from (5) in (3) and changing the order of integration which is justifiable under the above conditions, we obtain

$$\phi(t)=2^{\nu_1}\int_0^{\infty}z^{-\nu_1+\frac{1}{2}}f(z)\,dz$$
$$\int_0^{\infty}\int_0^{\infty}x^{\alpha-\nu_1+\frac{1}{2}-1}y^{\beta-1}(x+y)^{\sigma}$$
$$\times\,H_{u,v}^{f,g}\left[\lambda(x+y)\left|\begin{matrix}\{(A_u,\eta_u)\}\\\{(B_v,\xi_v)\}\end{matrix}\right.\right]$$
$$\times\,H_{p,q}^{m,n}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\left|\begin{matrix}\{(c_p,\gamma_p)\}\\\{(d_q,\delta_q)\}\end{matrix}\right.\right]$$
$$\times\,H_{2,4}^{2,1}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(k_1-m_1-\frac{1}{2},1\right),\left(\nu_1-k_1+m_1+\frac{1}{2},1\right)\\\left(\nu_1,1\right),\left(\nu_1+2m_1,1\right),\left(-2m_1,1\right),\left(0,1\right)\end{matrix}\right.\right]$$
$$\times\,dx\,dy \tag{6}$$

But from the power series expansion due to Mukherjee and Prasad [22]

$$H_{p,q+1}^{m+1,n}\left[ax^{\sigma}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\(b_0,\beta_0),\{(b_q,\beta_q)\}\end{matrix}\right.\right]=\frac{1}{\beta_0}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}$$
$$\times\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\rho_r)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\rho_r)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\rho_r)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\rho_r)}$$
$$\times a^{\rho r}x^{\sigma\rho r} \tag{7}$$

Where,

$$\rho_r=\frac{b_0+r}{\beta_0},\ \beta<R\left(\frac{b_0}{\beta_0}\right)<\delta,\ |arga|<\frac{1}{2}\lambda\pi,\lambda>0,A>0,$$

We have

$$H_{2,4}^{2,1}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(k_1-m_1-\frac{1}{2},1\right),\left(\nu_1-k_1+m_1+\frac{1}{2},1\right)\\(\nu_1,1),(\nu_1+2m_1,1),(-2m_1,1),(0,1)\end{matrix}\right.\right]$$
$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\times$$
$$\frac{\Gamma(2m_1-r)\Gamma\left(\frac{1}{2}-k_1+m_1+\nu_1+r\right)}{\Gamma(1+2m_1+\nu_1+r)\Gamma(1+v_1+r)\Gamma\left(-k_1+m_1+\frac{1}{2}-r\right)}$$
$$\times\left(\frac{xz}{2}\right)^{2\nu_1+2r} \tag{8}$$

Provided that
$R\left(v_1-k_1+m_1+\frac{1}{2}\right)>0,\ R(m_1)>0,$
$v_1<0,-v_1<-2m_1,$ and $2m_1$ is not a positive integer.
By substituting the results (8) in (6) in a switch of the order of integration and summation, we obtain the theorem.
Example: Let us consider $f(x)=e^{-x^2}$
Where the function $f(x)$ is a Gaussian function that often satisfies self-reciprocal properties in integral transforms.
Since $f(x)$ is self-reciprocal in the ψ_{ν_1,k_1,m_1} transform. On expressing $f(x)$ in an integral form involving Fox's H-function, we obtain equation (5) as
$f(x)=2^{\nu_1}\int_0^{\infty}(xz)^{-\nu_1+\frac{1}{2}}$
$\times H_{2,4}^{2,1}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(k_1-m_1-\frac{1}{2},1\right),\left(\nu_1-k_1+2m_1+\frac{1}{2},1\right)\\(\nu_1,1),(\nu_1+2m_1,1),(-2m_1,1),(0,1)\end{matrix}\right.\right]$
$\times f(z)\,dz$
Using $\alpha=1,\beta=1,\sigma=0;\nu_1=-1,k_1=0,m_1=1$, and $\lambda=1$, we get

$$f(x)=2^{-1}\int_0^{\infty}(xz)^1\times H_{2,4}^{2,1}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(-\frac{1}{2},1\right),\left(\frac{1}{2},1\right)\\(-1,1),(1,1),(-2,1),(0,1)\end{matrix}\right.\right]e^{-z^2}\,dz \tag{9}$$

Further on substituting equation (9) in (3), we get

$$\phi(t)=2^{-1}\int_0^{\infty}ze^{-z^2}\,dz\int_0^{\infty}\int_0^{\infty}x^{1/2}y^0(x+y)^0$$
$$H_{u,v}^{f,g}\left(\lambda(x+y)\left|\begin{matrix}(A_u,\eta_u)\\(B_v,\xi_v)\end{matrix}\right.\right)H_{p,q}^{m,n}\left(txy\left|\begin{matrix}(c_p,\gamma_p)\\(d_q,\delta_q)\end{matrix}\right.\right)$$
$$H_{2,4}^{2,1}\left(\frac{x^2z^2}{4}\left|\begin{matrix}\left(-\frac{1}{2},1\right),\left(\frac{1}{2},1\right)\\(-1,1),(1,1),(-2,1),(0,1)\end{matrix}\right.\right)dx\,dy. \tag{10}$$

Using the series expansion for Fox's H-function as provided by Mukherji and Prasad [22], we get $H_{2,4}^{2,1}\left(\frac{x^2z^2}{4}\left|\begin{matrix}\left(-\frac{1}{2},1\right),\left(\frac{1}{2},1\right)\\(-1,1),(1,1),(-2,1),(0,1)\end{matrix}\right.\right)=$ $\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\times\frac{\Gamma(2m-r)\Gamma\left(\frac{1}{2}-k_1+m_1+\nu_1+r\right)}{\Gamma(1+2m_1+\nu_1+r)\Gamma(1+\nu_1+r)\Gamma\left(-k_1+m_1+\frac{1}{2}-r\right)}$
$\times\left(\frac{xz}{2}\right)^{2\nu_1+2r}$

On substituting the above expression into equation (10), on interchanging the summation, and on simplifying each term by performing the integrations involving x and y separately we shall obtain the final expression in the form of equation (4). This result demonstrates the use of the series representation and convergence properties of Fox's H-function in the context of integral transforms.

3 Theorem 2

If
$$\phi(\mathrm{t})=\mathrm{DT}\left[f(x)\right]$$
and $f(x)$ is the Laplace transform of $g(z)$ then

$$\phi(t)=\lambda^{-\alpha-\beta-\sigma}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\lambda^{-r}\ H_{p+v+2,\ \ q+u+1}^{m+g,\ \ 2+f+n}$$
$$\left[t\lambda^{-\theta}\left|\begin{matrix}(1-\alpha-r,\ \alpha_1),\ (1-\beta,\beta_1),\\\{(\epsilon_f,\theta\xi_f)\},\{(c_p,\gamma_p)\},\\(\epsilon_{f+1},\theta\xi_{f+1}),\ldots,\ (\epsilon_v,\theta\xi_v)\\ \\\{(\Psi_g,\theta\eta_g)\},\{(d_q,\delta_q)\},\\(1-\alpha-\beta-r,\alpha_1+\beta_1),\\(\Psi_{g+1},\ \theta\eta_{g+1}),\ \ \ldots,(\Psi_u,\ \theta\eta_u)\end{matrix}\right.\right] \tag{11}$$
$$\times\int_0^{\infty}z^r g(z)\,dz$$

Where,

$$\begin{cases}\theta=\alpha_1+\beta_1+\sigma_1,\\\epsilon_j=1-B_j-(\alpha+\beta+\sigma)\,\xi_j,\quad j=1,\ldots,v,\\\Psi_j=1-A_j-(\alpha+\beta+\sigma)\,\eta_j,\quad j=1,\ldots,u,\end{cases}$$

Provided that the integrals $\int_0^{\infty}g(z)\,dz$ and $\int_0^{\infty}z^r g(z)\,dz$, $r\geq 1$, exist and all other conditions given with (2) are satisfied.
Proof: We have

$$f(x)=\int_0^{\infty}e^{-xz}g(z)\,dz$$

On substituting for $f(x)$ in (2) and changing the order of integration which is justifiable under the given conditions, we have

$$\phi(t)=\int_0^\infty g(z)\,dz\int_0^\infty\int_0^\infty e^{-xz}x^{\alpha-1}y^{\beta-1}(x+y)^\sigma \times H_{u,v}^{f,g}\left[\lambda(x+y)\,\middle|\,\begin{matrix}\{(A_u,\eta_u)\}\\ \{(B_v,\xi_v)\}\end{matrix}\right] \times H_{p,q}^{m,n}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\,\middle|\,\begin{matrix}\{(c_p,\gamma_p)\}\\ \{(d_q,\delta_q)\}\end{matrix}\right]dx\,dy$$

By the definition of Mellin transform $F(s)$ of a function $f(x)$ as

$$F(s)=M\{f(x):s\}=\int_0^\infty x^{s-1}f(x)\,dx$$

Where s is a complex number.
For convenience, we abbreviate the first member of equation (1) by

$$H_{p,q}^{f,g}[x]$$

Then by the Mellin inverse theorem [23], it follows from (1) that

$$M\left\{H_{p,q}^{f,g}[x]:s\right\}=\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}\,z^{-s}$$

Provided that

$$-\min_{1\le j<m}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)<\mathrm{Re}(s)<\min_{1\le j\le n}\mathrm{Re}\left(\frac{1-a_j}{\alpha_j}\right)$$

By using [21] the relation $\phi(x)=x^\sigma H_{p,q}^{f,g}[\lambda x]$ and expanding e^{-xz} in powers of xz and integrating term by term by using the following known integral due to Srivastava and Panda [21], we obtain

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\sigma H_{u,v}^{f,g}\left[\lambda(x+y)\,\middle|\,\begin{matrix}\{(A_u,\ \eta_u)\}\\ \{(B_v,\ \xi_v)\}\end{matrix}\right] \times H_{p,q}^{m,n}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\,\middle|\,\begin{matrix}\{(c_p,\gamma_p)\}\\ \{(d_q,\delta_q)\}\end{matrix}\right]dx\,dy$$
$$=\lambda^{-\alpha-\beta-\sigma}H_{p+v+2,\ q+u+1}^{m+g,\ 2+f+n} \times\left[t\lambda^{-\theta}\,\middle|\,\begin{matrix}(1-\alpha,\ \alpha_1),\ (1-\beta,\beta_1),\\ \{(\epsilon_f,\ \theta\xi_f)\},\ \{(c_p,\gamma_p)\},\\ (\epsilon_{f+1},\theta\xi_{f+1}),\dots,\ (\epsilon_v,\theta\xi_v)\\ \\ \{(\Psi_g,\theta\eta_g)\},\ \{(d_q,\delta_q)\},\\ (1-\alpha-\beta,\ \alpha_1+\beta_1),\\ (\Psi_{g+1},\theta\eta_{g+1}),\dots,(\Psi_u,\theta\eta_u)\end{matrix}\right] \tag{12}$$

Where all the conditions given in (11) are satisfied, we obtain the theorem.
Example: Let

$$g(z)=z^{-\mu-\frac{1}{2}}\,K_{\nu+\frac{1}{2}}(bz)$$

By using [24], we obtain

$$f(x)=\frac{\sqrt{\pi}\,\Gamma(-\mu+\nu+1)\,\Gamma(-\mu-\nu)}{(2b)^{\frac{1}{2}}}\left(x^2-b^2\right)_\nu^{\mu/2}\ P_\nu^\mu\left(\frac{x}{b}\right) \tag{13}$$

Where $P_\nu^\mu(z)$ is the associated Legendre function with$R(\mu)-1<R(\nu)<-R(\mu)$
By putting $s=r-\mu+\frac{1}{2}$ in [25], we obtain

$$\int_0^\infty Z^r g(z)\,dz=b^{-r+\mu-\frac{1}{2}}\,2^{r-\mu-\frac{3}{2}}\Gamma\left(\frac{1}{2}r-\frac{1}{2}\mu+\frac{1}{2}-\frac{1}{2}v\right) \times\Gamma\left(\frac{1}{2}r-\frac{1}{2}\mu+\frac{1}{2}+\frac{1}{2}\nu\right) \tag{14}$$

Finally, by putting the values from (13) and (14) in the theorem, we obtain

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\left(x^2-b^2\right)^{\frac{1}{2}\mu}P_\nu^\mu\left(\frac{x}{b}\right)(x+y)^\sigma \times H_{u,v}^{f,g}\left[\lambda(x+y)\,\middle|\,\begin{matrix}\{(A_u,\eta_u)\}\\ \{(B_v,\xi_v)\}\end{matrix}\right] \times H_{p,q}^{m,n}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\,\middle|\,\begin{matrix}\{(c_p,\gamma_p)\}\\ \{(d_q,\delta_q)\}\end{matrix}\right]dxdy$$
$$=\frac{\lambda^{-\alpha-\beta-\sigma}\,2^{-\mu-1}b^\mu}{\sqrt{\pi}\,\Gamma(-\mu+\nu+1)\ \Gamma(-\mu-\nu)} \sum_{r=0}^\infty\frac{(-2/b\lambda)^r}{r!}\Gamma\left(\frac{1}{2}r-\frac{1}{2}\mu+\frac{1}{2}-\frac{1}{2}\nu\right) \times\Gamma\left(\frac{1}{2}r-\frac{1}{2}\mu+\frac{1}{2}+\frac{1}{2}\nu\right)H_{p+v+2,\ q+u+1}^{m+n,\ 2+f+n} \left[t\lambda^{-\theta}\,\middle|\,\begin{matrix}(1-\alpha-r,\alpha_1),(1-\beta,\beta_1),\\ \{(\epsilon_f,\theta\xi_f)\},\{(c_p,\gamma_p)\},\\ (\epsilon_{f+1},\theta\xi_{f+1}),\dots,(\epsilon_v,\theta\xi_v)\\ \\ \{(\Psi_g,\theta\eta_g)\},\{(d_g,\delta_g)\},\\ (1-\alpha-\beta-r,\alpha_1+\beta_1),\\ (\Psi_{g+1},\theta\eta_{g+1}),\dots,(\Psi_u,\theta\eta_u)\end{matrix}\right] \tag{15}$$

Provided that

$$R(\mu)-1<R(\nu)<-R(\mu)$$

and the conditions given in (11) are satisfied.

4 Theorem 3

If

$$\phi(t)=DT[f(x)] \tag{16}$$

and

$$f(x)=J_{\nu,\lambda_1}^{\mu} \tag{17}$$

transform of $g(z)$.
Where,

$$J_{\nu,\lambda_1}^{\mu}(x)=\sum_{r=0}^\infty\frac{(-1)^r\left(\frac{1}{2}r\right)^{\nu+2r+2\lambda_1}}{\Gamma(1+\lambda_1+r)\,\Gamma(1+\lambda_1+\nu+\mu_r)},\quad \mu>0 \tag{18}$$

Then

$$\phi(t) = \frac{\lambda^{-\alpha-\beta-\sigma-\nu-2\lambda_1+\frac{1}{2}}}{2^{\nu+2\lambda_1}}$$
$$\sum_{r=0}^{\infty} \frac{(-1)^r(2\lambda)^{-2r}}{\Gamma(1+\lambda_1+r)\Gamma(1+\lambda_1+\nu+\mu_r)} H^{g+m,\,2+f+n}_{p+v+2,\,q+u+1}$$
$$\times \left[t\lambda^{-\theta} \left| \begin{array}{c} \left(\frac{1}{2}-\alpha-\nu-2\lambda-2r,\alpha_1\right), \\ (1-\beta,\beta_1), \{(\epsilon_f,\theta\xi_f)\}, \{(c_p,\gamma_p)\}, \\ (\epsilon_{f+1},\theta\xi_{f+1}),\ldots,(\epsilon_v,\theta\xi_v) \\ \\ \{(\Psi_g,\theta\eta_g)\}, \{(d_g,\delta_g)\}, \\ \left(\frac{1}{2}-\alpha-\beta-\nu-2\lambda_1-2r,\alpha_1+\beta_1\right), \\ (\Psi_{g+1},\theta\eta_{g+1}),\ldots,(\Psi_u,\theta\eta_u) \end{array} \right. \right]$$
$$\times \int_0^\infty z^{\nu+2r+2\lambda_1+\frac{1}{2}} g(z)\,dz \tag{19}$$

Provided that the integrals

$$\int_0^\infty z^{\frac{1}{2}} g(z)\,dz \;\; and \;\; \int_0^\infty z^{\nu+2\lambda_1+2r+\frac{1}{2}} g(z)\,dz$$

exist.
Where

$$\alpha_1.\beta_1,\sigma_1,\mu \geq 0, \quad R(\alpha+\nu+2\lambda_1+\frac{1}{2}) > 0, \quad R(\beta) > 0,$$
$$-\min_{1\leq j\leq f} R\left(\frac{B_j}{\xi_j}\right) < R(\alpha+\beta+\sigma+\nu+2\lambda_1+\frac{1}{2})$$
$$< -\max_{1\leq i\leq g} R\left(\frac{A_i-1}{\eta_i}\right)$$

and $\epsilon_j (j=1,2,\ldots,v)$ and $\varphi_j (j=1,2,\ldots,u)$ are the same as in (11).

Proof: On substituting for

$$f(x) = \int_0^\infty (xz)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda_1}(xz)\, g(z)\, dz \tag{20}$$

The expression for $\phi(t)$ given by (2) and changing the order of integration which is suitable under the given conditions, we have

$$\phi(t) = \int_0^\infty z^{\frac{1}{2}} g(z)\,dz \int_0^\infty \int_0^\infty x^{\alpha-\frac{1}{2}} y^{\beta-1} J^{\mu}_{\nu,\lambda_1}(xz)(x+y)^{\sigma}$$
$$\times H^{f,g}_{u,v}\left[\lambda(x+y) \left| \begin{array}{c} \{(A_u,\eta_u)\} \\ \{(B_v,\xi_v)\} \end{array} \right. \right]$$
$$\times H^{m,n}_{p,q}\left[t x^{\alpha_1} y^{\beta_1}(x+y)^{\sigma_1} \left| \begin{array}{c} \{(c_p,\gamma_p)\} \\ \{(d_q,\delta_q)\} \end{array} \right. \right] dx\,dy \tag{21}$$

Now on substituting the series expansion for $J^{\mu}_{\nu,\lambda_1}(xz)$ from (18) and evaluating the inner integral with the help of (12) we obtain the theorem stated above.

Example: Let

$$g(z) = z^{-\frac{1}{2}}\, e^{-az} J_\nu(bz)$$

On using [24], we get

$$f(x) = \int_0^\infty g(z)\, J\nu(zx)\,(zx)^{1/2} dz$$

$$f(x) = \int_0^\infty z^{-1/2}\, e^{-az} J\nu(bz)\, J_\nu(zx)\,(zx)^{1/2} dz$$

$$f(x) = \frac{1}{\pi} b^{-\frac{1}{2}} Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+x^2}{2bx}\right) \tag{22}$$

Provided that

$$R(a) > Im(b) > 0$$

and

$$R(\nu) > -\frac{1}{2}$$

By using [25], we obtain

$$\int_0^\infty z^{\nu+2r+\frac{1}{2}} g(z)\,dz = \frac{b^\nu\,\Gamma(2r+2\nu+1)}{2^\nu\, a^{2\nu+2r+1}\,\Gamma(\nu+1)}$$
$$\times\, {}_2F_1\left(\nu+r+\frac{1}{2};\nu+r+1;\nu+1;-\frac{b^2}{a^2}\right)$$

Provided that

$$R(a) > Im(b) > 0$$

and

$$R(\nu) > -\frac{1}{2}$$

Hence using the result (19) with $\lambda_1 = 0$ and $\mu = 1$, we have

$$\int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} (x+y)^{\sigma}\, Q_{\nu-1/2}\left(\frac{a^2+b^2+x^2}{2bx}\right)$$
$$\times H^{f,g}_{u,v}\left[\lambda(x+y) \left| \begin{array}{c} \{(A_u,\eta_u)\} \\ \{(B_v,\xi_v)\} \end{array} \right. \right]$$
$$\times H^{m,n}_{p,q}\left[t x^{\alpha_1} y^{\beta_1}(x+y)^{\sigma_1} \left| \begin{array}{c} \{(c_p,\gamma_p)\} \\ \{(d_q,\delta_q)\} \end{array} \right. \right] dx\,dy$$
$$= \frac{\lambda^{-\alpha-\beta-\sigma-\nu+\frac{1}{2}} b^{\nu+\frac{1}{2}\pi}}{2^{2\nu}\Gamma(1+\nu)a^{2\nu+1}}$$
$$\times \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(2\lambda a)^{-2r}\Gamma(2\nu+2r+1)}{\Gamma(\nu+r+1)}$$
$$\times\, {}_2F_1\left(\nu+r+\frac{1}{2},\nu+r+1;\nu+1;\frac{-b^2}{a^2}\right)$$
$$\times H^{m+g,2+f+n}_{p+v+2,q+u+1}$$
$$\times \left[t\lambda^{-\theta} \left| \begin{array}{c} \left(\frac{1}{2}-\alpha-\nu-2r,\alpha_1\right), (1-\beta,\beta_1), \\ \{(\epsilon_f,\theta\xi_f)\}, \{(c_p,\gamma_p)\}, \\ (\epsilon_{f+1},\theta\xi_{f+1}),\ldots,(\epsilon_v,\theta\xi_v) \\ \\ \{(\Psi_g,\theta\eta_g)\}, \{(d_q,\delta_q)\}, \\ \left(\frac{1}{2}-\alpha-\beta-\nu-2r,\alpha_1+\beta_1\right), \\ (\Psi_{g+1},\theta\eta_{g+1}),\ldots,(\Psi_u,\theta\eta_u) \end{array} \right. \right] \tag{23}$$

Provided that

$$R(a) > Im(b) > 0, R(\nu) > -\frac{1}{2},\ \alpha_1,\beta_1,\sigma_1 \geq 0$$

and the other conditions given in (11) are satisfied.

5 Conclusions

This paper contributes to the theoretical framework of Double Integral Transform (DIT), through the lens of Fox's H-Function, highlighting its crucial role in shaping and understanding these mathematical tools. At first, we have shown that the DIT, denoted as $\phi(t) = DT\ [f(x,y)]$ is intricately connected to established transforms such as the Laplace and Hankel transforms. The given problem is based on certain conditions which are solved by making use of Fox's H-function, defined as a Mellin-Barnes-type contour integral which is symbolically denoted as in equation (1). Next, we discussed the chain properties connecting the DIT as given in equation (2). Further, we established three key theorems that demonstrate the correlation between the DIT, the Laplace transform, the Hankel transform, and the specialized transforms given by Pathak and Narain. The established theorems one, two, and three are then proven analytically, and the corresponding examples of theorems one and two have been given to validate the results. The three presented theorems provide valuable insights for researchers dealing with crack problems and working in the field of integral transformation with special functions.

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