Mathematics and Statistics 13(2): 82-88, 2025 http://www.hrpub.org
DOI: 10.13189/ms.2025.130202

Second Order Unstable Type Difference Equations with Deviating Arguments: New Asymptotic Results

Sudha B[1], **Srinivasan K**[2], **Thandapani E**[3,*]

[1]Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603 203, Chengalpattu, Tamil Nadu, India

[2]Department of Mathematics, R.M.K Engineering College, Kavaraipettai, Tamil Nadu, India

[3]Department of Mathematics, University of Madras, Ramanujan Institute for Advanced Study in Mathematics, Chennai - 600 005, Tamil Nadu, India

Received December 1, 2024; Revised March 5, 2025; Accepted March 20, 2025

Cite This Paper in the Following Citation Styles

***(a)**: [1] Sudha B, Srinivasan K, Thandapani E, "Second Order Unstable Type Difference Equations with Deviating Arguments: New Asymptotic Results," Mathematics and Statistics, Vol.13, No.2, pp. 82-88, 2025. DOI: 10.13189/ms.2025.130202*

***(b)**: Sudha B, Srinivasan K, Thandapani E (2025). Second Order Unstable Type Difference Equations with Deviating Arguments: New Asymptotic Results, Mathematics and Statistics, 13(2), 82-88. DOI: 10.13189/ms.2025.130202*

Abstract Differential equations with deviating arguments serve as a fundamental cornerstone in mathematical modeling, offering a robust framework for characterizing the dynamics of various systems across multiple disciplines. Meanwhile, oscillatory theorems are essential in analyzing the intrinsic vibrational patterns within dynamic systems, providing critical insights into their stability and periodicity. This study focuses on examining the behavior of half-linear second-order difference equations of an unstable type when their arguments are altered from the form

$$\triangle\ (\lambda(\flat)(\triangle\ \rho(\flat))^{a}) = \eta(\flat)\rho^{a}(\sigma(\flat)).$$

Utilizing the summation averaging method alongside the generalized Riccati transformations, we initiate new properties of monotonic to non-oscillatory solutions, enabling us to establish conditions that eliminate specific types of non-oscillatory behavior.

These findings lead to novel oscillation criteria applicable to second-order difference equations with both advanced and delayed arguments. A key application of these results is the analysis of the oscillatory nature of difference equations arising in the Thomas–Fermi (T–F) model, a fundamental equation in physics. The T–F equation represents the simplest formulation for modeling the screened electrostatic Coulomb potential around a highly charged nucleus and its surrounding electron cloud. Beyond atomic physics, this equation finds broad applicability across numerous physical domains. The examples presented at the conclusion of this work illustrate the enhanced results achieved, demonstrating improvements over previously established findings.

Keywords Oscillation, Third-order, Neutral Differential Equation, Mixed Nonlinearities

---

1 Introduction

This paper explores the asymptotic and oscillatory characteristics of solutions to the half-linear second-order difference equation given by

$$\triangle\ (\lambda(\flat)(\triangle\ \rho(\flat))^{a}) = \eta(\flat)\rho^{a}(\sigma(\flat)),\ \ \flat \geq \flat_0. \tag{1.1}$$

We make the following assumptions:

(H_1) The sequences $\{\lambda(\flat)\}$ and $\{\rho(\flat)\}$ consist of positive real numbers and a be expressed as the ratio of two positive odd integers.

(H_2) The sequence [(H_2)] $\{\sigma(\flat)\}$ is strictly increasing, with $\lim\limits_{\flat\to\infty} \sigma(\flat) = \infty$.

A solution to equation (1.1), is a real sequence $\{\rho(\flat)\}$ defined for $\flat \geq \flat_0 - \inf\sigma(\flat)$ and satisfying (1.1) for all $\flat \geq \flat_0$, and fulfilling the condition $\sup\{|\rho(s)| : s \geq \flat\} > 0$ for all $\flat \geq \flat_0$. Such a solution $\{\rho(\flat)\}$ is termed oscillatory if, for any $\flat_1 \geq \flat_0$, there exist integers $\flat_2, \flat_3 \geq \flat_1$ such that

$\rho(\flat_2)\rho(\flat_3) \leq 0$; otherwise, it is referred to as nonoscillatory. Throughout this study, we examine equation (1.1) in its canonical form, which satisfies the condition

$$\aleph(\flat) = \sum_{s=\flat_0}^{\flat-1} \lambda^{-\frac{1}{s}}(s) \to \infty \; as \; \flat \to \infty.$$

Difference equations frequently emerge in discrete data sets, playing a crucial role in various fields such as applied mathematics, electrical engineering, cardiology, optics, and management sciences. In recent years, significant attention has been devoted to both linear and nonlinear difference equations with deviating arguments due to their broad spectrum of applications. Beyond their practical relevance, the study of difference equations—particularly functional difference equations—holds intrinsic mathematical significance.

In recent years, numerous researchers ([1]-[11]) have investigated the bounded oscillations of various forms of 2nd order delay difference equations. Notably, the works in [6, 10] have examined the existence of positive solutions and the bounded oscillatory behavior of 2nd order delay difference equations of both weighted and unstable types. In [6], it was demonstrated that for $\sigma(\flat) = \flat - k$, the condition

$$\lim_{\flat\to\infty} \sup \sum_{s=\flat-k}^{\flat} \left(\frac{1}{\lambda(s)} \sum_{t=s}^{\flat} \eta(t) \right)^{\frac{1}{a}} > 1$$

implies (1.1) has no positive decreasing solutions.
Conversely, equation (1.1) does not have positive solutions. if $\sigma(\flat) = \flat + k$ and

$$\lim_{\flat\to\infty} \sup \sum_{s=\flat}^{\flat+k-1} \left(\frac{1}{\lambda(s)} \sum_{t=s}^{s-1} \eta(t) \right)^{\frac{1}{a}} > 1$$

hold.

In spite of this investigations, we see that relatively less attention has been given to second-order linear/ nonlinear delay difference equations of unstable type and discussed the sufficient conditions for the bounded oscillation of this equations.

Based on the aforementioned observations, this paper establishes new monotonic properties of nonoscillatory solutions of (1.1). Utilizing these properties, we derive novel criteria for ruling out specific types of nonoscillatory solutions of (1.1). Furthermore, these findings enable us to formulate conditions ensuring the oscillation of all solutions of second-order difference equations involving both delay and advanced arguments. To highlight the significance of the main results, several illustrative examples are provided. It is worth mentioning that this research is partially influenced by recent findings in [2] on functional differential equations.

Note:

* Eventually positive solution: A solution $\{\rho(\flat)\}$ of 1.1 is said to be eventually positive if there exists an integer $\flat_* \in \mathbb{N}(\flat_0)$ such that $\rho(\flat) > 0$ for all $\flat \in \mathbb{N}(\flat_*)$

* The symbols s, j, l, ω are positive integers, belongs to $\mathbb{N}(\flat_0)$

2 Main Results

In view of the discrete generalized kiguradze's Lemma [12], the set of positive solutions of (1.1) has the following structure.

Lemma 2.1 *Suppose that $\{\rho(\flat)\}$ is an eventually positive solution of (1.1). Then, it must satisfy one of the following conditions:*
(S_0) *:* $\triangle\,(\lambda(\flat)(\triangle\,\rho(\flat))^a) > 0,\ \lambda(\flat)(\triangle\,\rho(\flat))^a < 0,$
(S_2) *:* $\triangle\,(\lambda(\flat)(\triangle\,\rho(\flat))^a) > 0,\ \lambda(\flat)(\triangle\,\rho(\flat))^a > 0$ *for all* $\flat \geq \flat_1 \geq \flat_0$.
If we let S, set of all positive solutions of (1.1), then we have

$$S = S_0 \cup S_2,$$

The class S_j consists of solutions satisfy the conditions (S_j), for $j = 0, 2$.

We start with a fundamental lemma that describes the monotonic properties of nonoscillatory solutions of (1.1).

Lemma 2.2 *Let $\sigma(\flat) = \flat - \tau$, τ is a positive integer. Presume that $\{\rho(\flat)\}$ is a positive solution of (1.1) belongs to (S_0). If there exists a positive constant $\delta \in (0, 1)$ such that*

$$\aleph(\flat) \left(\sum_{s=\flat}^{\flat+\tau} \eta(s) \right)^{\frac{1}{a}} \geq \delta, \; for \; \flat \geq \flat_0 \tag{2.1}$$

then $\left\{\aleph^\delta(\flat)\rho(\flat)\right\}$ is decreasing.

Proof. Let $\{\rho(\flat)\}$ be a positive solution of (1.1) belongs to (S_0). In view of $\lambda(\flat)(\triangle\,\rho(\flat))^a < 0$, we see that $\rho(\flat)$ is decreasing. So, a summation of (1.1) from $\flat - \tau$ to $\flat$ yields

$$-\lambda^{\frac{1}{a}}(\flat - \tau) \triangle \rho(\flat - \tau) \geq \rho(\flat - \tau) \left(\sum_{s=\flat-\tau}^{\flat} \eta(s) \right)^{\frac{1}{a}}.$$

That is, replacing $\flat$ by τ

$$-\aleph(\flat)\lambda^{\frac{1}{a}}(\flat) \triangle \rho(\flat) \geq \rho(\flat + 1)\aleph(\flat) \left(\sum_{s=\flat}^{\flat+\tau} \eta(s) \right)^{\frac{1}{a}}.$$

Using (2.1), we are led to

$$-\aleph(\flat)\lambda^{\frac{1}{a}}(\flat) \triangle \rho(\flat) \geq \delta\rho(\flat + 1). \tag{2.2}$$

Now

$$\triangle\,(\aleph^\delta(\flat)\rho(\flat)) = \rho(\flat + 1) \triangle \aleph^\delta(\flat) + \aleph^\delta(\flat) \triangle \rho(\flat). \tag{2.3}$$

By discrete Mean-value theorem, we have

$$\triangle\,\aleph^\delta(\flat) \leq \frac{\delta\aleph^{\delta-1}(\flat)}{\lambda^{\frac{1}{a}}(\flat)}$$

and using these in (2.3), we get

$$\triangle\left(\aleph^{\delta}(\flat)\rho(\flat)\right)\le\frac{\aleph^{\delta-1}(\flat)}{\lambda^{\frac{1}{a}}(\flat)}\left[\delta\rho(\flat+1)+\aleph(\flat)\lambda^{\frac{1}{a}}(\flat)\triangle\rho(\flat)\right]\le 0$$

by (2.2). Hence, we conclude that the sequence $\left\{\aleph^{\delta}(\flat)\rho(\flat)\right\}$ is decreasing.

Theorem 2.3 *Let $\sigma(\flat)=\flat-\tau$, τ is a positive integer and (2.1) hold. If*

$$\lim_{\flat\to\infty}\sup\aleph^{\delta}(\flat-\tau)\sum_{s=\flat-\tau}^{\flat}\frac{1}{\lambda^{\frac{1}{a}}(s)}\left(\sum_{t=s}^{\flat}\frac{\eta(t)}{\aleph^{\delta a}(t-\tau)}\right)^{\frac{1}{a}}>1 \tag{2.4}$$

then the class S_0 is empty.

Proof. To arrive at a contradiction, suppose that (1.1) has an eventually positive solution. $\{\rho(\flat)\}$ belongs to S_0. Summing up (1.1) from j to $\flat$ and using monotonic property of $\left\{\aleph^{\delta}(\flat)\rho(\flat)\right\}$, we have

$$-\lambda(j)(\triangle\rho(j))^{a}\ge\aleph^{\delta a}(\flat-\tau)\rho^{a}(\flat-\tau)\sum_{s=j}^{\flat}\frac{\eta(s)}{\aleph^{\delta a}(s-\tau)},\ j\in\mathbb{N}(\flat_0).$$

Summing up once more from j to $\flat$ we obtain

$$\rho(j)\ge\aleph^{\delta}(\flat-\tau)\rho(\flat-\tau)\sum_{s=j}^{\flat}\frac{1}{\lambda^{\frac{1}{a}}(s)}\left(\sum_{t=s}^{\flat}\frac{\eta(t)}{\aleph^{\delta a}(t-\tau)}\right)^{\frac{1}{a}}.$$

Letting $j=\flat-\tau$, we have

$$\rho(\flat-\tau)\ge\aleph^{\delta}(\flat-\tau)\rho(\flat-\tau)\sum_{s=\flat-\tau}^{\flat}\frac{1}{\lambda^{\frac{1}{a}}(s)}\left(\sum_{t=s}^{\flat}\frac{\eta(t)}{\aleph^{\delta a}(t-\tau)}\right)^{\frac{1}{a}}$$

which contradicts to (2.4) and we conclude that S_0 is empty.

Next, we examine the monotonic behavior of potential positive increasing solutions of (1.1).

Lemma 2.4 *Let $\sigma(\flat)=\flat+k$, k is a positive integer. Presume that $\{\rho(\flat)\}$ is a positive solution of (1.1) belongs to (S_2). If there exists a positive constant $\omega\in(0,1)$ such that*

$$\aleph(\flat)\left(\sum_{s=\flat}^{\flat+k-1}\eta(s)\right)^{\frac{1}{a}}\ge\omega,\ for\ \flat\ge\flat_0 \tag{2.5}$$

then $\left\{\frac{\rho(\flat)}{\aleph^{\omega}(\flat)}\right\}$ is increasing.

Proof. Let $\{\rho(\flat)\}$ be a positive solution of (1.1) belongs to (S_2). In view of $\lambda(\flat)(\triangle\rho(\flat))^{a}>0$, we have $\rho(\flat)$ is increasing. Using this property in (1.1) and then summing up from $\flat$ to $\flat+k-1$, we have

$$-\lambda^{\frac{1}{a}}(\flat+k)\triangle\rho(\flat+k)\ge\rho(\flat+k)\left(\sum_{s=\flat}^{\flat+k-1}\eta(s)\right)^{\frac{1}{a}},$$

and so in view of (2.5)

$$\aleph(\flat)\lambda^{\frac{1}{a}}(\flat)\triangle\rho(\flat)\ge\rho(\flat)\aleph(\flat)\left(\sum_{s=\flat}^{\flat+k-1}\eta(s)\right)^{\frac{1}{a}}\ge\rho(\flat)\omega. \tag{2.6}$$

Now, using the division rule of difference calculus [9], we have

$$\triangle\left(\frac{\rho(\flat)}{\aleph^{\omega}(\flat)}\right)=\frac{\aleph(\flat)\triangle\rho(\flat)-\rho(\flat)\triangle\aleph^{\omega}(\flat)}{\aleph^{\omega}(\flat)\aleph^{\omega}(\flat+1)}. \tag{2.7}$$

By the discrete Mean-value theorem, we have [9]

$$\triangle\left(\aleph^{\omega}(\flat)\right)\le\omega\aleph^{\omega-1}(\flat)\lambda^{-\frac{1}{a}}(\flat),\qquad\omega\in(0,1),$$

and using these in (2.7), we obtain

$$\triangle\left(\frac{\rho(\flat)}{\aleph^{\omega}(\flat)}\right)\ge\frac{1}{\aleph^{\omega+1}(\flat)\lambda^{\frac{1}{a}}(\flat)}\left[\aleph(\flat)\lambda^{\frac{1}{a}}(\flat)\triangle\rho(\flat)-\omega\rho(\flat)\right]\ge 0$$

by (2.6) and so the sequence $\left\{\frac{\rho(\flat)}{\aleph^{\omega}(\flat)}\right\}$ is increasing.

Theorem 2.5 *Let $\sigma(\flat)=\flat+k$, k is a positive integer and (2.5) hold. If*

$$\lim_{\flat\to\infty}\sup\aleph^{\omega}(\flat+k)\sum_{s=\flat}^{\flat+k}\frac{1}{\lambda^{\frac{1}{a}}(s)}\left(\sum_{t=l}^{s-1}\eta(t)\aleph^{\omega a}(t+k)\right)^{\frac{1}{a}}>1 \tag{2.8}$$

then the class S_2 is empty.

Proof. Presume on the contrary that (1.1) has a positive solution $\{\rho(\flat)\}$ belongs to (S_2). Summing up from $\flat$ to $j-1, j\in\mathbb{N}(\flat_0)$ and using monotonic property of $\left\{\frac{\rho(\flat)}{\aleph^{\omega}(\flat)}\right\}$ give

$$\begin{aligned}\lambda(j)(\triangle\rho(j))^{a}&\ge\sum_{s=\flat}^{j-1}\eta(s)\rho^{a}(s+k)\\&\ge\frac{\rho^{a}(\flat+k)}{\aleph^{\omega a}(\flat+k)}\sum_{s=\flat}^{j-1}\eta(s)\aleph^{\omega a}(s+k).\end{aligned}$$

Summing up once more from $\flat$ to $j-1$, we obtain

$$\rho(j)\ge\frac{\rho(\flat+k)}{\aleph^{\omega}(\flat+k)}\sum_{s=\flat}^{j-1}\frac{1}{\lambda^{\frac{1}{a}}(s)}\left(\sum_{t=l}^{s-1}\eta(t)\aleph^{\omega a}(t+k)\right)^{\frac{1}{a}}.$$

Now letting $j=\flat+k$, we obtain

$$\rho(\flat+k)\ge\frac{\rho(\flat+k)}{\aleph^{\omega}(\flat+k)}\sum_{s=\flat}^{\flat+k-1}\frac{1}{\lambda^{\frac{1}{a}}(s)}\left(\sum_{t=l}^{s-1}\eta(t)\aleph^{\omega a}(t+k)\right)^{\frac{1}{a}}.$$

which contradicts to (2.8) and we conclude that the class S_2 is empty.

Remark 2.1 *In Theorems 2.3 and 2.5, we have derived new criteria to eliminate solutions in the class S_0 and S_2. Further the conditions in Theorem 2.3 show that every bounded solution of (1.1) is oscillatory where as conditions in Theorem 2.5 proved that every unbounded solution of equation (1.1) is oscillatory.*

In the following, we proceed new monotony properties. The initial $\triangle-$ derivatives of nonoscillatory solutions will be employed to get another criteria to those given in Theorem 2.3 and Theorem 2.5.

Lemma 2.6 *Let $\sigma(\flat)=\flat-\tau$, τ is a positive integer. Presume that $\{\rho(\flat)\}$ is a positive solution of (1.1) belongs to (S_0). If there exists a positive constant $\gamma\in(0,1)$ such that*

$$\eta(\flat)\left[\aleph(\flat+1)-\aleph(\flat-\tau)\right]^{a}\aleph(\flat)\lambda^{\frac{1}{a}}(\flat)\ge\gamma,\ for\ \flat\ge\flat_0 \tag{2.9}$$

then $\left\{-\aleph^{\gamma}(\flat)\lambda(\flat)(\triangle\rho(\flat))^{a}\right\}$ is decreasing.

Proof. Let $\{\rho(\flat)\}$ be a positive solution of (1.1) belongs to (S_0). Since $-\lambda^{\frac{1}{a}}(\flat)\ \triangle\ \rho(\flat)$ is positive and decreasing, it is possible to view

$$\begin{aligned}\rho(\flat-\tau)&\ge\sum_{s=\flat-\tau}^{\flat}-\triangle\rho(s)\ge-\triangle\rho(\flat)\lambda^{\frac{1}{a}}(\flat)\sum_{s=\flat-\tau}^{\flat}\frac{1}{\lambda^{\frac{1}{a}}(s)}\\&=-\triangle\rho(\flat+1)\lambda^{\frac{1}{a}}(\flat+1)(\aleph(\flat+1)-\aleph(\flat-\tau)).\end{aligned} \tag{2.10}$$

Using (2.10) in (1.1), we get

$$\begin{aligned}&\triangle\left(\lambda(\flat)(\triangle\,(\rho(\flat)))^{a}\right)\\&\qquad\geq \eta(\flat)(-\triangle\,(\rho(\flat+1)))^{a}\lambda(\flat+1)(\aleph(\flat+1)-\aleph(\flat-\tau))^{a}\end{aligned}$$

which in the view of (2.9) yields

$$\triangle\left(\lambda(\flat)(\triangle\,\rho(\flat))^{a}\right)\aleph(\flat)\lambda^{\frac{1}{a}}(\flat)\geq\gamma(-\triangle\,(\rho(\flat+1)))^{a}\lambda(\flat+1). \tag{2.11}$$

Now

$$\begin{aligned}&\triangle\left(-\aleph^{\gamma}(\flat)\lambda(\flat)(\triangle\,(\rho(\flat)))^{a}\right)=-\\&\aleph^{\gamma}(\flat)\triangle\left(\lambda(\flat)(\triangle\,\rho(\flat))^{a}-\lambda(\flat+1)(\triangle\,\rho(\flat+1))^{a}\triangle\left(\aleph^{\gamma}(\flat)\right)\right.\end{aligned} \tag{2.12}$$

From the discrete Mean-value theorem, we have

$$\triangle\,\aleph^{\gamma}(\flat)\leq\gamma\frac{\aleph^{\gamma-1}(\flat)}{\lambda^{\frac{1}{a}}(\flat)} \tag{2.13}$$

and using the (2.13) in (2.12), we have

$$\begin{aligned}&\triangle\left(-\aleph^{\gamma}(\flat)\lambda(\flat)(\triangle\,(\rho(\flat)))^{a}\right)\\&\quad\leq\frac{\aleph^{\gamma-1}(\flat)}{\lambda^{\frac{1}{a}}(\flat)}\Big[(-\gamma\rho(\flat+1))^{a}\lambda(\flat+1)\gamma\\&\quad-\triangle\left[\lambda(\flat)(\triangle\,\rho(\flat))^{a}\right]\lambda^{\frac{1}{a}}(\flat)\aleph(\flat+1)\Big]\leq 0\end{aligned}$$

which shows that the sequence $\{-\aleph^{\gamma}(\flat)\lambda(\flat)(\triangle\,\rho(\flat))^{a}\}$ is decreasing, and the proof is complete.

Theorem 2.7 *Let $\sigma(\flat)=\flat-\tau$, τ is a positive integer and (2.9) hold. If*

$$\lim_{\flat\to\infty}\sup\aleph^{\gamma}(\flat-\tau)\sum_{s=\flat-\tau}^{\flat}\eta(s)\left(\sum_{u=s-\tau}^{\flat-\tau}\aleph^{-\frac{\gamma}{a}}(u)\lambda^{-\frac{1}{a}}(u)\right)^{a}>1 \tag{2.14}$$

then the class S_0 is empty.

Proof. Let $\{\rho(\flat)\}$ be a positive solution of (1.1) belongs to (S_0). Using the fact that $\{-\aleph^{\gamma}(\flat)\rho(\flat)(\triangle\,\rho(\flat))^{a}\}$ is positive and decreasing, we have

$$\begin{aligned}\rho(s-\gamma)&\geq\sum_{u=s-\tau}^{\flat-\tau}-\triangle\,\rho(u)\frac{\aleph^{\frac{\gamma}{a}}(u)\lambda^{\frac{1}{a}}(u)}{\aleph^{\frac{\gamma}{a}}(u)\lambda^{\frac{1}{a}}(u)}\\&\geq-\triangle\,\rho(\flat-\tau)\aleph^{\frac{\gamma}{a}}(\flat-\tau)\lambda^{\frac{1}{a}}(\flat-\tau)\sum_{u=s-\tau}^{\flat-\tau}\frac{1}{\aleph^{\frac{\gamma}{a}}(u)\lambda^{\frac{1}{a}}(u)}.\end{aligned}$$

Therefore

$$\begin{aligned}\rho^{a}(s-\gamma)&\geq(-\triangle\,\rho(\flat-\tau))^{a}\aleph^{\gamma}(\flat-\tau)\lambda(\flat-\tau)\\&\qquad\left(\sum_{u=s-\tau}^{\flat-\tau}\frac{1}{\aleph^{\frac{\gamma}{a}}(u)\lambda^{\frac{1}{a}}(u)}\right)^{\frac{1}{a}}.\end{aligned}$$

Summing up (1.1) from $\flat-\tau$ to $l, l\in\mathbb{N}(\flat_0)$ and then using the last estimate, we have

$$\begin{aligned}&-\lambda(\flat-\tau)(\triangle\,\rho(\flat-\tau))^{a}\\&\quad\geq\sum_{s=\flat-\tau}^{\flat}\eta(s)\rho^{a}(s-\tau)\\&\quad\geq\sum_{s=\flat-\tau}^{\flat}\eta(s)(-\triangle\,\rho(\flat-\tau))^{a}\aleph^{\gamma}(\flat-\tau)\lambda(\flat-\tau)\\&\qquad\qquad\left(\sum_{u=s-\tau}^{\flat-\tau}\aleph^{\frac{-\gamma}{a}}(u)\lambda^{-\frac{1}{a}}(u)\right)^{a}\\&\quad\geq(-\triangle\,\rho(\flat-\tau))^{a}\aleph^{\gamma}(\flat-\tau)\lambda(\flat-\tau)\sum_{s=\flat-\tau}^{\flat}\eta(s)\\&\qquad\qquad\left(\sum_{u=s-\tau}^{\flat-\tau}\aleph^{\frac{-\gamma}{a}}(u)\lambda^{-\frac{1}{a}}(u)\right)^{a}\end{aligned}$$

which contradicts (2.14) and so we conclude that the class S_0 is empty.

Lemma 2.8 *Let $\sigma(\flat)=\flat+k$, k is a positive integer. Presume that $\{\rho(\flat)\}$ is a positive solution of (1.1) belongs to (S_2). If there exists a positive constant $d\in(0,1)$ such that*

$$\eta(\flat)\left[\aleph(\flat+k)-\aleph(\flat)\right]^{a}\aleph(\flat)\lambda^{\frac{1}{a}}(\flat)\geq d,\ for\ \flat\geq\flat_0, \tag{2.15}$$

then $\left\{\frac{\lambda(\flat)(\triangle\rho(\flat))^{a}}{\aleph^{d}(\flat)}\right\}$ is increasing.

Proof. Let $\{\rho(\flat)\}$ be a positive solution of (1.1) belongs to (S_2). Since $\lambda^{\frac{1}{a}}(\flat)\triangle\,\rho(\flat)$ is positive and decreasing, it is easy to see that

$$\begin{aligned}\rho(\flat+k)&\geq\sum_{s=\flat}^{\flat+k-1}\triangle\,\rho(s)\geq\triangle\,\rho(\flat)\lambda^{\frac{1}{a}}(\flat)\sum_{s=\flat}^{\flat+k-1}\frac{1}{\lambda^{\frac{1}{a}}(s)}\\&=\triangle\,\rho(\flat)\lambda^{\frac{1}{a}}(\flat)(\aleph(\flat+k)-\aleph(\flat)).\end{aligned} \tag{2.16}$$

Substituting (2.16) into (1.1), we get

$$\triangle\left(\lambda(\flat)(\triangle\,\rho(\flat))^{a}\right)\geq\eta(\flat)(\triangle\,\rho(\flat))^{a}\lambda(\flat)(\aleph(\flat+k)-\aleph(\flat))^{a}$$

In view of (2.15), we have

$$\triangle\left(\lambda(\flat)(\triangle\,\rho(\flat))^{a}\right)\aleph(\flat)\lambda^{\frac{1}{a}}(\flat)\geq d(\triangle\,\rho(\flat))^{a}\lambda(\flat). \tag{2.17}$$

Thus,

$$\begin{aligned}&\triangle\left(\frac{\lambda(\flat)(\triangle\,\rho(\flat))^{a}}{\aleph^{d}(\flat)}\right)\\&\quad=\frac{\aleph^{d}(\flat)\triangle\left(\lambda(\flat)(\triangle\,\rho(\flat))^{a}-\lambda(\flat)(\triangle\,\rho(\flat))^{a}\triangle\left(\aleph^{d}(\flat)\right)\right.}{\aleph^{d}(\flat)\aleph^{d}(\flat+1)}\end{aligned}$$

Now by discrete Mean-value theorem

$$\triangle\left(\aleph^{d}(\flat)\right)\leq d\aleph^{d-1}(\flat)\frac{1}{\lambda^{\frac{1}{a}}(\flat)}$$

and using this, we have

$$\begin{aligned}&\triangle\left(\frac{\lambda(\flat)(\triangle\,\rho(\flat))^{a})}{\aleph^{d}(\flat)}\right)\geq\frac{1}{\aleph^{d+1}(\flat)\lambda^{\frac{1}{a}}(\flat)}\\&\quad\left[\aleph(\flat)\lambda^{\frac{1}{a}}(\flat)\triangle\left(\lambda(\flat)(\triangle\,\rho(\flat))\right)^{a}-d\lambda(\flat)(\triangle\,\rho(\flat))^{a}\right]\geq 0.\end{aligned}$$

by (2.17). Hence we conclude that $\left\{\frac{\lambda(\flat)(\triangle\rho(\flat))^{a}}{\aleph^{d}(\flat)}\right\}$ is increasing.

Theorem 2.9 *Let $\sigma(\flat) = \flat + k$, k is a positive integer and (2.15) hold. If*

$$\lim_{\flat\to\infty} \sup \frac{1}{\aleph^d(\flat+k)} \sum_{s=\flat}^{\flat+k-1} \eta(s) \left(\sum_{u=\flat+k}^{s+k-1} \frac{\aleph^{\frac{d}{a}}(u)}{\lambda^{\frac{1}{a}}(u)} \right)^a > 1 \tag{2.18}$$

then the class S_2 is empty.

Proof. Let $\{\rho(\flat)\}$ be a positive solution of (1.1) belongs to (S_2). In view of $\left\{ \frac{\lambda(\flat)(\triangle \rho(\flat))^a}{\aleph^d(\flat)} \right\}$ is increasing, we have

$$\begin{aligned} \rho(s+k) &\geq \sum_{u=\flat+k}^{s+k-1} \triangle\, \rho(u) \frac{\aleph^{\frac{d}{a}}(u)\lambda^{\frac{1}{a}}(u)}{\aleph^{\frac{d}{a}}(u)\lambda^{\frac{1}{a}}(u)} \\ &\geq \frac{\triangle\, \rho(\flat+k)\lambda^{\frac{1}{a}}(\flat+k)}{\aleph^{\frac{d}{a}}(\flat+k)} \sum_{u=\flat+k}^{s+k-1} \frac{\aleph^{\frac{d}{a}}(u)}{\lambda^{\frac{1}{a}}(u)}. \end{aligned}$$

Summing up (1.1) from $\flat$ to $\flat + k - 1$ and using the above inequality, we get

$$\begin{aligned} \lambda(\flat+k)(\triangle\, \rho(\flat+k))^a &\geq \sum_{s=\flat}^{\flat+k-1} \eta(s)\rho^a(s+k) \\ &\geq \frac{\lambda(\flat+k)(\triangle\, \rho(\flat+k))^a}{\aleph^d(\flat+k)} \\ &\quad \sum_{s=\flat}^{\flat+k-1} \eta(s) \left(\sum_{u=\flat+k}^{s+k-1} \frac{\aleph^{\frac{d}{a}}(u)}{\lambda^{\frac{1}{a}}(u)} \right)^a \end{aligned}$$

which contradicts (2.18) and so we conclude that the class S_2 is empty.

Remark 2.2 *The methods employed in Theorems 2.7 and 2.9 rely on the properties of the $\triangle$ − derivative of solutions of (1.1). Consequently, the results in these theorems differ from those in Theorems 2.3 and 2.5.*

All solutions to specific functional difference equations involving delayed and advanced arguments will inevitably be oscillatory, as was previously discussed. We demonstrate this for the equation that follows.

$$\triangle\, (\lambda(\flat)(\triangle\, \rho(\flat))^a) = \eta(\flat)\rho^a(\flat - \tau) + \zeta(\flat)\rho^a(\flat + k) \tag{2.19}$$

where $\{\lambda(\flat)\}$, $\{\zeta(\flat)\}$ and a are subject to the conditions (H_1) and (H_2) and moreover:
(H_3) $\{\zeta(\flat)\}$ is a sequence of positive reals;
(H_4) τ and k are positive integers.

Theorem 2.10 *Let conditions (2.1) and (2.4) hold. Presume that there exists a positive constant $\omega_0 \in (0,1)$ such that*

$$\aleph(\flat) \left(\sum_{s=\flat-k}^{\flat-1} \zeta(s) \right)^a > \omega_0, \ \ \flat \geq \flat_0. \tag{2.20}$$

If

$$\lim_{\flat\to\infty} \sup \aleph^{-\omega_0}(\flat+k) \sum_{s=\flat}^{\flat+k} \frac{1}{\lambda^{\frac{1}{a}}(s)} \left(\sum_{t=l}^{s-1} \zeta(t)\aleph^{a\omega_0}(t+k) \right)^{\frac{1}{a}} > 1 \tag{2.21}$$

then equation (2.19) is oscillatory.

Proof. let $\{\rho(\flat)\}$ be an eventually positive solution of (2.19). Then $\{\rho(\flat)\}$ belongs to either S_0 or S_2. Presume first that $\{\rho(\flat)\}$ is from the class S_0. From (2.19), it is easy to find that

$$\triangle\, (\lambda(\flat)(\triangle\, \rho(\flat))^a) \geq \eta(\flat)\rho^a(\flat - \tau)$$

and then proceeding exactly as in the proof of Theorem 2.3, we see that (2.4) guarantees that the class S_0 is empty.
Next, assume that $\{\rho(\flat)\}$ belongs to S_2, then it follows from (2.19) that

$$\triangle\, (\lambda(\flat)(\triangle\, \rho(\flat))^a) \geq \zeta(\flat)\rho^a(\flat + k).$$

Proceeding as in the proof of Theorem 2.9, we see that the class S_2 is empty. This completes the proof.

Theorem 2.11 *Let (2.9) and (2.14) hold. Presume that there exists a positive constant $d_0 \in (0,1)$ such that*

$$\aleph(\flat)\zeta(\flat)(\aleph(\flat+k) - \aleph(\flat))^a\lambda^{\frac{1}{a}}(\flat) \geq d_0, \ \ \flat \geq \flat_0. \tag{2.22}$$

If

$$\lim_{\flat\to\infty} \sup \aleph^{-d_0}(\flat+k) \sum_{s=\flat}^{\flat+k-1} \zeta(s) \left(\sum_{u=\flat+k}^{s+k-1} \frac{\aleph^{\frac{d_0}{a}}(u)}{\lambda^{\frac{1}{a}}(u)} \right)^a > 1 \tag{2.23}$$

then equation (2.19) is oscillatory.

Since the proof follows a similar approach as in Theorem 2.10, we omit the details.

3 Examples

In this segment , three examples are presented to illustrate the main results.

Example 3.1 *Consider the second order functional difference equation*

$$\triangle\, (\flat^{\frac{-1}{3}}(\triangle\, \rho(\flat))^{\frac{1}{3}}) = 2\flat^{\frac{1}{3}}\rho^{\frac{1}{3}}(\flat - 3) + 2\flat^{\frac{5}{6}}\rho^{\frac{1}{3}}(\flat + 2), \tag{3.1}$$

where $\flat \geq 3$. Here $\lambda(\flat) = \flat^{\frac{-1}{3}}$, $\eta(\flat) = 2\flat^{\frac{1}{3}}$, $\zeta(\flat) = 2\flat^{\frac{5}{6}}$, $\tau = 3$, $k = 2$ and $a = \frac{1}{3}$. Since $\aleph(\flat) = \flat^2$, by choosing $\delta = \frac{1}{2}$, the condition (2.1) is clearly satisfied. Further the condition (2.4) becomes

$$\begin{aligned} &\lim_{\flat\to\infty} \sup(\flat - 3) \sum_{s=\flat-3}^{\flat} \frac{1}{s} \left(\sum_{t=s}^{\flat} \frac{2t^{\frac{1}{3}}}{(t-3)^{\frac{1}{3}}} \right)^3 \\ &\simeq \lim_{\flat\to\infty} \sup \flat \sum_{s=\flat-3}^{\flat} \frac{8}{s}(\flat - s)^3 \geq 8 > 1 \end{aligned}$$

and so condition (2.4) holds. Next choosing $\omega_0 = \frac{1}{2}$, we see that condition (2.20) holds. The condition (2.21) becomes

$$\begin{aligned} &\lim_{\flat\to\infty} \sup \frac{1}{(\flat - 3)} \sum_{s=\flat}^{\flat+2} \frac{1}{s} \left(\sum_{t=\flat}^{s-1} 2t^{\frac{5}{3}}(t+2)^{\frac{1}{6}} \right)^3 \\ &\simeq \lim_{\flat\to\infty} \sup \frac{1}{\flat} \sum_{s=\flat}^{\flat+2} \frac{1}{s} \left(\sum_{t=\flat}^{s-1} 2t \right)^3 \\ &\simeq \lim_{\flat\to\infty} \sup \frac{1}{\flat} \sum_{s=\flat}^{\flat+2} \frac{1}{s} \left(s^2 - \flat^2 \right)^3 \\ &\geq \lim_{\flat\to\infty} \sup \frac{1}{\flat^2} (2\flat + 1)^3 = \infty > 1 \end{aligned}$$

and so condition (2.21) holds. Therefore by Theorem 2.10, the equation (3.1) is oscillatory.

Example 3.2 *Consider the second-order functional difference equation*

$$\triangle^2\ \rho(\flat) = \flat\rho(\flat-2) + \flat^2\rho(\flat+1),\ \ \flat \geq 3. \tag{3.2}$$

Here $\lambda(\flat) = \flat$, $\eta(\flat) = \flat, \zeta(\flat) = \flat^2$, $\tau = 2$, $k = 1$ *and* $a = 1$. *Simple calculation shows that* $\aleph(\flat) = \flat$, *and by choosing* $\delta = \frac{1}{2}$, *the condition (2.1) holds. The condition (2.4) becomes*

$$\begin{aligned}&\lim_{\flat\to\infty} \sup(\flat-2)^{\frac{1}{2}} \sum_{s=\flat-2}^{\flat}\left(\sum_{t=s}^{\flat}\frac{t}{(t-2)^{\frac{1}{2}}}\right)\\ &\simeq \lim_{\flat\to\infty}\sup \flat^{\frac{1}{2}} \sum_{s=\flat-2}^{\flat}\left(\sum_{t=s}^{\flat} t^{\frac{1}{2}}\right) = \infty > 1\end{aligned}$$

and so condition (2.4) holds.
Again choosing $\omega_0 = \frac{1}{2}$, *the condition (2.20) clearly holds. The condition (2.21) becomes*

$$\begin{aligned}&\lim_{\flat\to\infty}\sup\frac{1}{(\flat+1)^{\frac{1}{2}}}\sum_{s=\flat}^{\flat+1}\left(\sum_{t=\flat}^{s-1} t^2(t+1)^{\frac{1}{2}}\right)\\ &\qquad\simeq \lim_{\flat\to\infty}\sup\frac{1}{\flat^{\frac{1}{2}}}\flat^2(\flat+1)^{\frac{1}{2}} = \infty > 1\end{aligned}$$

and so condition (2.21) holds. Therefore by Theorem 2.10, the equation (3.2) is oscillatory.

Example 3.3 *Consider the second-order functional difference equation*

$$\triangle\left((\triangle\ \rho(\flat))^3\right) = \frac{2\flat}{\flat+1}\rho^3(\flat-2) + \frac{3\flat}{\flat+1}\rho^3(\flat+1),\ \ \flat \geq 2. \tag{3.3}$$

Here $\lambda(\flat) = 1$, $\eta(\flat) = \frac{2\flat}{\flat+1}$, $\zeta(\flat) = \frac{3\flat}{\flat+1}$, $\tau = 2$, $k = 1$ *and* $a = 3$. *So* $\aleph(\flat) = \flat$, *the condition (2.9) becomes*

$$\frac{2\flat}{\flat+1}(8)(\flat+1) = 16\flat > \frac{1}{2},$$

that is, we can choose $\gamma = \frac{1}{2}$, *the condition (2.9) to hold. The condition (2.14) becomes*

$$\begin{aligned}&\lim_{\flat\to\infty}\sup(\flat-1)^{\frac{1}{2}}\sum_{s=\flat-1}^{\flat}\frac{2s}{s+1}\left(\sum_{u=s-1}^{\flat-1}\frac{1}{u^{\frac{1}{6}}}\right)^3\\ &\qquad > \lim_{\flat\to\infty}\sup(\flat-1)^{\frac{1}{2}}\sum_{s=\flat-1}^{\flat}\frac{2s}{s+1}\frac{1}{(\flat-1)^{\frac{1}{2}}}\\ &\qquad > \lim_{\flat\to\infty}\left(\frac{2(\flat-1)}{\flat} + 2\frac{\flat}{(\flat+1)}\right) = 4 > 1\end{aligned}$$

so condition (2.14) holds, The condition (2.22) becomes

$$3\frac{\flat^2}{\flat+1} > \frac{1}{2}, for\ \ \flat \geq 2,$$

that is, by taking $d_0 = \frac{1}{2}$, *we see that (2.22) holds. The condition (2.23) becomes*

$$\begin{aligned}&\lim_{\flat\to\infty}\sup\frac{1}{(\flat+2)^{\frac{1}{2}}}\sum_{s=\flat}^{\flat+1}3\frac{2s}{s+1}\left(\sum_{u=\flat+2}^{s+1}u^{\frac{1}{6}}\right)^3\\ &\qquad \geq \lim_{\flat\to\infty}\sup\frac{3}{(\flat+2)^{\frac{1}{2}}}\frac{\flat+1}{\flat+2}(\flat+2)^{\frac{1}{2}} = 3 > 1\end{aligned}$$

that is, condition (2.23) holds. Therefore by the Theorem 2.11, the equation (26) is oscillatory.

4 Conclusion

In this work, by using new monotonic properties we are able to present criteria for the elimination of certain class of nonoscillatory solutions of the studied equation (1.1). Using this we establish new criteria for the ocillation of all solutions of the second-order difference equations with delay and advanced arguments. Therefore the results produced in this paper are new and compliment to the existing literature. Further, note that none of the results reported in the literature ([1],[13]-[18]) can be applied to the equations [(3.1)-(3.3)] since these equations contain both delay and advanced terms. It is interesting to extend the results of this paper to higher equations of the form $\Delta^{m-1}\{\lambda(\flat)(\Delta\rho(\flat))\} = \eta(\flat)\rho^a(\sigma(\flat))$ where $m \geq 3$ is a positive integer.

REFERENCES

[1] Agarwal R. P., Wong P. J. Y., "Advanced Topics in Difference Equations," Kluwar Publication, Dordrecht, 1997. `https://doi.org/10.1007/978-94-015-8899-7`

[2] B. Baculikova B., Dzurina J., "New asymptotic results for half-linear differential equations with deviating arguments," Carpathian, Journal of Mathematics, vol.38, pp.327-335, 2022. `https://doi.org/10.37193/CJM.2022.02.05`

[3] B.Baculikova., "Monotonic properties of Kneser solutions of second-order linear differential equations with delayed argument," Opuscula Math. vol.45, no.1, pp.27-38, 2025. `https://doi.org/10.7494/OpMath.2025.45.1.27`

[4] Tang X. H., "Bounded oscillation of second-order delay difference equations of unstable type," Computers and Mathematics with Application, vol.44, pp.1147-1156, 2002. `https://doi.org/10.1016/S0898-1221(02)00222-5`

[5] Lalli B. S., Zhang B. G., "On existence of positive solution and bounded oscillations for neutral difference equations," Journal of Mathematical Analysis Applications, vol.166, pp.272-287, 1992. `https://doi.org/10.1016/0022-247X(92)90342-B`

[6] Thandapani E., Pandian S., Balasubbramanian R.K., "Oscillatory behavior of second order unstable type neutral difference equation," Tamkang Journal of Mathematics, vol.36, pp.57-68, 2005. `https://doi.org/10.5556/j.tkjm.36.2005.136`

[7] Wong P. J. Y., Agarwal R. P., "Oscillation and nonoscillation of half-linear difference equations generated by deviating arguments, Computer and Mathematics Application, vol.36, pp.11-26, 1998.

[8] Zhang Z., Ping B., Dong W., "Oscillatory of unstable type second-order neutral difference equations," Korean Journal Computational and Applied Mathematics, vol.9, pp.87-99, 2002.

[9] Agarwal R. P., "Difference Equations and Inequalities," Marcel Dekker, New York, 2000.

[10] Thandapani E., Arul R., Raja P. S., "Bounded oscillation of second order unstable neutral type difference equations," Journal of Applied Mathematics and Computing, vol.16, pp.79-90, 2004. `https://doi.org/10.1007/bf02936152`

[11] Debnath L., Jiang J. C., “Bounded oscillation criteria for second-order nonlinear delay difference equations of unstable type,” Computers and Mathematics with Applications, vol.56, pp.1797-1807, 2008. `https://doi.org/10.1016/j.camwa.2008.04.006`

[12] Migda, M., “On the discrete version of generalized Kiguradze’s lemma,” Fasciculi Mathematici, vol. 35, pp.77-83, 2005. `https://fasciculi-mathematici.put.poznan.pl/artykuly/35/FM35-Mig.pdf`

[13] Agarwal R. P., Bohner M., Grace S. R., O’Regan D., “Discrete Oscillation Theory,” Hindawi Publications Corporation, New York, 2005. `http://dx.doi.org/10.1155/9789775945198`

[14] Chatzarakis G. E., Dzurina J., Jadlovska I., “New oscillation criteria for second-order half-linear advanced differential equations,” Applied Mathematics and Computation, vol.347, pp.404–416, 2019. `https://doi.org/10.1016/j.amc.2018.10.091`

[15] Chatzarakis G. E., Grace S. R., Jadlovska I., “A sharp oscillation criterion for second-order half-linear advanced differential equations,” Acta Mathematica Hungarica, vol.163, pp.552–562, 2021. `https://doi.org/10.1007/s10474-020-01110-w`

[16] Dosly O., Rehak P., “Half-linear Differential Equations,” North-Holland Mathematics Studies, vol.202, 2005.

[17] Ladde G. S., Lakshmikantham V., Zhang B. G., “Oscillation Theory of Differential Equations with Deviating Arguments,” Marcel Dekker, New York, 1987.

[18] Baculıkova B., “Oscillation of second-order nonlinear noncanonical differential equations with deviating argument,” Applied Mathematics Letters, vol.91, pp.68–75, 2019. `https://doi.org/10.1016/j.aml.2018.11.021`