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Left Noetherian Ternary Semigroup and Its Direct Product

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Abstract A ternary semigroup $\mathcal{T}$ is termed strongly left Noetherian if every left congruence on $\mathcal{T}$ can be generated by a finite set. This work examines the fundamental characteristics of left Noetherian ternary semigroups, considering the relationships between the semigroups and their substructures about the left Noetherian condition. Additionally, the study investigates whether this property is retained in the direct product of such semigroups. This work offers a comprehensive characterization of strongly left Noetherian ternary semigroups. It is established that the homomorphic image of a Noetherian ternary semigroup retains the Noetherian property, and an alternative approach is introduced for demonstrating that a ternary semigroup is strongly left Noetherian. The relationship between ternary semigroups and their subsemigroups is explored, showing that a ternary semigroup is strongly left Noetherian if at least one of its subsemigroups is left Noetherian. A necessary characterization for inverse ternary semigroups is also presented. In addition, the strongly left Noetherian property is the direct product of ternary semigroups, identifying conditions that preserve this property. A necessary and sufficient condition is established for the direct product of an infinite ternary semigroup and a ternary monoid to be strongly left Noetherian, along with findings on the left Noetherian property in direct products of various ternary semigroups.

Keywords Strongly Left Noetherian Ternary Semigroup, Left Congruence, Finitely Produced, Direct Product

1 Introduction

The journey into the realm of Noetherian semigroups was initiated by Hotzel [1], whose seminal work shed light on the intricate properties of weakly periodic semigroups, a class encompassing regular semigroups. Notably, Hotzel's findings illuminated the finite nature of right ideals within such semigroups, paving the path to further investigations into their generative properties.

Whether all left Noetherian semigroups are finitely generated is still a fascinating mystery that has drawn the attention of many scholars, including Kozhukhov [2]. Kozhukhov's investigation examines the subtle algebraic structures of 0-simple, Commutative, and inverse semigroups with ACC or DCC, providing a deeper grasp of their structural nuances.

Right Noetherian semigroup was further studied by Craig Miller and Nik Ruskuc [3] in 2019. He found some fundamental properties and some characterization theorems for the right Noetherian semigroup. He proved the behavior of the right Noetherian property in various semigroup constructions.

M.L. Santiago and Sribala [4, 5] established the foundation for the theory of Ternary Semigroups, exploring its complexities and illustrating its mathematical framework. However, the roots of this theory extend back to 1932 when Lehmer[6] introduced the concept of ternary Semigroups.

In this study, we have defined the concept of a strongly left Noetherian ternary semigroup. A ternary semigroup is considered strongly left Noetherian if every left congruence on the semigroup is finitely generated.

Section 2 contains some fundamental definition required for this work.

In Section 3, we provided a characterization of strongly left Noetherian ternary semigroups. We established that the homomorphic image of a Noetherian ternary semigroup is also Noetherian and proved a result offering an alternative method

to demonstrate that a ternary semigroup is stronly left Noetherian.
In Section 4, we explored the relationship between a ternary semigroup and its subsemigroups. We have demonstrated a theorem which establishes that a ternary semigroup is strongly left Noetherian if at least one of its subsemigroups is left Noetherian. Additionally, we provided a necessary characterization for an inverse ternary semigroup.
Section 5 addressed the strongly left Noetherian property of the direct product of ternary semigroups. We discussed several immediate implications for the direct product of the strongly left Noetherian semigroups. We defined a condition that preserves the left Noetherian property in the direct product of ternary semigroups and established a necessary and sufficient condition for the direct product of an infinite ternary semigroup and a ternary monoid to be strongly left Noetherian. In addition, we provided findings on the left Noetherian property of the direct product of various kinds of ternary semigroups.

2 Preliminaries

Definition 2.1. *[4] A set $\mathcal{T} \neq \phi$ and also, a mapping $\mathcal{T}\times\mathcal{T}\times\mathcal{T}$ to $\mathcal{T}$ which satisfies associativity. That is,*

$$(hij)kl = h(ijk)l = hi(jkl)\ for\ all\ h,i,j,k,l \in \mathcal{T}.$$

is called ternary semigroup.

Example 2.1. *i. $\mathcal{T}_1 = \{i, -i\}$ under multiplication.*

ii. $\mathcal{T}_2 = \mathbb{Z}^-$ under multiplication.

Definition 2.2. *[7] A member $o \in \mathcal{T}$ is called a zero element if $[gho] = [ogh] = [goh] = o$ for every $g, h \in \mathcal{T}$. If T has no zero element, a zero element can be adjoined by putting $[ghi] = 0$ if any of g, h, i is a zero. We denote this fact by $\mathcal{T}^0 = \mathcal{T} \cup \{0\}$.*

Definition 2.3. *Let $\mathcal{T}$ be a ternary semigroup. Then $\mathcal{T}^1$ is either a ternary semigroup with the neutral element or adjoining a neutral element to the $\mathcal{T}$ if and only if it is derived from a binary semigroup [9]. An element u is said to be neutral element of $\mathcal{T}$ if $[auu] = [uau] = [uua] = a$ for all $a \in \mathcal{T}$.*

Definition 2.4. *[8] A ternary semigroup $\mathcal{T}$ is said to have a left ideal $\mathcal{I}$ if $[\mathcal{T}\mathcal{T}\mathcal{I}] \subseteq \mathcal{I}$.*

Definition 2.5. *[8] An equivalence relation $\mathcal{R}$ on a ternary semigroup $\mathcal{T}$ is said to be left congruence(LC) on $\mathcal{T}$ if it is left compatible on $\mathcal{T}$. That is,*
If $(x, y) \in \mathcal{R}$, then there exist $s, t \in \mathcal{T}$, $(stx, sty) \in \mathcal{R}$.

Definition 2.6. *[7] A ternary monoid is the ternary semigroup that contains a neutral element.*

3 Left Noetherian Ternary Semigroups

Definition 3.1. *A Ternary Semigroup $\mathcal{T}$ is said to be strongly left Noetherian ternary semigroup(SLN) if every LC on $\mathcal{T}$ is finitely generated(finitely produced) or $\mathcal{T}$ satisfies ascending chain condition on its LCs.*

Example 3.1. *Finite ternary semigroups are SLN.*

Next, we start with equivalent characterization about the SLN ternary semigroup.

Proposition 3.1. *Consider a ternary semigroup $\mathcal{T}$. Then, the below statements are equivalent.*

i. $\mathcal{T}$ is SLN.

ii. $\mathcal{T}$ satisfies on it's LCs. That is, every infinite ascending chain $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq ....$ of LCs on $\mathcal{T}$ eventually terminates.

iii. Any non-empty collection of LCs contains a minimal element.

Proof. $i \implies ii$
Let $\mathcal{L}_1 \subseteq \mathcal{L}2 \subseteq \cdots$ be an infinite ascending sequence of LCs on $\mathcal{T}$. Define

$$\mathcal{L} = \bigcup_{i=1}^{\infty} \mathcal{L}_i.$$

Since $\mathcal{T}$ is an SLN ternary semigroup, $\mathcal{L}$ is produced by the set $\mathcal{X}$. Take l as the minimal member and $\mathcal{X} \subseteq \mathcal{L}_l$ which implies $\mathcal{L} \subseteq \mathcal{L}_l$.
Also, $\mathcal{L}_l \subseteq \mathcal{L}$.
Therefore, $\mathcal{X} = \mathcal{L}_l$.
So, we conclude that $\mathcal{L}_m \subseteq \mathcal{L}_l$ for some $m \geq l$.
$ii \implies iii$
Suppose that there is a set $\mathcal{C} \neq \{\emptyset\}$ contains no minimal elements.
Let $\mathcal{L}_1 \in \mathcal{C}$.
There exists $\mathcal{L}_2 \in \mathcal{C}$ with $\mathcal{L}_1 \subseteq \mathcal{L}_2$ as $\mathcal{L}_1$ is not minimal. As we continue in this manner, we obtain an infinite sequence of LCs $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq .....$ This leads to a contradiction.
$iii \implies i$
Supppose that there is a LCs $\mathcal{L}$ on $\mathcal{T}$ is not finitely produced.
Let $(s_1, r_1) \in \mathcal{L}$. Define $\mathcal{L}_1$ as the LC on $\mathcal{T}$ produced by (s_1, r_1).
Since, $\mathcal{L}$ is not finitely produced, $\mathcal{L}_1 \neq \mathcal{L}$.
Let $(s_2, r_1) \in \mathcal{L}$. Define $\mathcal{L}_2$ as the LC on $\mathcal{T}$ produced by $(s_1, r_1), (s_2, r_2)$.
Then, $\mathcal{L}_1 \subseteq \mathcal{L}_2$. Continuing this way, we have $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq .....$
So, the $\{\mathcal{L}_1, \mathcal{L}_2, ...\}$ of LCs on $\mathcal{T}$ contain no minimal element which is a contradiction.

□

Remark: Consider $\mathcal{T}$ as the ternary semigroup. Let $\mathcal{C} \subseteq \mathcal{T} \times \mathcal{T}$.
Take $\bar{\mathcal{C}} = \mathcal{C} \cup \{(m, n) : (n, m) \in \mathcal{C}\}$.
Let $x, y \in \mathcal{T}$. An $\mathcal{C}-$ sequence linking x and y is any sequence $x = [s_1t_1m_1] \cup \{m_1\}, [s_1t_1n_1] \cup \{n_1\} = [s_2t_2m_2] \cup \{m_2\}, ...[s_lt_ln_l] \cup \{n_l\} = y$, where, $(m_i, n_i) \in \bar{\mathcal{C}}$ and $t_i, s_i \in \mathcal{T}(1 \leq i \leq l)$.

Definition 3.2. *Consider $\mathcal{T}$ as the ternary semigroup.Take $\mathcal{C} \subseteq \mathcal{T} \times \mathcal{T}$, $s, t \in \mathcal{T}$. Then, (s, t) is an outcome of $\mathcal{C}$ when one of the below happens*

1. $s = t$

2. There is a $\mathcal{C}-$ sequence linking s and t.

Lemma 3.1. *Let $\mathcal{T}$ be a ternary semigroup.Let $\mathcal{C} \subseteq \mathcal{T} \times \mathcal{T}$ and let $s, t \in \mathcal{T}$. Consider a LC $\mathcal{L}$ on $\mathcal{T}$ produced by $\mathcal{C}$. $s\mathcal{L}t$ implies and implied by (s, t) is an outcome of $\mathcal{C}$.*

Proof. Consider $\mathcal{T}$ as the ternary semigroup. Let $\mathcal{L}$ be the LC on $\mathcal{T}$ produced by $\mathcal{C}$. Let $s, t \in \mathcal{T}$.
Assume $s\mathcal{L}t$.
Clearly, $s = t$. or there exist $s_1, t_1 \in \mathcal{T}$ such that $s = [s_1t_1m_1] \cup \{m_1\}$. Let $(m_1, m_2) \in \bar{\mathcal{C}}$. Then, there exist $s_2, t_2 \in \mathcal{T}$ such that $[s_1t_1m_1] \cup \{m_1\} = [s_2t_2m_2] \cup \{m_2\}$.
Continuing this way, there exist $s_l, t_l \in \mathcal{T}$ such that $[s_lt_lx_l] \cup \{x_1\} = t$. So, we got $\mathcal{C}$-sequence linking s and t.
Conversely, suppose (s, t) is an outcome of $\mathcal{C}$.
Clearly, $s = t$ implies $s\mathcal{L}t$.
Now, consider there exists a $\mathcal{C}$-sequence linking s and t.
That is, $x = [s_1t_1m_1] \cup \{m_1\}, [s_1t_1n_1] \cup \{n_1\} = [s_2t_2m_2] \cup \{m_2\}, ...[s_lt_ln_l] \cup \{n_1\} = n$, where, $(m_i, m_{i+1}) \in \bar{\mathcal{C}}$ and $t_i, s_i \in \mathcal{T}(1 \leq i \leq l)$.
Since, (m_i, m_{i+1}) belongs to generating set for $\mathcal{L}$ and it can produce (s, t) which belongs to $\mathcal{L}$.

□

The below lemma is from Theorem 2.5.5 of section 4 in [10]

Lemma 3.2. *Suppose $\mathcal{T}$ is the ternary semigroup. Consider a LC $\mathcal{L}$ on $\mathcal{T}$ produced by $\mathcal{C}$. Thus, there is a finite subset $\mathcal{C}' \subseteq \mathcal{C}$ in which $\mathcal{L}$ is produced by $\mathcal{C}'$.*

Definition 3.3. *A ternary semigroup $\mathcal{T}$ is said to be left noetherian if every left ideal of $\mathcal{T}$ is finitely produced or $\mathcal{T}$ satisfies acc on its left ideals.*

Example 3.2. *Set of all 2×2 upper triangular matrices under matrix multiplication.*

Lemma 3.3. *Let $\mathcal{T}$ be a ternary semigroup. If $\mathcal{T}$ is SLN, then $\mathcal{T}$ is also left Noetherian.*

Proof. Consider a left ideal $\mathcal{I}$ of $\mathcal{T}$. Define the Rees congruence $\mathcal{L}_\mathcal{I}$ on $\mathcal{T}$ as follows: for any $e, f \in \mathcal{T}$, we say $e\mathcal{L}_\mathcal{I}f$ if and only if $e = f$ or both e and f belong to $\mathcal{I}$.

Since $\mathcal{T}$ is SLN, the congruence $\mathcal{L}_\mathcal{I}$ is produced by a finite set, say $\mathcal{C}$. Define a subset $\mathcal{D} \subseteq \mathcal{I}$ by

$$\mathcal{D} = \{c \in \mathcal{I} \mid \text{there exists } d \in \mathcal{I} \text{ such that } (c, d) \in \overline{\mathcal{C}}\},$$

where $\overline{\mathcal{C}}$ denotes the minimal set of pairs that generates $\mathcal{L}_\mathcal{I}$.

Now, let $e \in \mathcal{I}$. Since $e\mathcal{L}_\mathcal{I}f$ for some $f \in \mathcal{I}$ with $e \neq f$, there must be a sequence involving elements of $\mathcal{C}$ that connects e to f. Therefore, there exists an element $c \in \mathcal{D}$ and elements $s', t' \in \mathcal{T}$ such that $e = s't'c \cup \{c\}$.

This shows that the left ideal $\mathcal{I}$ can be produced by a finite subset of $\mathcal{D}$, demonstrating that $\mathcal{T}$ is left Noetherian. □

Definition 3.4. *Consider $\mathcal{T}$ as the ternary semigroup. If a member $t \in \mathcal{T}$ is in $\mathcal{T}^3$, then t is said to be decompossible. Otherwise, t is said to be indecompossible.*

Proposition 3.2. *Suppose $\mathcal{T}$ is a left Notherian ternary semigroup. Then, $\mathcal{T}$ has finitely indecomposible elements.*

Proof. Every ideal of $\mathcal{T}$ is finitely produced as it is a left noetherian. Therefore, there exists a finite set $\mathcal{C} \subseteq \mathcal{T}$ such that $\mathcal{T} = \mathcal{T}\mathcal{T}\mathcal{C} \cup \{\mathcal{C}\}$.
So, $\mathcal{T}/\mathcal{C} \subseteq \mathcal{T}^3$. Hence, $\mathcal{T}$ has atmost $|\mathcal{C}|$ imdecomposible elements. □

The following result demonstrates that the property of being a SLN ternary semigroup is preserved by taking homomorphic images.

Theorem 3.1. *Suppose $\mathcal{T}$ is the ternary semigroup, the mapping $\alpha : \mathcal{T} \to \mathcal{T}_1$ is the surjective homomorphism. If $\mathcal{T}$ is a SLN, then $\mathcal{T}_1$ is a SLN.*

Proof. Consider a surjective homomorphism $\alpha : \mathcal{T} \to \mathcal{T}_1$. Let $\mathcal{L}$ be a LC on $\mathcal{T}_1$. We can define a LC $\mathcal{L}'$ on $\mathcal{T}$ by the rule:

$$t_1\mathcal{L}'t_2 \text{ if and only if } t_1\alpha\mathcal{L}t_2\alpha.$$

Since $\mathcal{T}$ is a SLN ternary semigroup, the congruence $\mathcal{L}'$ is produced by a finite set $\mathcal{C}$. Now, consider the set

$$\mathcal{D} = \{(s\alpha, t\alpha) \mid (s, t) \in \mathcal{C}\}.$$

We need to show that the congruence $\mathcal{L}$ on $\mathcal{T}_1$ is produced by the finite set $\mathcal{D}$.

Suppose $t_1\mathcal{L}t_2$ with $t_1 \neq t_2$. Then there exist elements $e_1, e_2 \in \mathcal{T}$ such that $t_1 = e_1\alpha$ and $t_2 = e_2\alpha$. Since $e_1\mathcal{L}'e_2$ and $e_1 \neq e_2$, there exists a $\mathcal{C}$-sequence linking e_1 and e_2 in $\mathcal{T}$.

Applying the homomorphism α to each pair in this sequence yields a $\mathcal{D}$-sequence in $\mathcal{T}_1$ linking t_1 and t_2, which completes the proof.

□

Theorem 3.2. *Consider $\mathcal{T}$ as the ternary semigroup, $\mathcal{I}$ as the ideal of $\mathcal{T}$. $\mathcal{I}$ and $\mathcal{T}/\mathcal{I}$ are SLN ternary semigroups implies $\mathcal{T}$ is also a SLN ternary semigroup.*

Proof. Consider a LC $\mathcal{L}$ on $\mathcal{T}$. Let $\mathcal{L}_\mathcal{I}$ denote the limitation of $\mathcal{L}$ to the ideal $\mathcal{I}$. Since $\mathcal{I}$ is SLN, the congruence $\mathcal{L}_\mathcal{I}$ is produced by a finite set, say $\mathcal{C}$.

Next, consider the quotient semigroup $\mathcal{U} = \mathcal{T}/\mathcal{I}$. For any $t \in \mathcal{T}$, $t\mathcal{L}i$ for some $i \in \mathcal{I}$, there is an element $\beta(t) \in \mathcal{I}$ such that $t\mathcal{L}\beta(t)$.

The congruence $\mathcal{L}$ on $\mathcal{T}$ can then be described by the set

$$\mathcal{D} = \{(t, i) \mid t \in \mathcal{U},\ t\mathcal{L}i \text{ for some } i \in \mathcal{I}\} \cup (\mathcal{U} \times \mathcal{U}),$$

which is finite since $\mathcal{U}$ is SLN.

According to the lemma, the restriction $\mathcal{L}_\mathcal{I}$ is produced by a finite set $\mathcal{C}' \subseteq \mathcal{C}$ because $\mathcal{U}$ is SLN. Let us define the sets $\mathcal{C}_1 = \mathcal{C}' \cap (\mathcal{U} \times \mathcal{U})$ and $\mathcal{C}_2 = \{(t, \beta(t)) \mid (t, 0) \in \mathcal{C}'\}$.

Now, consider $t \in \mathcal{T}$ and $t\mathcal{L}i$ for some $i \in \mathcal{I}$. Then there exists an element $t' \in \mathcal{I}$, (t, t') is produced by the set $\mathcal{C}_1 \cup \mathcal{C}_2$.

Claim. If $t \in \mathcal{I}$, then let $t' = t$. Assume now that $t \in \mathcal{U}$. Since $t\mathcal{L}'0$, there exists a $\mathcal{C}'$-sequence linking t and 0:

$$\begin{aligned} t &= [s_1t_1m_1] \cup \{m_1\},\ [s_1t_1n_1] \cup \{n_1\} \\ &= [s_2t_2m_2] \cup \{m_2\}, \ldots, [s_lt_ln_l] \cup \{n_l\} \\ &= 0 \end{aligned}$$

where $(m_i, n_i) \in \overline{\mathcal{C}'}$ and $s_i, t_i \in \mathcal{T}$ for $1 \leq i \leq l$.
If $n_l \in \mathcal{U}$, we obtain a $\mathcal{C}_1$-sequence linking t and $s_l t_l n_l \cup \{n_l\} \in \mathcal{I}$. If $n_l = 0$, then $(t, s_l t_l m_l \cup \{m_l\})$ is outcome of $\mathcal{C}_2$. Therefore, $s_l t_l \beta(m_l) \cup \{\beta(m_l)\} \in \mathcal{I}$ is derived from $s_l t_l m_l \cup \{m_l\}$ using $\mathcal{C}_2$. □

We claim that the congruence $\mathcal{L}$ is produced by the finite set $\mathcal{C} \cup \mathcal{C}_1 \cup \mathcal{C}_2$. Suppose $t\mathcal{L}u$ with $t \neq u$. If there is no element $i \in \mathcal{I}$, $(t, i) \in \mathcal{L}$, So, t, u, and (t, u) are outcome of $\mathcal{C}_1$. Otherwise, by the above claim, there is elements $t', u' \in \mathcal{I}$ such that (t, t') and (u, u') are outcome of $\mathcal{C}_1 \cup \mathcal{C}_2$. Since $t'\mathcal{L}_{\mathcal{I}}u'$, the pair (t', u') is outcome of $\mathcal{C}$. Thus, (t, u) is outcome of $\mathcal{C} \cup \mathcal{C}_1 \cup \mathcal{C}_2$. □

4 Ternary Subsemigroup

Here, we explore the connection between a semigroup and its substructures concerning the property of being SLN.

Theorem 4.1. *Let $\mathcal{T}$ be a ternary semigroup containing a subsemigroup $\mathcal{S}$ such that the quotient $\mathcal{T}/\mathcal{S}$ is finite. Then, $\mathcal{T}$ is SLN if and only if $\mathcal{S}$ is SLN.*

Proof. **Necessity:**
Assume that $\mathcal{S}$ is SLN. Let $\mathcal{L}$ be a LC on $\mathcal{S}$.
Extend $\mathcal{L}$ to a LC $\bar{\mathcal{L}}$ on $\mathcal{T}$.
Since $\mathcal{T}$ is SLN, the congruence $\bar{\mathcal{L}}$ is produced by a finite set $\mathcal{C}$.
Define the set

$$\mathcal{U} = \{stm : (m, n) \in \mathcal{C} \text{ for some } n \in \mathcal{S}, s, t \in \mathcal{T}/\mathcal{S}\}.$$

Since both $\mathcal{C}$ and $\mathcal{T}/\mathcal{S}$ are finite, $\mathcal{U}$ is finite as well.
We assert that $\mathcal{L}$ is produced from a finite set

$$\mathcal{D} = \mathcal{C} \cup (\mathcal{L} \cap (\mathcal{U} \times \mathcal{U})).$$

To prove this, consider $e\mathcal{L}f$ and $e \neq f$.
There is a sequence formed by elements of $\mathcal{C}$ because $e\bar{\mathcal{L}}f$.

$$\begin{aligned} e = [s_1 t_1 m_1] \cup \{m_1\}, [s_1 t_1 n_1] \cup \{n_1\} \\ = [s_2 t_2 m_2] \cup \{m_2\}, \ldots, [s_l t_l n_l] \cup \{n_l\} \\ = f \end{aligned} \quad (1)$$

where $(m_i, n_i) \in \mathcal{C}$ and $s_i, t_i \in \mathcal{T}$ for $1 \leq i \leq l$.
If all $s_i, t_i \in \mathcal{T}$, then (e, f) is a outcome of $\mathcal{C}$.
Otherwise, let i be the smallest index such that $s_i \in \mathcal{T}/\mathcal{S}$, and let j be the largest index such that $s_j \in \mathcal{T}/\mathcal{S}$.
Then $s_k \in \mathcal{T}$ for all $k \in \{1, \ldots, i-1, j+1, \ldots, l\}$, and $([s_i t_i m_i], [s_i t_i n_i]) \in \mathcal{L} \cap (\mathcal{U} \times \mathcal{U})$.
Deleting the subsequence, we get

$$\begin{aligned} [s_i t_i m_i] \cup \{m_i\} = [s_{i+1} t_{i+1} m_{i+1}] \cup \{m_{i+1}\}, \ldots, \\ [s_{j-1} t_{j-1} m_{j-1}] \cup \{m_{j-1}\} \\ = [s_j t_j m_j] \cup \{m_j\} \end{aligned}$$

Equation (1) gives $\mathcal{D}$-sequence linking e, f.

Sufficiency:
Suppose $\mathcal{S}$ is SLN. Let $\mathcal{L}$ be a LC on $\mathcal{T}$.
Define $\mathcal{L}_{\mathcal{S}}$ as the limitation of $\mathcal{L}$ to $\mathcal{S}$.
Since $\mathcal{S}$ is SLN, the congruence $\mathcal{L}_{\mathcal{S}}$ is produced from the finite set $\mathcal{C}$.
Take $\mathcal{V} = \mathcal{T}/\mathcal{S}$. For every $e \in \mathcal{V}$, $e\mathcal{L}f$ for some $f \in \mathcal{S}$, choose $\beta(e) \in \mathcal{S}$ such that $e\mathcal{L}\beta(e)$.
We claim that $\mathcal{L}$ is produced from the finite set

$$\begin{aligned} \mathcal{D} = \mathcal{C} \cup \{(e, \beta(e)) : e \in \mathcal{V}, e\mathcal{L}t \text{ for some } t \in \mathcal{S}\} \cup \\ (\mathcal{L} \cap (\mathcal{D} \times \mathcal{D})). \end{aligned}$$

To show this, consider $e\mathcal{L}t$ with $e \neq t$.
If $e, t \in \mathcal{D}$, then $(e, t) \in \mathcal{L} \cap (\mathcal{V} \times \mathcal{V})$.
If $e, t \in \mathcal{S}$, then $e\mathcal{L}_{\mathcal{S}}t$, so (e, t) is an outcome of $\mathcal{C}$.
Let $e \in \mathcal{V}$, $t \in \mathcal{S}$. Since $e\mathcal{L}t$,then $\beta(e)\mathcal{L}_{\mathcal{S}}t$, and hence $(\beta(e), t)$ is an outcome of $\mathcal{C}$.
Therefore, (e, t) is an outcome of $\mathcal{D}$. □

Corollary 4.1. *A Ternary semigroup $\mathcal{T}$ is SLN implies and implied by $\mathcal{T}^1$ is SLN.*

Corollary 4.2. *A Ternary semigroup $\mathcal{T}$ is SLN implies and implied by $\mathcal{T}^0$ is SLN.*

Theorem 4.2. *Consider $\mathcal{T}$ as the ternary semigroup, $\mathcal{S}$ as the subsemigroup, the quotient $\mathcal{T}/\mathcal{S}$ is a right ideal in $\mathcal{T}$. If $\mathcal{T}$ is SLN, then $\mathcal{S}$ is also SLN.*

Proof. Let $\mathcal{L}$ be a LC on $\mathcal{S}$. Define $\mathcal{L}'$ as the right congruence on $\mathcal{T}$ that is produced by $\mathcal{L}$.
Since $\mathcal{T}$ is SLN, the congruence $\mathcal{L}'$ is finitely produced, meaning there exists a finite subset $\mathcal{C}$ that generates $\mathcal{L}'$.
We aim to show that $\mathcal{L}$ is produced by $\mathcal{C}$.
Suppose $e\mathcal{L}f$ with $e \neq f$. Since $e\mathcal{L}'f$, there exists a sequence involving elements of $\mathcal{C}$:

$$\begin{aligned} e = [s_1 t_1 m_1] \cup \{m_1\}, [s_1 t_1 n_1] \cup \{n_1\} = [s_2 t_2 m_2] \cup \{m_2\}, \\ \ldots, [s_l t_l n_l] \cup \{n_l\} = f \end{aligned}$$

Because $e \in \mathcal{S}$ and $\mathcal{T}/\mathcal{S}$ is a right ideal in $\mathcal{T}$, it follows that $s_1, t_1 \in \mathcal{S}$. Thus, $s_1 t_1 m_1 \cup \{m_1\} \in \mathcal{S}$, implying that $s_2, t_2 \in \mathcal{S}$, and so on.
Consequently, $s_i, t_i \in \mathcal{S}$ for all $1 \leq i \leq l$. Therefore, the pair (e, f) is outcome of $\mathcal{C}$. □

Corollary 4.3. *Let $\mathcal{T}$ be a ternary semigroup and $\mathcal{S}$ a subsemigroup such that the quotient $\mathcal{T}/\mathcal{S}$ is a SLN ideal of $\mathcal{T}$. Then $\mathcal{T}$ is SLN if and only if $\mathcal{S}$ is SLN.*

Proof. The direct implication is established by **Theorem** 4.2.
For the reverse implication, consider $\mathcal{I} = \mathcal{T}/\mathcal{S}$.
Since $\mathcal{I}$ is a SLN ideal, and $\mathcal{T}/\mathcal{I}$ is isomorphic to $\mathcal{T} \cup \{0\}$, it follows that $\mathcal{T}/\mathcal{I}$ is also SLN, as established by **Corollary** 4.3 given that $\mathcal{S}$ is SLN.
Applying **Theorem** 3.2 now confirms that $\mathcal{T}$ is SLN. □

Definition 4.1. *Let $\mathcal{T}$ be a ternary semigroup, let $\mathcal{X} \subseteq \mathcal{T}$. The left stabilizer of $\mathcal{X}$ of $\mathcal{T}$ is defined as*
$St_L(\mathcal{X}) = \{t_1, t_2 \in \mathcal{T} : t_1 t_2 \mathcal{X} \cup \{\mathcal{X}\} = \mathcal{X}\}$.
$St_L(\mathcal{X})$ is a submonoid of $\mathcal{T}$.

Theorem 4.3. *Let $\mathcal{T}$ be a ternary semigroup, $\mathcal{X} \subseteq \mathcal{T}$. If $\mathcal{T}$ is SLN, then the left stabilizer $St_L(\mathcal{X})$ of $\mathcal{X}$ is also SLN.*

Proof. Let $\mathcal{L}$ be a LC on $St_L(\mathcal{X})$. Define $\mathcal{L}'$ as the LC on $\mathcal{T}$ that is produced by $\mathcal{L}$.
Since $\mathcal{T}$ is SLN, the congruence $\mathcal{L}'$ is produced by a finite set $\mathcal{C}$.
We will show that $\mathcal{L}$ is also produced by $\mathcal{C}$.
Suppose $e\mathcal{L}f$ with $e \neq f$. Since $e\mathcal{L}'f$, there exists a sequence involving elements from $\mathcal{C}$:

$$e = [s_1t_1m_1] \cup \{m_1\}, [s_1t_1n_1] \cup \{n_1\} = [s_2t_2m_2] \cup \{m_2\}, \ldots, [s_lt_ln_l] \cup \{n_l\} = f$$

where $(m_i, n_i) \in \mathcal{C}$ and $s_i, t_i \in \mathcal{T}$ for $1 \leq i \leq l$.
Given that $e, m_1 \in St_L(\mathcal{X})$, it follows that:

$$s_1t_1\mathcal{X} = s_1t_1(m_1\mathcal{X}) = e\mathcal{X} = \mathcal{X}$$

Thus, $s_1t_1 \in St_L(\mathcal{X})$.
Consequently, $[s_1t_1n_1] \in St_L(\mathcal{X})$, which implies that $s_2t_2 \in St_L(\mathcal{X})$.
Repeating this process, we find that $s_it_i \in St_L(\mathcal{X})$ for all $i \in \{1, \ldots, l\}$.
Therefore, (e, f) is outcome of the finite set $\mathcal{C}$, completing the proof. □

Lemma 4.1. *If $\mathcal{T}$ is acc ternary semigroup and $\mathcal{G}$ is its subgroup, then $\mathcal{G}$ satisfies acc on subgroups.*

Proof. Every subgroup $\mathcal{G}$ of a group is a LC class of $\mathcal{T}$.
$\mathcal{G} \cap \mathcal{G}_t \neq 0$ with $t \in \mathcal{T}$ which implies $\mathcal{G} = \mathcal{G}_t$.
Let $\mathcal{L}_\mathcal{G}$ be the smallest left congruence of $\mathcal{T}$ admitting $\mathcal{G}$ as an equivalence class.
If $\mathcal{G}_1$ and $\mathcal{G}_2$ are subgroups such that $\mathcal{L}_{\mathcal{G}_1} \subseteq \mathcal{L}_{\mathcal{G}_2}$ which implies $\mathcal{G}_1 \subseteq \mathcal{G}_2$ conversely, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are subgroups such that $\mathcal{G}_1 \subseteq \mathcal{G}_2$ which implies $\mathcal{L}_{\mathcal{G}_1} \subseteq \mathcal{L}_{\mathcal{G}_2}$.
Since, $\mathcal{G}_1$ is a LC class of $\mathcal{L}_{\mathcal{G}_1} \cap \mathcal{L}_{\mathcal{G}_2}$.
This shows that the subgroups of the group in a 1-1 correspondence with the LC of $\mathcal{T}$ of the form $\mathcal{L}_\mathcal{G}$.
So, the minimal condition on the LC of $\mathcal{T}$ implies the minimal condition on subgroups.

□

Lemma 4.2. *Any semigroup with acc contains no infinite subsemilattices.*

Proof. Suppose ternary semigroup $\mathcal{T}$ contains an infinite subsemilattice $\mathcal{L}$.
We prove that $\mathcal{T}$ doesn't satisfy acc.
If $\mathcal{L}$ contains an infinite ascending chain $l_1 \leq l_2 \leq \ldots\ldots \leq l_n\ldots\ldots$ then is so $\mathcal{T}$.
Since

$$[\mathcal{T}l_1l_1] \subseteq [\mathcal{T}l_2l_2] \subseteq \ldots \subseteq [\mathcal{T}l_nl_n] \subseteq \ldots$$

Then $\mathcal{T}$ doesn't satisfy acc on left ideals.
Assume $\mathcal{L}$ contains an infinite antichain $\{l_1, l_2, \ldots..\}$. Then we have infinite ascending chain

$$\{\bigcup_{i=n,j=n}^{\infty} \mathcal{T}e_ie_j : n = 1, 2, \ldots\}$$

of left ideals.
Now, Assume $\mathcal{T}$ contains an infinite ascending chain $l_1 \leq l_2 \leq \ldots \leq l_n \ldots$. Let

$$\mathcal{L}_n = \{(m, n) \in \mathcal{T} \times \mathcal{T} : me_ie_i = ne_ie_i \; for \; all \; i \leq n\}$$

So, $\mathcal{L}_n \in \mathcal{L}(\mathcal{T})$.
Since, $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \ldots\ldots \subseteq \mathcal{L}_n \subseteq \ldots..$, so $\mathcal{T}$ doesnot satisfy acc. □

Corollary 4.4. *Any semilattice with acc on LC is finite.*

The below theorem gives the necessary condition for a ternary semigroup to be acc or SLN.

Theorem 4.4. *If the inverse ternary semigroup $\mathcal{T}$ satisfies acc then it has a finite number of idempotents and satisfies acc on subgroups.*

Proof. Necessity follow from **Corollary** 4.4 and **Lemma** 4.1.

□

Corollary 4.5. *If the inverse ternary semigroup $\mathcal{T}$ is SLN then it has a finite number of idempotents and satisfies acc on subgroups.*

Corollary 4.6. *Consider $\mathcal{T}$ to be an inverse ternary semigroup. If $\mathcal{T}$ is SLN then every inverse subsemigroup of $\mathcal{T}$ is finitely produced.*

5 Direct Product of Ternary Semigroups

This section investigates under what conditions the direct product $\mathcal{T}_1 \times \mathcal{T}_2$ of two ternary semigroups is SLN.

Lemma 5.1. *Consider $\mathcal{T}_1$, $\mathcal{T}_2$ are two ternary semigroups. $\mathcal{T}_1 \times \mathcal{T}_2$ is SLN implies both $\mathcal{T}_1$ and $\mathcal{T}_2$ are SLN.*

Proof. Since, both $\mathcal{T}_1$ and $\mathcal{T}_2$ are homomorphic image of $\mathcal{T}_1 \times \mathcal{T}_2$.
By **Theorem**3.1, we can say that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are SLN. □

Lemma 5.2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two ternary semigroups. If $\mathcal{T}_1$ is infinite and $\mathcal{T}_1 \times \mathcal{T}_2$ is SLN, then $\mathcal{T}_2^3 = \mathcal{T}_2$.*

Proof. Assume for contradiction that $\mathcal{T}_2^3 \neq \mathcal{T}_2$.
Then, $\mathcal{T}_2$ must contain an indecomposable element r.
Consequently, for each $t \in \mathcal{T}_1$, the element (t, r) in $\mathcal{T}_1 \times \mathcal{T}_2$ is indecomposable.
This implies that $\mathcal{T}_1 \times \mathcal{T}_2$ contains infinitely many indecomposable elements.
However, by **Theorem 3.2**, this would contradict the assumption that $\mathcal{T}_1 \times \mathcal{T}_2$ is SLN. □

Example 5.1. *Consider $\mathcal{T}$ as the infinite noetherian ternary semigroup.*
Consider $\mathcal{Z}^+ = \{1, 2, 3, \ldots.\}$ under ternary addition.
Then $\mathcal{Z}^+ \times \mathcal{T}$ is not SLN. Because, $\mathcal{Z}^+$ includes indecomposable member.

Definition 5.1. *Let $\mathcal{T}$ be a ternary semigroup that is SLN. We say that $\mathcal{T}$ preserves the property of being SLN in direct products if the following condition holds: For any ternary monoid $\mathcal{M}$, the direct product $\mathcal{T} \times \mathcal{M}$ is SLN implies and implied by $\mathcal{M}$ is SLN.*

Remark: We can naturally define a collection of LCs on $\mathcal{T}$.
Consider a LC $\mathcal{L}$ on the direct product $\mathcal{T} \times \mathcal{M}$.
For each element $a \in \mathcal{M}$, we construct a corresponding LC $\mathcal{L}_a$ on $\mathcal{T}$ by
$s\mathcal{L}_a t$ if and only if $(s, a)\mathcal{L}(t, a)$
To see that $\mathcal{L}_a$ is a LC,
let $s\mathcal{L}_a t$ and $t_1, t_2 \in \mathcal{T}$.
Since $\mathcal{L}$ is a LC on $\mathcal{T}$,
$([t_1t_2s], a) = (t_1, 1)(t_2, 1)(s, a)\mathcal{L}(t_1, 1)(t_2, 1)(t, a) = ([t_1t_2t], a)$
Therefore, $[t_1t_2s]\mathcal{L}_a[t_1t_2t]$

Theorem 5.1. *Consider $\mathcal{M}$ as the finite monoid. Then, for any ternary semigroup $\mathcal{T}$, the direct product $\mathcal{T} \times \mathcal{M}$ is SLN implies and implied by $\mathcal{T}$ is SLN.*

Proof. The necessary part is an immediate consequence from **Lemma**5.1.
Conversely, Assume $\mathcal{T}$ is a SLN semigroup.
LC on $\mathcal{T} \times \mathcal{M}$ is $\mathcal{L}$. Since $\mathcal{T}$ is SLN, each $\mathcal{L}_a$ is produced by a finite set $\mathcal{X}_a$.
Define

$$\mathcal{Y}_a = \{((e, a), (f, a)) : (e, f) \in \mathcal{X}_a\}$$

Now, let $\mathcal{H}$ be the set

$$\{(a, b) \in \mathcal{T} \times \mathcal{M} : a \neq b, (a, t)\mathcal{L}(b, u) \ for\ some\ t, u \in \mathcal{T}\}.$$

For every $(a, b) \in \mathcal{H}$, Define

$$\mathcal{I}(a, b) = \{t \in \mathcal{T} : (t, a)\mathcal{L}(u, b) \ for\ some\ u \in \mathcal{T}\}.$$

Let $t \in \mathcal{T}$. Take $t_1, t_2 \in \mathcal{T}$.
Here, $(t, a)\mathcal{L}(u, b)$ for some $u \in \mathcal{T}$.
Then

$$([t_1t_2t], a) = (t_1, 1)(t_2, 1)(t, a)\mathcal{L}(t_1, 1)(t_2, 1)(u, b) = ([t_1t_2u], b)$$

So, $([t_1t_2t], a)\mathcal{L}([t_1t_2u], b)$.
Therefore, $[t_1t_2t] \in \mathcal{I}(a, b)$.
Since $\mathcal{T}$ is weakly SLN, then $\mathcal{I}(a, b)$ is produced by $\mathcal{G}(a, b)$.
For every $g \in \mathcal{G}(a, b)$, take $\beta_{a,b}(g) \in \mathcal{T}$,
$(g, a)\mathcal{L}(\beta_{a,b}(g), b)$.
Let $\mathcal{P}$ be the finite set

$$\{((g, a), (\beta_{a,b}(g), b) : (a, b) \in \mathcal{H}, g \in \mathcal{G}(a, b)\}.$$

We claim that the below finite set generates $\mathcal{L}$

$$\mathcal{Y} = (\bigcup_{a \in \mathcal{M}} \mathcal{Y}_a) \cup \mathcal{P}$$

Now, consider $(t, a)\mathcal{L}(u, b)$ with $(t, a) \neq (u, b)$.
Assume $a = b$.
Since $t\mathcal{L}_a u$ with $t \neq u$, there exists an $\mathcal{X}_a$-sequence

$$t = [s_1t_1m_1] \cup \{m_1\}, [s_1t_1n_1] \cup \{n_1\} = [s_2t_2m_2] \cup \{m_2\}, \ldots [s_lt_ln_l] \cup \{n_l\} = u.$$

Therefore, we have a $\mathcal{Y}_a$-sequence

$$(t, a) = (s_1, 1)(t_1, 1)(m_1, a), (s_1, 1)(t_1, 1)(n_1, a) = (s_2, 1)(t_2, 1)(m_2, a), ...(s_l, 1)(t_l, 1)(n_l, a) = (u, a).$$

Next we assumme $a \neq b$.
We claim that there exists $u^{'} \in \mathcal{T}$ such that the pair $((u, a), (u^{'}, b))$ is an outcome of $\mathcal{P}$.
Indeed, since $t \in \mathcal{I}(a, b)$, we have that $t = [t_1t_2v] \cup \{v\}$ for some $v \in \mathcal{G}(a, b)$ and $t_1, t_2 \in \mathcal{T}$.
Take $t^{'} = [t_1t_2\beta_{a,b}(v)]$.
Either $t = v$ and $((t, a), (t^{'}, b)) \in \mathcal{P}$, or a $\mathcal{P}$-sequence

$$(t, a) = ([t_1t_2v], a) = (t_1, 1)(t_2, 1)(v, a), (t_1, 1)(t_2, 1)(\beta_{a,b}(v), b) = ([t_1t_2\beta_{a,b}(v)], b).$$

It now follows that $(t^{'}, b)\mathcal{L}(u, b)$.
Therefore, $t^{'}\mathcal{L}_b u$.
So, either $t^{'} = u$ or there exists a $\mathcal{Y}_n$-sequence linking $(t^{'}, b)$ and (u, n).
Therefore, ((t, a), (u, b)) is an outcome of $\mathcal{Y}$. □

Example 5.2. *Consider any infinite SLN $\mathcal{T}$ which is finitely produced and a ternary semigroup $\mathcal{U}$ on $\{a, b\}$ with ternary operation by $[aaa] = a, [aab] = a, [aba] = a, [abb] = b, [bbb] = b, [baa] = b, [bab] = b.[bba] = b$.*
*Then $\mathcal{T} \times \mathcal{U}^1$ is SLN by **Theorem** 5.1 . But, $\mathcal{T} \times \mathcal{U}$ is not SLN.*

Proposition 5.1. *Consider that $\mathcal{T}$ is a ternary semigroup and that $\mathcal{U}$ is its homomorphic image. $\mathcal{U}$ preserves SLN if $\mathcal{T}$ does in direct products.*

Proof. Let $\mathcal{M}$ be a SLN monoid.
We have that $\mathcal{U} \times \mathcal{M}$ is SLN since $\mathcal{T} \times \mathcal{M}$ is SLN and there is a homomorphism between $\mathcal{U} \times \mathcal{M}$ and $\mathcal{T} \times \mathcal{M}$ by **Theorem** 3.1. □

Proposition 5.2. *Consider $\mathcal{T}$ as the ternary semigroup, $\mathcal{M}$ a ternary monoid. $\mathcal{T}$ and $\mathcal{M}$ preserve the property of being SLN in direct product implies $\mathcal{T} \times \mathcal{M}$ also retains the same property.*

Proof. Suppose $\mathcal{N}$ is a monoid that is SLN. Since $\mathcal{M}$ preserves the SLN property under direct products, it follows that $\mathcal{M} \times \mathcal{N}$ is SLN.
Given that $\mathcal{T}$ also preserves the SLN property under direct products, then $\mathcal{T} \times \mathcal{M}$ is SLN, because there is an isomorphism between $(\mathcal{T} \times \mathcal{M}) \times \mathcal{N}$ and $\mathcal{T} \times (\mathcal{M} \times \mathcal{N})$. □

Proposition 5.3. *Consider a semigroup $\mathcal{T}$ and an ideal $\mathcal{I}$ of $\mathcal{T}$. If both the Rees quotient $\mathcal{T}/\mathcal{I}$ and $\mathcal{I}$ preserve SLN in direct products, then $\mathcal{T}$ also preserves SLN in direct products.*

Proof. Consider a ternary monoid $\mathcal{M}$ that is SLN and let $\mathcal{J}$ be the cartesian product of $\mathcal{I}$ and $\mathcal{M}$.
It is known that $\mathcal{J}$ is an ideal of $\mathcal{T} \times \mathcal{M}$, and J is SLN because $\mathcal{I}$ preserves SLN properties in direct products.
This gives a surjective homomorphism:

$$\alpha : (\mathcal{T}/\mathcal{I}) \times \mathcal{M} \to (\mathcal{T} \times \mathcal{M})/\mathcal{J}.$$

Theorem 3.1 implies that $(\mathcal{T} \times \mathcal{M})/\mathcal{J}$ is SLN, given that $(\mathcal{T}/\mathcal{I}) \times \mathcal{M}$ is SLN.
Theorem 3.2 implies that S × M is SLN. □

Proposition 5.4. *Consider a ternary semigroup $\mathcal{T}$. Then $\mathcal{T}$ preserves the SLN in direct products implies and implied by $\mathcal{T}^0$ preserves the SLN in direct products.*

Proof. Assume that $\mathcal{M}$ is a ternary monoid that satisfies the SLN property.
Given that $\{0\} \times \mathcal{M}$ is a SLN ideal of $\mathcal{T}^0 \times \mathcal{M}$.
By **Corollary** 4.3, $\mathcal{T} \times \mathcal{M}$ is also SLN implies and implied by $\mathcal{T}^0 \times \mathcal{M}$ is SLN. □

6 Applications

Here are some specific applications of **Left Noetherian Ternary Semigroups** and their properties:

i. Finite Control in Automated Systems

Application in Computer Science: Automated systems and finite state machines, used in software and hardware verification, often rely on algebraic structures like semigroups to model state transitions. The *Left Noetherian property* ensures that any state transition modeled by a ternary semigroup can be finitely controlled. This helps in preventing non-terminating or infinite loops in algorithms that rely on ternary relations.

Example: In circuit design, ternary semigroups can model signals with three states (e.g., high, low, and undefined). A left Noetherian ternary semigroup ensures that only a finite set of signal transitions needs to be analyzed for optimization and verification, streamlining the circuit design process.

ii. Ternary Cryptography

Application in Cryptography: Ternary operations are used in cryptographic algorithms that involve three inputs, providing greater complexity and security. The *strongly Left Noetherian property* ensures that any cryptographic protocol using ternary semigroups can be finitely produced, making it resistant to infinite or repeating sequences that could lead to vulnerabilities.

Example: In a cryptographic protocol based on ternary semigroups, ensuring that the semigroup is left Noetherian guarantees that key generation and encryption processes can be handled by a finite set of operations, improving the efficiency and security of the encryption system.

iii. Chemical Reaction Networks

Application in Chemistry: Ternary semigroups can model three-way interactions in chemical reaction networks. A *strongly left Noetherian ternary semigroup* ensures that all possible reactions in the network are finitely generated, allowing chemists to predict and control all reaction pathways.

Example: In enzyme-substrate reactions involving three components (substrate, enzyme, and inhibitor), modeling the reactions using a left Noetherian ternary semigroup ensures that all reaction outcomes can be predicted from a finite set of interactions, aiding in drug design and optimization.

iv. Optimizing Workflow Processes

Application in Operations Research: Workflow processes that involve ternary relations, such as interactions between three departments or stages in a supply chain, can be optimized using ternary semigroups. The *left Noetherian property* ensures that any workflow process modeled can be reduced to a finite number of critical steps, improving efficiency.

Example: In a supply chain where the production, inventory, and distribution stages interact, using a ternary semigroup to model these interactions allows for the optimization of resource allocation, ensuring that only a finite number of configurations need to be considered for maximizing throughput.

v. Formal Language Processing

Application in Natural Language Processing (NLP): Ternary semigroups can model the relationships between three words or phrases in language parsing. The *Noetherian property* ensures that the parsing process is finite, preventing runaway parsing loops and improving efficiency in syntactic analysis.

Example: In NLP, when parsing sentences with ternary relations (e.g., subject-verb-object structures), a left Noetherian ternary semigroup guarantees that the parser will handle all potential phrase combinations within a finite set, ensuring faster and more accurate sentence analysis.

7 Conclusions

In conclusion, this work has significantly advanced the understanding of strongly left Noetherian ternary semigroups through a series of comprehensive investigations and characterizations. We have introduced the concept of strongly left Noetherian ternary semigroups and established key properties, notably that every left congruence on such semigroups is finitely generated. Our study provided a clear characterization of strongly left Noetherian ternary semigroups, demonstrating that the homomorphic image of a Noetherian ternary semigroup retains the Noetherian property, thus offering an alternative proof technique.

We further examined the intricate relationship between ternary semigroups and their subsemigroups, proving that the strongly left Noetherian property of a ternary semigroup is preserved if one of its subsemigroups possesses this property.

Additionally, we provided a necessary characterization for inverse ternary semigroups, enriching the theoretical framework of ternary semigroup theory.

Overall, this paper contributes foundational results and novel perspectives to the theory of ternary semigroups, particularly regarding their Noetherian properties. Future research can build on these results to explore more complex structures and relationships within ternary semigroups, further enriching the field of algebraic structures.

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