

Comparison on Contraction Conditions and Conjecture in G-Metric Space

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Abstract Fixed point theory is an important discipline in Mathematics because of its results which are utilized to investigate the existence of solutions for the problems in applied sciences and engineering. The theorems in fixed point theory assured that a self-map on a metric space has an invariant point that can be noticed as the limit of an iterative scheme obtained by repeated elements under the contraction mapping of an initial point in the metric space. Many fixed-point results have been widely generalized throughout the years in various directions by introducing new metric spaces and setting of new contraction mappings. The results in fixed point theory can be noticed in geometry, computational algorithms, economics, fluid dynamics, micro-structures, nonlinear sciences, medical fields and optimization theory. Recently many Mathematicians have developed various contractions in G-metric spaces. In this paper, brief comparison was done on G-contractions developed by Mohanta, Vats et. al. and Phaneendra et. al. With this comparison, we realize that some more terms need to be added and also need to change the restricted range of constant in the condition, to get the generalized contraction condition. Finally, conjecture was made based on the comparison of these existed G-contractions. It is observed that, conjecture gives the generalization of results established by Mohanta, Vats et. al. and Phaneendra et. al.

Keywords G-Metric Space, Fixed Point, G-Convergent, G-Complete Orbit, Conjecture

1. Introduction

Let Y be a nonempty set. A mapping $g: Y \rightarrow Y$ is called a self-map on Y . A point $y \in Y$ is called a fixed point of a mapping g if $gy = y$. For instance, 0 and 2 are fixed points of the self-maps $gy = 4y$ and $gy = \frac{y^2+4}{4}$ for all $y \in \mathbb{R}$ respectively. The mapping $gy = y^2 + 10y + 9$ for all $y \in \mathbb{R}$ has a fixed point, namely -3 which is unique. Also every real number is a fixed point for the mapping $gy = y$ for all $y \in \mathbb{R}$. However, $gy = y + k$ for all $y \in \mathbb{R}$ and $k \neq 0$, has no fixed points.

Not every self-map guarantees a fixed point. A fundamental question arises: Under what conditions can we assure the existence of one or more fixed points? Fixed-point theorems provide the answers, outlining the necessary properties of the self-map and its domain. These theorems are explored in various mathematical structures, such as metric spaces, partial metric spaces, and normed linear spaces. The conditions often involve metrical properties or compactness. Fixed-point theory has become indispensable in applied sciences and engineering. Its core theorems are instrumental in proving the existence and uniqueness of solutions to differential and integral equations, prevalent in physics, engineering, and optimization (See [1-5]). Beyond these, the principles of fixed-point theory find applications in diverse fields, including computer science, medicine, and economics. Many recent advancements have been significant in the field of fixed point theory.

In 2009, Abbas and Rhoades [6] established fixed-point results for non-commuting mappings in G-metric spaces, without requiring continuity. Aydi and Shatanawi [7] further extended this work by deriving fixed-point results for weakly contractive mappings in G-metric spaces. Samet et al. [8,9] provided insights into the connection between G-metric spaces and quasi-metric spaces. More recently, Reddy et al. [10] developed coupled fixed-point theorems for JS-G contractions in G-metric spaces. The authors [11,12] have also made substantial contributions to this field.

In this paper, a brief comparison was done on G-contractions developed by Mohanta, Vats et al. and Phaneendra et al. and conjecture was made based on the comparison on existed fixed point results.

The following basic definitions and preliminaries are used to establish the results in our paper:

Definition 1.1 Let Y be a set which is non empty and $G: Y \times Y \times Y \rightarrow [0, \infty)$ such that

(G1) $G(\tau, v, \varphi) \geq 0$ for all $\tau, v, \varphi \in Y$ with $G(\tau, v, \varphi) = 0$ if $\tau = v = \varphi$,

(G2) $G(\tau, \tau, v) > 0$ for all $\tau, v \in Y$ with $\tau \neq v$,

(G3) $G(\tau, \tau, v) \leq G(\tau, v, \varphi)$ for all $\tau, v \in Y$ with $\varphi \neq v$,

(G4) $G(\tau, v, \varphi) = G(\tau, \varphi, v) = G(v, \tau, \varphi) = G(\varphi, \tau, v) = G(v, \varphi, \tau) = G(\varphi, v, \tau)$ for all $\tau, v, \varphi \in Y$

(G5) $G(\tau, v, \varphi) \leq G(\tau, \varpi, \varpi) + G(\varpi, v, \varphi)$ for all $\tau, v, \varphi, \varpi \in Y$.

Then (Y, G) is called a G-metric space with G-metric 'G'. The notion of G-metric was initiated by Mustafa et al. [13] in 2006.

Proposition 1.1 If (Y, G) is a G-metric space, then we

have

$$G(\tau, v, v) \leq 2G(\tau, \tau, v) \text{ for } \tau, v \in Y. \quad (1.1)$$

Proof. From (G5) in Definition 1.1, we have

$$\begin{aligned} G(\tau, v, v) &\leq G(\tau, v, \tau) + G(\tau, \tau, v) \\ &= G(\tau, \tau, v) + G(\tau, \tau, v) \quad (\text{By (G4)}) \\ &= 2G(\tau, \tau, v) \end{aligned}$$

Proposition 1.2 Suppose that (Y, G) is a G-metric space such that

$$G(\tau, v, v) = 0 \text{ for } \tau, v \in Y, \text{ then } \tau = v. \quad (1.2)$$

Proof. From (G5) in Definition 1.1, we have

$$G(\tau, v, \varphi) = 0 \text{ if } \tau = v = \varphi,$$

which implies that $G(\tau, v, v) = 0$ only when $\tau = v$.

That is, $G(\tau, v, v) = 0$ for $\tau, v \in Y$, then $\tau = v$

Mustafa et al. [14] developed the following notions in G-metric space:

Lemma 1.1 The following are equivalent in (Y, G) :

(a) $\langle \tau_n \rangle_{n=1}^{\infty} \subset Y$ converges to an element $\rho \in Y$

(b) $\lim_{n \rightarrow \infty} G(\tau_n, \tau_n, \rho) = 0$,

(c) $\lim_{n \rightarrow \infty} G(\tau_n, \rho, \rho) = 0$.

Definition 1.2 In (Y, G) , $\langle \tau_n \rangle_{n=1}^{\infty}$ is said to G-Cauchy, if $\lim_{n, m \rightarrow \infty} G(\tau_n, \tau_m, \tau_m) = 0$ and $\langle \tau_n \rangle_{n=1}^{\infty} \subset Y$ is G-convergent with limit $\rho \in Y$, if $\lim_{n, m \rightarrow \infty} G(\rho, \tau_n, \tau_m) = 0$.

Definition 1.3 If every Cauchy sequence in Y is G-convergent in Y , then (Y, G) , is called G-complete.

Definition 1.4 If $\tau_0 \in Y$ and g is a self-map in (Y, G) , then the orbit $O_f(\tau_0)$ at τ_0 is defined the sequence of g-iterates $\langle \tau_0, g\tau_0, \dots, g^n\tau_0, \dots \rangle$.

2. Main Results

A Brief Comparison of G-contractions

In [15], Mohanta proved the following result:

Theorem 2.1. Let be a f be a self-map in a complete G-metric space (Y, G) with

$$\begin{aligned} G(f\tau, f\vartheta, f\varphi) &\leq \alpha G(\tau, \vartheta, \varphi) + \beta G(\tau, f\tau, f\tau) + \gamma G(\vartheta, f\vartheta, f\vartheta) + \delta G(\varphi, f\varphi, f\varphi) \\ &\quad + \varpi \max\{G(\tau, f\vartheta, f\vartheta), G(\vartheta, f\tau, f\tau), G(\vartheta, f\varphi, f\varphi), \\ &\quad G(\varphi, f\vartheta, f\vartheta), G(\varphi, f\tau, f\tau), G(\tau, f\varphi, f\varphi)\} \end{aligned} \quad (2.1)$$

for all $\tau, v, \varphi \in Y$, where $\alpha, \beta, \gamma, \delta, \varpi \geq 0$ not all zero, with $\alpha + \beta + \gamma + \delta + 2\varpi < 1$. Then, f has a unique fixed point ρ .

Writing $\varpi = 0$ in Theorem 2.1, we have

Corollary 2.1. Suppose that f is a self-map on complete G-metric space (Y, G) satisfying

$$G(f\tau, f\vartheta, f\varphi) \leq \alpha G(\tau, \vartheta, \varphi) + \beta G(\tau, f\tau, f\tau) + \gamma G(\vartheta, f\vartheta, f\vartheta) + \delta G(\varphi, f\varphi, f\varphi) \quad (2.2)$$

for all $\tau, v, \varphi \in Y$, where α, β, γ and δ are nonzero and positive real numbers with $\alpha + \beta + \gamma + \delta < 1$. Then f has a fixed point ρ which is unique.

Note that Corollary 2.1. was given by Mustafa et al. [16] as a fixed point result.

In [15], Mohanta also proved the following:

Theorem 2.2. Suppose that f is a mapping on complete G-metric space (Y, G) with

$$G(f\tau, f\vartheta, f\varphi) \leq \kappa \max\{G(\tau, f\nu, f\nu) + G(v, f\tau, f\tau) + G(\varphi, f\varphi, f\varphi), G(v, f\varphi, f\varphi) + G(\varphi, f\nu, f\nu) + G(\tau, f\tau, f\tau), G(\varphi, f\tau, f\tau) + G(\tau, f\varphi, f\varphi) + G(\vartheta, f\vartheta, f\vartheta)\} \tag{2.3}$$

for all $\tau, \nu, \varphi \in Y$, and $0 < \kappa < 1/3$. Then, f has a fixed point ρ which is unique.

Remark 2.1. Since $\alpha + \beta + \gamma \leq 3 \max\{\alpha, \beta, \gamma\}$ for $\alpha \geq 0, \beta \geq 0$ and $\gamma \geq 0$, (2.3) can be written as

$$G(f\tau, f\vartheta, f\varphi) \leq \mu \max\{G(\tau, f\nu, f\nu), G(v, f\tau, f\tau), G(\varphi, f\varphi, f\varphi), G(v, f\varphi, f\varphi), G(\varphi, f\nu, f\nu), G(\tau, f\tau, f\tau), G(\varphi, f\tau, f\tau), G(\tau, f\varphi, f\varphi), G(\vartheta, f\vartheta, f\vartheta)\}$$

for all $\tau, \nu, \varphi \in Y$, where $\mu = 3\kappa$. Now for $0 < \kappa < 1/9$, we see that $0 < \mu < 1/3$. Therefore, under the restricted range of the constant, we have the following generalization of Theorem 2.2:

Theorem 2.3. Let f denote a self-map in a complete G -metric space (Y, G) satisfying

$$G(f\tau, f\vartheta, f\varphi) \leq \mu \max\{G(\tau, f\nu, f\nu), G(v, f\tau, f\tau), G(\varphi, f\varphi, f\varphi), G(v, f\varphi, f\varphi), G(\varphi, f\nu, f\nu), G(\tau, f\tau, f\tau), G(\varphi, f\tau, f\tau), G(\tau, f\varphi, f\varphi), G(\vartheta, f\vartheta, f\vartheta)\} \tag{2.4}$$

Then f has a fixed point ρ which is unique.

In [17], Vats et al. proved the result as follows:

Theorem 2.4. Suppose that f be a self-map in a complete G -metric space (Y, G) with the condition

$$G(f\tau, f\vartheta, f\varphi) \leq \kappa \max\{G(\tau, f\tau, f\tau) + G(\vartheta, f\vartheta, f\vartheta) + G(\varphi, f\varphi, f\varphi), G(\tau, f\vartheta, f\vartheta) + G(\vartheta, f\tau, f\tau) + G(\varphi, f\vartheta, f\vartheta), G(\tau, f\varphi, f\varphi) + G(v, f\varphi, f\varphi) + G(\varphi, f\tau, f\tau)\} \tag{2.5}$$

for all $\tau, \nu, \varphi \in Y$, where $0 < \kappa < 1/4$. Then f has a fixed point ρ which is unique.

Remark 2.2. Again, since $\alpha + \beta + \gamma \leq 3 \max\{\alpha, \beta, \gamma\}$ for $\alpha \geq 0, \beta \geq 0$ and $\gamma \geq 0$, (2.5) can be written as

$$G(f\tau, f\vartheta, f\varphi) \leq \nu \max\{G(\tau, f\tau, f\tau), G(\vartheta, f\vartheta, f\vartheta), G(\varphi, f\varphi, f\varphi), G(\tau, f\vartheta, f\vartheta), G(\vartheta, f\tau, f\tau), G(\vartheta, f\tau, f\tau), G(\tau, f\varphi, f\varphi), G(v, f\varphi, f\varphi), G(\varphi, f\tau, f\tau)\} \tag{2.6}$$

for all $\tau, \nu, \varphi \in Y$, where $\nu = 3\kappa$. For $0 < \kappa < 1/12$, we see that $0 < \nu < 1/4$.

Therefore, we have the following generalization of Theorem 2.4:

Theorem 2.5. If f denotes a self-map on complete G -metric space (Y, G) satisfying (2.6), then f has a fixed point ρ which is unique

In [18], Phaneendra with second author generalized the Theorem 2.3 and Theorem 2.5 is obtained by using wider inequality.

Given $\tau, \nu, \varphi \in Y$, define $S(\tau, \nu, \varphi) = \max T_f(\tau, \nu, \varphi)$, where

$$T_f(\tau, \nu, \varphi) = \left\{ G(f^i \sigma, f^j \mu, f^k \epsilon) : 0 \leq i, j, k \leq 1; \sigma, \mu, \epsilon \in \{\tau, \nu, \varphi\} \right\} \tag{2.7}$$

It may be noted that $T_f(\tau, \nu, \varphi)$ has 36 elements.

Theorem 2.6. If f is a mapping in (Y, G) such that

$$G(f\tau, f\vartheta, f\varphi) \leq \theta S(\tau, \nu, \varphi) \text{ for all } \tau, \nu, \varphi \in Y, \tag{2.8}$$

and $0 < \theta < 1/3$. If Y is G -complete, then f has a fixed point ρ which is unique.

Also, we observed that

Given $\tau, \nu, \varphi \in X$, write

$$B_f(\tau, \nu, \varphi) = \{G(\tau, f\tau, f\tau), G(\vartheta, f\vartheta, f\vartheta), G(\varphi, f\varphi, f\varphi), G(\tau, f\vartheta, f\vartheta), G(v, f\varphi, f\varphi), G(\varphi, f\tau, f\tau), G(\tau, f\varphi, f\varphi), G(\vartheta, f\tau, f\tau), G(\varphi, f\vartheta, f\vartheta), G(\tau, f\nu, f\nu), G(v, f\varphi, f\varphi), G(v, f\varphi, f\varphi), G(\varphi, f\tau, f\tau), G(\tau, \nu, f\varphi), G(v, \varphi, f\tau), G(\varphi, \tau, f\vartheta), G(\tau, \nu, \varphi)\}.$$

Then it is clear that $B_f(\tau, \nu, \varphi) \subset T_f(\tau, \nu, \varphi)$ for all $\tau, \nu, \varphi \in Y$.

Therefore, (2.4) implies (2.8).

Hence, Theorem 2.6 is a generalization of Theorem 2.5.

While, if we write

$$B_f(\tau, \nu, \varphi) = \{G(\tau, f\tau, f\tau), G(\tau, f\vartheta, f\vartheta), G(\tau, f\varphi, f\varphi), G(\vartheta, f\vartheta, f\vartheta), G(\vartheta, f\tau, f\tau), G(v, f\varphi, f\varphi), G(\varphi, f\varphi, f\varphi), G(\varphi, f\tau, f\tau), G(\varphi, f\vartheta, f\vartheta)\}$$

then it follows that $B_f(\tau, \nu, \varphi) \subset T_f(\tau, \nu, \varphi)$ for all $\tau, \nu, \varphi \in Y$. Therefore, (2.6) implies (2.8). Hence, Theorem 2.6 is a generalization of Theorem 2.3.

3. Conjecture

If several non-negative terms are included into the set $T_f(\tau, v, \varphi)$ and the choice of θ is extended, it is natural to impose one or more additional conditions, to obtain a fixed point. In this context, we make the following conjecture:

Conjecture 3 Let (Y, G) be a G -metric space and $f : Y \rightarrow Y$ satisfying the following inequality:

$$G(f\tau, f\vartheta, f\varphi) \leq \mu \max T_f^{(s)}(\tau, v, \varphi) \text{ for all } \tau, v, \varphi \in Y, s \geq 1 \quad (3.1)$$

where

$$T_f^{(s)}(\tau, v, \varphi) = \left\{ G(f^i \sigma, f^j \mu, f^k \epsilon) : 0 \leq i, j, k \leq s; \sigma, \mu, \epsilon \in \{\tau, v, \varphi\} \right\} \quad (3.2)$$

and $0 < \mu < 1$. Given $\tau_0 \in X$, suppose that $O_f(\tau)$ is bounded. If Y is G -complete, then f has a fixed point ρ which is unique.

Remark 3.1. We see that Theorem 2.6 follows from Conjecture 3. It is obvious that by restricting the contraction constant θ in Theorem 2.6 to $[0, 1/3)$, (3.1) reduces to (2.8). Now for each $\varphi_0 \in Y$, $O_f(\varphi_0)$ is bounded under (2.8). Suppose that $O_f(\tau_0)$ is unbounded. Then there is a positive integer n so that

$$G(\varphi_1, \varphi_n, \varphi_n) > \mu \max C_n, \quad (3.3)$$

Where

$$C_n = \{G(\varphi_1, \varphi_r, \varphi_r) : 0 \leq r \leq n-1\}, n \geq 2 \quad (3.4)$$

and

$$\mu = \max \left\{ \frac{4\theta}{1-5\theta}, 6\theta \right\} \quad (3.5)$$

Now from the inequality (2.8) with $\varphi = \varphi_0$, $y = \varphi = \varphi_{n-1}$, we have

$$G(\varphi_1, \varphi_n, \varphi_n) = G(f\varphi_0, f\varphi_{n-1}, f\varphi_{n-1}) \leq \theta \max \{G(\varphi_0, \varphi_0, \varphi_1), G(\varphi_0, \varphi_0, \varphi_{n-1}), G(\varphi_0, \varphi_0, \varphi_n)\},$$

$$\begin{aligned} & G(\varphi_0, \varphi_1, \varphi_1), G(\varphi_0, \varphi_1, \varphi_{n-1}), G(\varphi_0, \varphi_1, \varphi_n), \\ & G(\varphi_0, \varphi_{n-1}, \varphi_{n-1}), G(\varphi_0, \varphi_{n-1}, \varphi_n), G(\varphi_0, \varphi_n, \varphi_n), \\ & G(\varphi_1, \varphi_1, \varphi_{n-1}), G(\varphi_1, \varphi_1, \varphi_n), G(\varphi_1, \varphi_{n-1}, \varphi_{n-1}), \\ & G(\varphi_1, \varphi_{n-1}, \varphi_n), G(\varphi_1, \varphi_n, \varphi_n) \} \\ & = \theta M, \end{aligned} \quad (3.6)$$

where

$$M = \max S(\varphi_0, \varphi_{n-1}, \varphi_{n-1}) \quad (3.7)$$

Now we present different cases:

▪ Case (i): Suppose that $M = G(\varphi_0, \varphi_0, \varphi_1)$. In view of (3.3), (3.4) and (3.5), it follows from (3.6) that

$$G(\varphi_1, \varphi_n, \varphi_n) \leq \theta G(\varphi_0, \varphi_0, \varphi_1) \leq \mu \max C_n < G(\varphi_1, \varphi_n, \varphi_n),$$

which is a contradiction.

▪ Case (ii): Suppose that $M = G(\varphi_0, \varphi_1, \varphi_1)$. In view of (1.1), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$G(\varphi_1, \varphi_n, \varphi_n) \leq \theta G(\varphi_0, \varphi_1, \varphi_1) \leq \theta [2G(\varphi_1, \varphi_0, \varphi_0)] \leq \max C_n < G(\varphi_1, \varphi_n, \varphi_n),$$

which is a contradiction.

▪ Case (iii): Suppose that $M = G(\varphi_0, \varphi_0, \varphi_n)$. In view of (G5), (1.1), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} & G(\varphi_1, \varphi_n, \varphi_n) \leq \theta G(\varphi_0, \varphi_0, \varphi_n) \\ & \leq \theta [2G(\varphi_n, \varphi_n, \varphi_0)] \\ & \leq 2\theta [G(\varphi_n, \varphi_1, \varphi_1) + G(\varphi_1, \varphi_n, \varphi_0)] \\ & \leq 2\theta [2G(\varphi_1, \varphi_n, \varphi_n) + G(\varphi_1, \varphi_n, \varphi_0)] \\ & \leq 4\theta G(\varphi_1, \varphi_n, \varphi_n) + 2\theta [G(\varphi_0, \varphi_1, \varphi_1) + G(\varphi_1, \varphi_1, \varphi_n)] \\ & = 8\theta G(\varphi_1, \varphi_n, \varphi_n) + 4\theta G(\varphi_1, \varphi_0, \varphi_0) \\ & \leq \left(\frac{4\theta}{1-8\theta} \right) G(\varphi_1, \varphi_0, \varphi_0) \\ & \leq \mu \max C_n < G(\varphi_1, \varphi_n, \varphi_n), \end{aligned}$$

which is a contradiction.

▪ Case (iv): Suppose that $M = G(\varphi_0, \varphi_n, \varphi_n)$. In view of (G5), (1.1), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} & G(\varphi_1, \varphi_n, \varphi_n) \leq \theta G(\varphi_0, \varphi_n, \varphi_n) \\ & \leq \theta [G(\varphi_0, \varphi_1, \varphi_1) + G(\varphi_1, \varphi_n, \varphi_n)] \\ & \leq 2\theta G(\varphi_1, \varphi_0, \varphi_0) + \omega G(\varphi_1, \varphi_n, \varphi_n) \\ & \leq \left(\frac{2\theta}{1-\theta} \right) G(\varphi_1, \varphi_0, \varphi_0) \\ & \leq \mu \max C_n < G(\varphi_1, \varphi_n, \varphi_n), \end{aligned}$$

which is a contradiction.

▪ Case (v): Suppose that $M = G(\varphi_0, \varphi_1, \varphi_n)$. In view of (G5), (1.1), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} & G(\varphi_1, \varphi_n, \varphi_n) \leq \theta G(\varphi_0, \varphi_1, \varphi_n) \\ & \leq \theta [G(\varphi_0, \varphi_1, \varphi_1) + G(\varphi_1, \varphi_1, \varphi_n)] \\ & \leq \theta [2G(\varphi_1, \varphi_0, \varphi_0) + 2G(\varphi_1, \varphi_n, \varphi_n)] \\ & \leq \left(\frac{2\theta}{1-2\theta} \right) G(\varphi_1, \varphi_0, \varphi_0) \\ & \leq \mu \max C_n < G(\varphi_1, \varphi_n, \varphi_n), \end{aligned}$$

which is a contradiction.

▪ Case (vi): Suppose that $M = G(\varphi_1, \varphi_n, \varphi_n)$. In view of (3.3), (3.4) and (3.5), it follows from (3.6) that

$$G(\varphi_1, \varphi_n, \varphi_n) \leq \theta G(\varphi_1, \varphi_n, \varphi_n) \leq \mu \max C_n < G(\varphi_1, \varphi_n, \varphi_n),$$

which is a contradiction.

- Case (vii): Suppose that $M = G(\phi_1, \phi_1, \phi_n)$. In view of (1.1), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_1, \phi_1, \phi_n) \\ &\leq 2\theta G(\phi_1, \phi_n, \phi_n) \\ &\leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n), \end{aligned}$$

which is a contradiction.

- Case (viii): Suppose that $M = G(\phi_0, \phi_1, \phi_{n-1})$. In view of (G5), (1.1), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_0, \phi_1, \phi_{n-1}) \\ &\leq \theta [G(\phi_0, \phi_1, \phi_1) + G(\phi_1, \phi_1, \phi_{n-1})] \\ &\leq \theta [2G(\phi_1, \phi_0, \phi_0) + 2G(\phi_1, \phi_{n-1}, \phi_{n-1})] \\ &\leq 4\theta \max \{G(\phi_1, \phi_r, \phi_r) : 0 \leq r \leq n - 1\} \\ &\leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n), \end{aligned}$$

which is a contradiction.

- Case (ix): Suppose that $M = G(\phi_{n-1}, \phi_1, \phi_1)$.

In view of (G5), (1.1), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_{n-1}, \phi_1, \phi_1) \\ &\leq \theta [G(\phi_{n-1}, \phi_n, \phi_n) + G(\phi_n, \phi_1, \phi_1)] \\ &\leq \theta [G(\phi_{n-1}, \phi_n, \phi_n) + 2G(\phi_1, \phi_n, \phi_n)] \\ &\leq \theta [G(\phi_{n-1}, \phi_1, \phi_1) + G(\phi_1, \phi_n, \phi_n) \\ &\quad + 2\theta G(\phi_1, \phi_n, \phi_n)] \\ &\leq \left(\frac{2\theta}{1-3\theta}\right) G(\phi_1, \phi_{n-1}, \phi_{n-1}) \\ &\leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n), \end{aligned}$$

which is a contradiction.

- Case (x): Suppose that $M = G(\phi_1, \phi_{n-1}, \phi_{n-1})$. In view of (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_1, \phi_{n-1}, \phi_{n-1}) \leq \\ &\mu \max C_n < G(\phi_1, \phi_n, \phi_n), \end{aligned}$$

which is a contradiction.

- Case (xi): Suppose that $M = G(\phi_1, \phi_{n-1}, \phi_n)$. In view of (G5), (1.3), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_1, \phi_{n-1}, \phi_n) \\ &\leq \theta [G(\phi_1, \phi_n, \phi_n) + G(\phi_n, \phi_{n-1}, \phi_n)] \\ &\leq \theta [G(\phi_1, \phi_n, \phi_n) + 2G(\phi_n, \phi_{n-1}, \phi_{n-1})] \\ &\leq \theta G(\phi_1, \phi_n, \phi_n) + 2[G(\phi_n, \phi_1, \phi_1) \\ &\quad + G(\phi_1, \phi_{n-1}, \phi_{n-1})] \end{aligned}$$

$$\begin{aligned} &\leq \theta G(\phi_1, \phi_n, \phi_n) + 4\theta G(\phi_1, \phi_n, \phi_n) \\ &\quad + 2\theta G(\phi_1, \phi_{n-1}, \phi_{n-1}) \end{aligned}$$

$$\leq \left(\frac{2\theta}{1-5\theta}\right) G(\phi_1, \phi_{n-1}, \phi_{n-1})$$

$$\leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n),$$

which is a contradiction.

- Case (xii): Suppose that $M = G(\phi_0, \phi_{n-1}, \phi_n)$. In view of (G5), (1.1), (3.3), (3.4), (3.5) and case (xi), it follows from (3.6) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_0, \phi_{n-1}, \phi_n) \\ &\leq \theta [G(\phi_0, \phi_1, \phi_1) + G(\phi_1, \phi_{n-1}, \phi_n)] \\ &\leq \theta [2G(\phi_1, \phi_0, \phi_0) + G(\phi_1, \phi_{n-1}, \phi_n)] \\ &\leq 2\theta G(\phi_1, \phi_0, \phi_0) + 5\theta G(\phi_1, \phi_n, \phi_n) \\ &\quad + 2\theta G(\phi_1, \phi_{n-1}, \phi_{n-1}) \\ &\leq 4\theta \max C_n \\ &\leq \left(\frac{4\theta}{1-5\theta}\right) \max C_n \leq \mu \max C_n \\ &\quad < G(\phi_1, \phi_n, \phi_n), \end{aligned}$$

which is a contradiction.

- Case (xiii): Suppose that $M = G(\phi_0, \phi_0, \phi_{n-1})$. In view of (G5), (1.1), (3.3), (3.4), (3.5) and case (xii), it follows from (3.6) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_0, \phi_0, \phi_{n-1}) \\ &\leq \theta [G(\phi_0, \phi_1, \phi_1) + G(\phi_1, \phi_0, \phi_{n-1})] \\ &\leq 2\theta G(\phi_1, \phi_0, \phi_0) + \theta G(\phi_1, \phi_0, \phi_{n-1}) \\ &\leq 2\theta G(\phi_1, \phi_0, \phi_0) + \theta [2G(\phi_1, \phi_0, \phi_0) \\ &\quad + 2G(\phi_1, \phi_{n-1}, \phi_{n-1})] \\ &= 4\theta G(\phi_1, \phi_0, \phi_0) + 2\theta G(\phi_1, \phi_{n-1}, \phi_{n-1}) \\ &\leq 6\theta \max C_n \leq \mu \max C_n \\ &\quad < G(\phi_1, \phi_n, \phi_n), \end{aligned}$$

which is a contradiction.

- Case (xiv): Suppose that $M = G(\phi_0, \phi_{n-1}, \phi_{n-1})$. In view of (G5), (1.1), (3.3), (3.4) and (3.5), it follows from (3.6) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_0, \phi_{n-1}, \phi_{n-1}) \\ &\leq \theta [G(\phi_0, \phi_1, \phi_1) + G(\phi_1, \phi_{n-1}, \phi_{n-1})] \\ &\leq \theta [2G(\phi_1, \phi_0, \phi_0) + G(\phi_1, \phi_{n-1}, \phi_{n-1})] \\ &\leq 3\theta \max \{G(\phi_1, \phi_r, \phi_r) : 0 \leq r \leq n - 1\} \\ &\leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n), \end{aligned}$$

which is a contradiction.

All these contradictions prove that $O_f(\phi_0)$ is bounded and $\sup\{O_f(\phi_0)\} < \infty$. Therefore the existence of a fixed point is ensured by Conjecture 3.

Remark 3.2. Suppose that f satisfies (2.1). Then we see that

$$\begin{aligned}
& G(f\tau, fv, f\varphi) \leq aG(\tau, v, \varphi) + bG(\tau, f\tau, f\tau) \\
& \quad + cG(v, fv, fv) \\
& \quad + dG(\varphi, f\varphi, f\varphi) + e \max\{G(\tau, fv, fv), \\
& \quad G(v, f\tau, f\tau), G(v, f\varphi, f\varphi), G(\varphi, fv, fv), \\
& \quad G(\varphi, f\tau, f\tau), G(\tau, f\varphi, f\varphi)\} \\
& \leq k \max\{G(\tau, v, \varphi), G(\tau, f\tau, f\tau), G(v, fv, fv), \\
& \quad G(\varphi, f\varphi, f\varphi), \\
& \quad \frac{G(\tau, fv, fv)}{2}, \frac{G(v, f\tau, f\tau)}{2}, \frac{G(v, f\varphi, f\varphi)}{2}, \\
& \quad \frac{G(\varphi, fv, fv)}{2}, \frac{G(\varphi, f\tau, f\tau)}{2}, \frac{G(\tau, f\varphi, f\varphi)}{2}\} \\
& \leq k \max\{G(\tau, v, \varphi), G(\tau, f\tau, f\tau), G(v, fv, fv), \\
& \quad G(\varphi, f\varphi, f\varphi), G(\tau, fv, fv), G(v, f\tau, f\tau), \\
& \quad G(v, f\varphi, f\varphi), G(\varphi, fv, fv), G(\varphi, f\tau, f\tau), \\
& \quad G(\tau, f\varphi, f\varphi)\} \\
& \leq k \max T_f^{(2)} \text{ for all } \tau, v, \varphi \in Y \quad (3.8)
\end{aligned}$$

Thus (3.8) is stronger than (3.1).

Let $\tau_0 \in Y$ be arbitrary. We prove that

$$O_f(\tau_0) = \{\tau_0\} \cup \langle \tau_n \rangle_{n=1}^\infty$$

is bounded, where $\tau_n = f \tau_{n-1}$ for all $n \geq 1$ under (2.1). If possible, suppose that $O_f(\tau_0)$ is unbounded. Then there exists a positive integer n such that

$$G(\tau_1, \tau_n, \tau_n) > \mu \max\{G(\tau_1, \tau_r, \tau_r) : 0 \leq r \leq n-1\}, \quad (3.9)$$

Where

$$\mu = \max\left\{\frac{2a + 2b + 2e}{1 - c - d - 2e}, \frac{a + 2c + 2d + 2e}{1 - c - d - 2e}\right\} \quad (3.10)$$

Writing $\tau = \tau_0$, $v = \varphi = \tau_{n-1}$ in (3.1) and then using (G5) and (2.3), we get

$$\begin{aligned}
& G(\tau_1, \tau_n, \tau_n) = G(f\tau_0, f\tau_{n-1}, f\tau_{n-1}) \\
& \leq aG(\tau_0, \tau_{n-1}, \tau_{n-1}) + bG(\tau_0, f\tau_0, f\tau_0) \\
& + cG(\tau_{n-1}, f\tau_{n-1}, f\tau_{n-1}) + dG(\tau_{n-1}, f\tau_{n-1}, f\tau_{n-1}) \\
& + e \max\{G(\tau_0, f\tau_{n-1}, f\tau_{n-1}), G(\tau_{n-1}, f\tau_0, f\tau_0), \\
& \quad G(\tau_{n-1}, f\tau_{n-1}, f\tau_{n-1}), G(\tau_{n-1}, f\tau_{n-1}, f\tau_{n-1}), \\
& \quad G(\tau_{n-1}, f\tau_0, f\tau_0), G(\tau_0, f\tau_{n-1}, f\tau_{n-1})\} \\
& = (2a + 2b)G(\tau_1, \tau_0, \tau_0) + (a + 2c \\
& \quad + 2d)G(\tau_{n-1}, \tau_1, \tau_1) \\
& + (c + d)G(\tau_1, \tau_n, \tau_n) + e[2G(\tau_1, \tau_0, \tau_0) \\
& \quad + G(\tau_1, \tau_n, \tau_n) \\
& \quad + 2G(\tau_1, \tau_{n-1}, \tau_{n-1}) + G(\tau_1, \tau_n, \tau_n)] \\
& \leq (2a + 2b + 2e)G(\tau_1, \tau_0, \tau_0) + (a + 2c \\
& \quad + 2d + 2e)G(\tau_1, \tau_{n-1}, \tau_{n-1}) \\
& \quad + (c + d + 2e)G(\tau_1, \tau_n, \tau_n)
\end{aligned}$$

or

$$\begin{aligned}
& (1 - c - d - 2e)G(\tau_1, \tau_n, \tau_n) \\
& \leq (2a + 2b + 2e)G(\tau_1, \tau_0, \tau_0) \\
& \quad + (a + 2c + 2d + 2e)G(\tau_1, \tau_{n-1}, \tau_{n-1})
\end{aligned}$$

so that

$$\begin{aligned}
& \leq \left(\frac{2a + 2b + 2e}{1 - c - d - 2e}\right)G(\tau_1, \tau_0, \tau_0) \\
& \quad + \left(\frac{a + 2c + 2d + 2e}{1 - c - d - 2e}\right)G(\tau_1, \tau_{n-1}, \tau_{n-1}),
\end{aligned}$$

which gives contradiction in view of (3.9) and (3.10) that

$$\begin{aligned}
& G(G(\tau_1, \tau_n, \tau_n) \leq \mu \max\{G(\tau_1, \tau_r, \tau_r) : 0 \\
& \leq r \leq n - 1\} \\
& < G(\tau_1, \tau_n, \tau_n).
\end{aligned}$$

Hence $O_f(\tau_0)$ is bounded. Therefore the existence of a fixed point is ensured by Conjecture 3.

4. Conclusions

We studied and compared the results in G-metric space introduced by Mohanta, Vats et al., and Phaneendra et al. Based on this comparison, we made the conjecture which is the generalization of many existed results in G-metric space.

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