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Super-convergence and Stability Analysis of the Finite Element Orthogonal Collocation Method for Time-Fractional Telegraph Equation

Ebimene James Mamadu[1,*], Henrietta Ify Ojarikre[1], Daniel Chinedu Iweobodo[2], Jude Chukwuyem Nwankwo[3], Ebikonbo-Owei Anthony Mamadu[1,4], Jonathan Tsetimi[1], Ignatius Nkonyeasua Njoseh[1]

[1]Department of Mathematics, Delta State University, Abraka, Nigeria
[2]Department of Mathematics, Dennis Osadebay University, Asaba, Nigeria
[3]Department of Mathematics, University of Delta, Agbor, Nigeria
[4]Department of Mathematics, Michael and Cecilia Ibru University, Agbarha-Otor, Delta State, Nigeria

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(b): *Ebimene James Mamadu, Henrietta Ify Ojarikre, Daniel Chinedu Iweobodo, Jude Chukwuyem Nwankwo, Ebikonbo-Owei Anthony Mamadu, Jonathan Tsetimi, Ignatius Nkonyeasua Njoseh (2024). Super-convergence and Stability Analysis of the Finite Element Orthogonal Collocation Method for Time-Fractional Telegraph Equation. Mathematics and Statistics, 12(6), 523 - 528. DOI: 10.13189/ms.2024.120602.*

Abstract Super-convergence and stability analysis are essential components for ensuring efficiency, reliability, and accuracy of numerical approximations to iterative methods and differential equations. Stability analysis highlights the mechanism of error control over iterations to ensure reliability and long-term accuracy of simulation. Super-convergence analysis offers some extra conditions to ensure faster convergence of numerical solutions than expected, enabling enhanced accuracy and strategies. These analyses together guarantee the effective development of complex numerical algorithms as the basis for error control and estimation. These analyses involve complex simulations and are thus relevant in fields such as physics, engineering, and weather modeling. Thus, this paper is an extension of the Finite Element Orthogonal Collocation Method (FEOCM) by Mamadu *et al*. (2023) for the numerical approximation of the time fractional telegraph equation with Mamadu-Njoseh polynomials as basis functions, where relevant numerical simulations were carried out. The present study offers a comprehensive super-convergence and stability analysis of solutions for the time-fractional telegraph equations via FEOCM. Here, the L_2 -norm, H^1- norm, interpolation theory, and Cauchy-Schwarz inequality are employed as optimal estimators to propose relevant theorems for the analysis of stability and super-convergence of solutions. The analysis shows that the solutions of the fully discretized scheme FEOCM are unconditionally stable and exhibit super-convergence, with the optimal error estimated as $O(h+T^{2-\alpha})$.

Keywords Super-convergence, Cauchy-Schwarz Inequality, Interpolation Theory, Telegraph Equation, Finite Element Method, Orthogonal Collocation Method, Stability

1. Introduction

Several frequency-related models have been designed in recent years through the notion of fractional differential equations. These models are applied relatively frequently to frequency efficiency problems such as turbulent flows in water hammers, signal processing, network analysis, and wave propagation, among others. Contrary to integral order equations in the classical category, fractional differential equations possess the attributes and characteristics of analyzing the hereditary and memory properties of many real-life phenomena. In this paper, the time-fractional telegraph equation of the form (1.1) is considered [1, 2]:

$$\begin{cases} D_t^{\alpha}u + u_t - u_{xx} = f(x,t) \quad \text{in } w_a \coloneqq \Omega \times (0,T], t > 0, x \in [0,T], \\ u(.,0) = u_0 \ \text{ in } \ w_a \coloneqq \Omega \times (0,T], \\ u(.,0) = u_1 \ \text{ in } \ w_a \coloneqq \Omega \times (0,T], \\ u(0,.) = u(T,t) = 0 \ \text{ in } \ w_a \coloneqq \Omega \times (0,T], \end{cases} \tag{1.1}$$

where $1.5 \leq \alpha \leq 2$, $w_a = (-1,1)^k$ for $k = 2,3$ and $u|_{\partial\Omega} = 0$ for $t \in [0,T]$. We also assume a convex domain of $\Omega \subset \mathbb{R}^2$. Again, $D_t^{\alpha}u$ is a functional derivative of the Caputo type, $f(x,t) \in C(\overline{w}_a)$ and $u_0 \in C(\overline{w}_a)$.

In (1.1), $D_t^{\alpha}u$ is defined as

$$D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-s)^{\alpha}}\frac{\partial u(x,s)}{\partial s}ds \tag{1.2}$$

The time-fractional telegraph has been known to be a reliable device in resolving difficult and robust complex physical systems, such as, electric charge carriers in power lines, analysis of pulses of direct current, frequency rouse dynamics in audio and radio frequencies, resonance frequency estimation, and others [3–5]. Most fractional differential models do not have analytic solutions. However, existing analytic solutions to fractional differential equations include the Sumudu, Laplace, and Elzaki transform methods, the d-expansion method, the differential transform method, the fractional complex transform, etc. [6–9].

Numerical methods have become extremely popular over the past decade for resolving time-fractional differential models. Several authors have proposed and developed different numerical techniques for these problems. These methods commonly include the finite element method [10–13], the finite difference method [14–17], the finite difference orthogonal collocation method [3], the finite element orthogonal collocation method [18], the spectral method [19], the Galerkin discontinuous method [20–23], the finite volume method [24–25], among others. The time-fractional telegraph equation (1.1) is very significant and quite interesting, and the approximate solutions of this equation are examined or scrutinized by a few researchers only. For example, Vieira *et al*. [26] considered a higher spatial dimension of the time-fractional telegraph with fractional order derivatives expressed in the Hilfer sense, which interpolates between the Caputo and Riemann-Liouville derivatives. The Fourier, Mellin, and Laplace transforms were employed to obtain convolutions of solutions belonging to the Laplace integral of Fox H-functions. Huang *et al.* [27] considered a more elaborate and efficient numerical technique for a time-fractional telegraph equation. The authors first converted the time telegraph equation into an integro-differential equation involving a weakly singular kernel. An integro-differential difference discretization scheme was then constructed on a gridded node to approximate the integro-differential equation. The authors also proved the convergence of the method by taking into consideration the singularity of the analytic solution. Tasbozan and Esen [28] investigated the numerical treatment of a time-fractional telegraph using the cubic-spline collocation algorithm. The fractional derivatives were defined in the Caputo fractional sense. The L_2 and L_∞ formulae were used to discretize the Caputo time fractional derivative. The resulting numerical evidence was expressed in L_2 and L_∞ error norms to investigate the rate of convergence of the method. Liu [29] investigated the approximate solution of the time-fractional telegraph equation using two approaches: the Caputo fractional difference method and the Grunwald formula. The authors presented the stability and convergence of the methods by analyzing the eigenvalues of the iterated matrices.

The work of Mamadu *et al.* [18] highlights the role of orthogonal polynomials as grid points in a finite element space corresponding to a new basis function (or a nodal point) in the approximation of the time fractional telegraph equation. Cardinal theorems on convergence and error analysis were equally presented. However, the subject of super-convergence and stability of the time fractional telegraph equation with regard to the role of orthogonal polynomials as grid or nodal points has not been explored in the literature. Thus, the present study extends the work of Mamadu *et al.* [18] to investigate the super-convergence and stability of the finite element orthogonal collocation method (FEOCM) for the approximate solution of the time-fractional telegraph equation. Consequently, an analysis of the FEOCM's super-convergence and stability using relevant norms will be established.

2. Preliminaries

i) Let y_n define a uniform quasi partition of Ω into $\aleph_n$ finite element for $n = 0(1)N$, and $h = \max_{1\leq n\leq N}\{\aleph_n\}$ denote the step size. Then, we define the bilinear element space as

$$U_h = \{u_h \in H_0^1(\Omega): u_h|_{\aleph_n} \in L\{1,x,y,xy\} \ \text{for} \ n = 1(2)N\},$$

and

$$U_{0h} = \{u_h \in U_h: u_h|_{\partial\Omega} = 0\}.$$

ii) A L_2 projector $B_h: L_2(\Omega) \to U_{0h}$ defined by $(B_h w, u_h) = (w, u_h)$ for all $u_h \in U_h$, satisfies the inequality (see [30-35])

$$||B_h w|| \leq \aleph ||B_h u|| \ \text{for all} \ u \in H_0^1(\Omega), \ \aleph > 0.$$

iii) Interpolation theory refers to the strategy for estimating the extent of generating new data points within a known discrete set of points.

iv) A Ritz projection with time dependent $\Psi_h(t)$: $H_0^1(\Omega) \to U_{0h}$ is defined by

$$(\Theta(.,t) \times B_h \Psi_h(t), u_h) = (\Theta(.,t) B_h w, B_h u_h),\ u \in H_0^1(\Omega), u_h \in U_h, t > 0.$$

v) The finite element orthogonal collocation method (FEOCM) as developed and implemented by Mamadu *et al*. [18] for the approximate solution of (1.1) is given in its abstract sense as

$$\left({}_0^R D_t^{\alpha}\ u\ (.,t_j), \wp\right) + B_h\ U_h = G_h f,\ t > 0, \quad (2.1)$$

With $(A_h V_h\ ,\beta) = (u_t,\beta) - (u_x,\beta_x),\ \alpha \in U_h,\ H_h : H \to U_h$ defined by

$$(B_h U_h,\ \wp) = \left(\frac{\partial u}{\partial t}, \wp\right) - \left(\frac{\partial u}{\partial x}, \frac{\partial \wp}{\partial x}\right),\ \wp \in U_h.$$

Again, $B_h\!: B \to U_h$ defined by

$$(B_h U_h,\ \wp) = (u, \wp),\ \forall\ u \in H_0^1,\ u \in L_2,$$

such that $\|B_j\| \le \aleph^{\alpha-2}\ sup_{t\in[0,T]} \left\|u''\left(t_j(1-t)\right)\right\|$ for n= $\frac{j}{t_j}$, $j = 1(2)n$.

The approximate solution to (2.1) is obtained by substituting (2.2) into (2.1)

$$u = U_j \approx U_h\ (t_j) = \sum_{j=1}^{N-1} a_j \phi_j(.,t), \quad (2.2)$$

where $\phi_j(.,t)$, $j = 0(1)N$, are sub-basis functions (see [30]), and $a_j's$ are constant parameters to be determined via interpolation and collocation for $N > 1$.

We consider the following Lemmas for the purpose of establishing the theoretical analysis of the super-convergence and stability of (2.1).

Lemma 2.1 [36]. Suppose that $u \in H_0^1(\Omega)$, then for $u_h \in U_h$, there exist

$$\left((B_h(u - w_h u), B_h U_h\right)_h \le \aleph h |u|_1 \|u_h\|_h,$$

where $\|.\|_h = \sqrt{\sum_j |.|_{1,j}^2}$, $\|.\|_0$ is L_2 −norm and $\aleph$ is a constant free of h and u.

Lemma 2.2 [37]. Let $\phi_n \ge 0$ for $n = 1(2)\infty$, $\phi_0 = 0, a > 0$, satisfying $\phi_n + \sum_{j=1}^{n-1} b_{n,j}\,\phi_j \le a$, then

$$\phi_n \le \aleph u^{-\alpha} a,$$

where $\aleph > 0$, and ϕ_n, $n = 0(1)N$, are sub-basis functions established in [30].

Lemma 2.3 [37]. For each $0 < t \le T$ with $u_t(x,t) \in L_2(\Omega)$ and $Q^n = D_t^{\alpha} u(t_n) - \mathfrak{H}$, then $\|Q^n\|_0 \le \aleph \max_{t\in[0,T]} \|u_t(.,t)\|_0 u^{2-\alpha}$ holds for $\mathfrak{H} = \frac{u^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n-1} b_j \partial_t\, u^{n-j}$.

Lemma 2.4 [36]. Suppose u is the analytic solution of (2.1), and $\in H_0^1(\Omega) \cap H^1(\Omega)$, $u_h \in U_h$, then we have,

$$\left|\sum_j \int_{\partial_j} \frac{\partial u}{\partial n} u_h ds\right| \le \aleph h |u|_1 \|u_h\|_h,$$

where the unit norm is denoted by n on ∂_j.

Lemma 2.5 [38, 39]. Let $\{\varkappa_n\}_{n=0}^N$ be functions on Ω, then we have,

$$\left(\varkappa_n, \sum_{j=0}^{n} b_{j,n}\,\varkappa_n\right) = \frac{1}{2}\left(\|\varkappa_n\|_0^2 + \sum_{j=0}^{n-1} b_{j,n}\,\|\varkappa_n\|_0^2 - \sum_{j=1}^{n-1} b_{j,n} \left\|e_j - e_n\right\|_0^2\right).$$

3. Stability and Super-convergence Analysis

In this section, we state and prove relevant theorems for the analysis of convergence and stability of the fully-discretized scheme (2.1). Basically, Theorem 3.1 shows that the scheme (2.1) is stable unconditionally.

Theorem 3.1. Suppose u_n is the approximate solution of (2.1), then there exist

$$\|u_n\|_0 \le \|u_0\|_0 + \left\{\max\left\{\Gamma(2-\alpha), \frac{\mathrm{T}^{\alpha}}{(1-\alpha)}\right\}\right\} \sup_{0<t\le T} \|f(t)\|_0.$$

Proof. Substituting $u_h = u_n$ in (2.1) gives us

$$(D_t^{\alpha}\ u_n, u_n)_h + \|\beta_h\|_0^2 = (f_n, u_n)_h\ . \quad (3.1)$$

By definition,

$$(D_t^{\alpha}\ u_n, u_n)_h = \frac{T^{-\alpha}}{\Gamma(2-\alpha)} \|u_n\|_0^2 + \frac{T^{-\alpha}}{\Gamma(2-\alpha)} \left(\sum_{j=0}^{n-1} b_{nj}\ \ u_j, u_n\right)_h. \quad (3.2)$$

Using the Cauchy-Schwarz inequality, (3.2) and $\|B_h u_n\|_0^2 \ge 0$, $b_{nj} < 0$ with $j \in [0,n]$, we have

$$\|u_n\|_0^2 + \left(\sum_{j=0}^{n-1} b_{nj}\ \ u_j, u_n\right)_h \le T^{\alpha}(f_n, u_n)\Gamma(2-\alpha) \le T^{\alpha} \|f_n\|_0 \|u_n\|_0 - \sum_{j=0}^{n-1} b_{nj}\ \ \|u_j\|_0 \|u_n\|_0.$$

Using Lemma 2.2, we have that

$$\le \|u_0\|_0 + \{\max\{\Gamma(2-\alpha),\}\} \sup_{0<t\le T} \|f(t)\|_0,$$

which completes the proof.

Theorem 3.2. Let $U(t_n)$ and $u(t_n)$ be solutions of (1.1) and (2.1), respectively. Now, for any $0 < t \le T$, $u(.,t) \in H^1(\Omega)$, $u_t(.,t) \in H^1(\Omega)$, $u_t(.,t) \in L_2(\Omega)$, then for each $n \in [1,N]$, we have

$$\|U_n - u_n\| = O(h + T^{2-\alpha}),\ \|U_n - w_h u_n\|_h = O(h + T^{2-\alpha}).$$

Proof. Let us denote

$$(u_n - w_h u_n) + (w_h u_n - U_n) = \varkappa_n + e_n, \ \|\phi(t)\|_{L_\infty\left(H^1(\Omega)\right)} = \max_{t\in(0,T]} |\phi(t)|_n.$$

Thus, the error equation from (1.1) and (2.1) is given as

$$(D_t^\alpha \ u_h, e_n)_h + B_h u_h = -(D_t^\alpha \ \varkappa_n, u_h)_h - (\varkappa_n, u_h)_h - (R_n, u_h)_h + \sum_j \int_{\partial_j} \frac{\partial u_n}{\partial n} u_h ds. \tag{3.1}$$

Let $u_h = e_n$ in (3.1), then we have,

$$(D_t^\alpha \ e_n, e_n)_h + \|B_h e_n\|_0 = -(D_t^\alpha \ \varkappa_n, e_n)_h - (\varkappa_n, e_n)_h - (R_n, e_n)_h + \sum_j \int_{\partial_j} \frac{\partial u_n}{\partial n} e_n ds \tag{3.2}$$

Using the definition of $D_t^\alpha \ e_n$ and (3.2), we obtain the equality,

$$\|e_n\|_0 + T^\alpha \Gamma(2-\alpha)\|B_h e_n\|_0 = -\sum_{j=0}^{n-1}\left(b_{nj} \ \ e_j, e_j\right)_h - T^\alpha\Gamma(2-\alpha)(D_t^\alpha \ \varkappa_n, e_n)_h + \ T^\alpha\Gamma(2-\alpha)\sum_j \int_{\partial_j} \frac{\partial u_n}{\partial n} e_n ds - T^\alpha\Gamma(2-\alpha)(\varkappa_n, e_n)_h - \ T^\alpha\Gamma(2-\alpha)(R_n, e_n)_h = \sum_{i=1}^{2} N_i. \tag{3.3}$$

Applying the condition $b_{nj} < 0$ for $j \in [0, n]$, together with the Cauchy-Schwarz inequality yields

$$\|N_0\| = \left|-\sum_{j=0}^{n-1}\left(b_{nj} \ \ e_j, e_j\right)_h\right| \le \sum_{j=0}^{n-1} b_{nj}\left\|e_j\right\|_0 \|e_n\|_0. \tag{3.4}$$

Since $n^{1-\alpha} = \sum_{j=0}^{n-1} \acute{b}_j$ and $\left\|\partial_j \varkappa_{n-j}\right\|_0 = \left\|\frac{1}{T}\int_{t_{n-j-1}}^{t_{n-j}} \varkappa_t dt\right\| \le g h \|u_t\|_{L_\infty\left(H^1(\Omega)\right)}$, we have

$$\|N_1\| = |-T^\alpha\Gamma(2-\alpha)(D_t^\alpha \ \varkappa_n, e_n)_h| \le g\Gamma(2-\alpha)T^{1-\alpha} h \|u_t\|_{L_\infty\left(H^1(\Omega)\right)} \|e_n\|_0. \tag{3.5}$$

Applying Lemma 2.4, we obtain

$$\|N_2\| = \left|-T^\alpha\Gamma(2-\alpha)\sum_j \int_{\partial_j} \frac{\partial u_n}{\partial n} e_n ds\right| \le g\Gamma(2-\alpha)T^\alpha h \|u_n\|_1 \|e_0\|_0. \tag{3.6}$$

By Lemma 2.1, we have

$$\|N_3\| = |-T^\alpha\Gamma(2-\alpha)(\varkappa_n, e_n)_h| \le g\Gamma(2-\alpha)T^\alpha h \|u_n\|_1 \|e_n\|_0 \tag{3.7}$$

Applying Lemma 2.3, we get

$$\|N_4\| = |-T^\alpha\Gamma(2-\alpha)(R_n, e_n)_h| \le g\Gamma(2-\alpha) \max_{t\in[0,T]} \|u_{tt}(.,t)\|_0 T^2 \|e_n\|_0 \tag{3.8}$$

Using Lemma 2.2, combing (3.2) – (3.8) and omitting the term $T^\alpha\Gamma(2-\alpha)\|B_h e_n\|_0$, we get

$$\|e_n\|_0 = O(h + T^{2-\alpha}). \tag{3.9}$$

The error estimate defined in L_2 is defined via interpolation theory to obtain

$$\|U_n - u_n\|_0 = O(h + T^{2-\alpha}).$$

We estimate $\|e_n\|_h$ next by choosing $u_h = D_t^\alpha e_n$ in (3.1) to obtain

$$\|D_t^\alpha e_n\|_0 + B_h e_n = -(D_t^\alpha \ \varkappa_n, e_n)_h - (\varkappa_n, e_n)_h - (R_n, e_n)_h + \sum_j \int_{\partial_j} \frac{\partial u_n}{\partial n} D_t^\alpha e_n ds. \tag{3.10}$$

Using Lemma 2.5, (3.10) can be written as

$$\|D_t^\alpha e_n\|_0 + \frac{T^{-\alpha}}{2\Gamma(2-\alpha)}\left(\|B_h e_n\|_0 + \sum_{j=0}^{n-1} b_{nj} \ \ \left\|B_h e_j\right\|_0 - \sum_{j=0}^{n-1} b_{nj} \ \ \left\|B_h e_j - B_h e_n\right\|_0\right). \tag{3.11}$$

Using the interpolation theory and Cauchy-Schwarz inequality, we get

$$\|-(D_t^\alpha \varkappa_n, D_t^\alpha e_n)_h\| \le \|D_t^\alpha \varkappa_n\|_0 + \frac{1}{2}\|D_t^\alpha e_n\|_0 \le g h T^{2(1-\alpha)} \|u_t\|_{L_\infty\left(H^1(\Omega)\right)} + \frac{1}{2}\|D_t^\alpha e_0\|_0. \tag{3.12}$$

Combining Lemma 2.1 and 2.4, we obtain

$$|-(B_h \varkappa_n, B_h D_t^\alpha e_n)_h| + \left|\sum_j \int_{\partial_j} \frac{\partial u_n}{\partial n} D_t^\alpha e_n ds\right| \le g h \|u_n\|_0 + \frac{1}{2}\|D_t^\alpha e_n\|_0. \tag{3.13}$$

Also, by Lemma 2.3, there exists

$$|-(R_h \varkappa_n, B_h D_t^\alpha e_n)_h| \le g \max_{t\in[0,T]} \|u_{tt}(.,t)\|_0 + \frac{1}{2}\|D_t^\alpha e_n\|_0 \|e_n\|_0. \tag{3.14}$$

Combining (3.11) – (3.14) and omitting the term $\sum_{j=0}^{n-1} b_{nj} \ \ \left\|B_h e_j - B_h e_n\right\|_0$, we obtain

$$\|B_h e_n\|_0 \le \sum_{j=0}^{n-1} b_{nj} \ \ \|B_h e_n\|_0 + g h T^\alpha \left(\|u_{tt}\|_{L_\infty\left(H^1(\Omega)\right)}\right) + g \max_{t\in[0,T]} \|u_{tt}(.,t)\|_0 T^\alpha. \tag{3.15}$$

Combining (3.15) and Lemma 2.2, the superclose result expressed in H^1- norm is given as

$$\|e_n\|_h = O(h + T^{2-\alpha}). \tag{3.16}$$

By the interpolation theory and (3.16), we obtain

$$\|U_n - u_n\|_0 = O(h + T^{2-\alpha}),$$

which completes the proof.

Remark: Discussions on the need for numerical results and satisfactory discussion of results in regard to this paper can be seen in Mamadu *et al.* [18].

4. Conclusions

In this paper, we have successfully shown that the approximate solutions of the time-fractional telegraph equation using the FEOCM are unconditionally stable and also exhibit super-convergence. To be precise, the optimal error estimated to guarantee super-convergence as expressed in the norm is $O(h + T^{2-\alpha})$.

REFERENCES

[1] Mamadu E.J., Njoseh I.N., Ojarikre H.I., “Space Discretization of Time-Fractional Telegraph Equation with Mamadu-Njoseh Basis Functions,” Applied Mathematics, Vol. 13, no. 9, pp. 760-773, 2022.

[2] Mamadu E.J., Ojarikre H.I., Njoseh I.N., “Convergence Analysis of Space Discretization of Time-Telegraph Equation,” Mathematics and Statistics, Vol. 11, no. 2, pp. 245-51, 2023a. DOI: 10.13189/ms.2023.110202

[3] Mamadu E.J., Ojarikre H.I, Njoseh I.N., “An Error Analysis of Implicit Finite Difference Method with Mamadu-Njoseh Basis Functions for Time Fractional Telegraph Equation,” Asian Res. J. Math., Vol. 19, no. 7, pp. 20-30, 2023b.

[4] Kamran M. U., Ali, A. “On the Approximation of Time-Fractionaal Telegraph Equations using Localized Kernel-Based Method,” Adavances in Differential Equations, Vol. 305, 2018.

[5] Wei L., Liu, L., Sun, H., “Numerical Methods for Solving the Time-Fractional Telegraph Equation,” Tawaiwanese J. Math., Vol. 22, no. 6, pp. 1509-1528, 2018.

[6] Ahmad, J., Mohyud-Din, S.T., “An Efficient Algorithm for some Highly Nonlinear Fractional PDEs In Mathematical Physics,” PLoS ONE, Vol. 9, no. 12, pp. 1–17, 2014.

[7] Merdan, M., Gökdogan, A., Yildirim, A., Mohyud-Din, S.T., “Solution of Time-Fractional Generalized Hirota–Satsuma Coupled Kdv Equation by Generalised Differential Transformation Method,” Int. J. Numer. Methods Heat Fluid Flow, Vol. 23 no. 5, pp. 927–940, 2013.

[8] Mohyud-Din, S.T., Bibi, S., Ahmed, N., Khan, U., “Some Exact Solutions of the Nonlinear Space-Time Fractional Differential Equations,” Waves Random Complex Media, Vol. 29, no. 4, pp. 645–664, 2019.

[9] Mohyuddin, S.T., Asad Iqbal, M., Hassan, S.M., “Modified Legendre Wavelets Technique for Fractional Oscillation Equations,” Entropy, Vol. 17, no. 10, pp. 6925–6936, 2015.

[10] Bu, W., Xiao, A., “An H–P Version of the Continuous Petrov–Galerkin Finite Element Method for Riemann–Liouville Fractional Differential Equation with Novel Test Basis Functions,” Numer. Algorithmsno, Vol. 81, no. 2, pp. 529–545, 2019.

[11] Natalia, K., “Error Analysis of the L1 Method on Graded and Uniform Meshes for a Fractional-Derivative Problem in Two and Three Dimensions,” Math. Comput., Vol. 88, no. 319, pp. 2135–2155, 2019.

[12] Wang, F., Chen, H., Wang, H., “Finite Element Simulation and Efficient Algorithm for Fractional Cahn–Hilliard Equation,” J. Comput. Appl. Math., Vol. 356, pp. 248–266, 2019.

[13] Njoseh, I.N., Ayoola, E.O., “On the Finite Element Analysis of the Stochastic Cahn-Hilliard Equation,” Journal of Mathematics and Computer Science, Vol. 21, pp. 47-53, 2008.

[14] Chen, J., Liu, F., Liu, Q., Chen, X., Anh, V., Turner, I., Burrage, K., “Numerical Simulation for the Three-Dimension Fractional Sub-Diffusion Equation,” Appl. Math. Model, Vol. 38, no. (15–16), pp. 3695–3705, 2014.

[15] Li, H., Wu, X., Zhang, J., “Numerical Solution of the Time-Fractional Sub-Diffusion Equation on an Unbounded Domain in Two-Dimensional Space,” East Asian J. Appl. Math., Vol. 7, no. 3. pp. 439–454, 2017

[16] Li, Z., Yan, Y., “Error Estimates of High-Order Numerical Methods for Solving Time Fractional Partial Differential Equations,” Fract. Calc. Appl. Anal., Vol. 21, no. 3, pp. 746–774, 2018.

[17] Mohyud-Din, S.T., Akram, T., Abbas, M., Ismail, A.I., Ali, N.H.M., “A Fully Implicit Finite Difference Scheme Based on Extended Cubic B-Splines for Time Fractional Advection–Diffusion Equation,” Adv. Differ. Equ., Vol. 109, no. 17, 2018.

[18] Mamadu, E., Ojarikre, H., Maduku, E., “Finite Element Orthogonal Collocation Approach for Time Fractional Telegraph Equation with Mamadu-Njoseh Polynomials,” Journal of Applied Mathematics and Physics, Vol. 11, pp. 2585-2596, 2023.

[19] Lin, Y., Xu, C., “Finite difference/spectral approximations for the time-fractional diffusion equation,” J. Comput. Phys. vol. 225, no. 2, pp. 1533–1552, 2007.

[20] An, N., Huang, C., Yu, X., “Error Analysis of Direct Discontinuous Galerkin Method for Two-Dimensional Fractional Diffusion-Wave Equation,” Appl. Math. Comput., Vol. 349, pp. 148–157, 2019.

[21] Huang, C., An, N., Yu, X., “A Fully Discretized Direct Discontinuous Galerkin Method for the Fractional Diffusion-Wave Equation,” Appl. Anal., Vol. 97, no. 4, pp. 659–675, 2018.

[22] An, N., Huang, C., Yu, X., “Error Analysis of Discontinuous Galerkin Method for the Time Fractional Kdv Equation with Weak Singularity Solution,” Discretized Contin. Dyn. Syst., Ser., Vol. 25, no. 1, pp. 321–334, 2020.

[23] Huang, C., An, N., Yu, X., "A Local Discontinuous Galerkin Method for Time-Fractional Diffusion Equation with Discontinuous Coefficient," Appl. Numer. Math., Vol. 151, pp. 367–379, 2020.

[24] Jia, J., Wang, H., "A Fast Finite Volume Method for Conservative Space-Time Fractional Diffusion Equations Discretized on Space-Time Locally Refined Meshes," Comput. Math. Appl., Vol. 78, no. 5, pp. 1345–1356, 2019.

[25] Zhao, J., Li, H., Fang, Z., Liu, Y., "A Mixed Finite Volume Element Method for Time-Fractional Reaction-Diffusion Equations on Triangular Grids," Mathematics 7, Vol. 600, 2019.

[26] Vieira, N., Rodrigues, M.M., Ferreira, M., "Time-Fractional Telegraph Equation of Distributed Order in Higher Dimensions with Hilfer Fractional Derivatives," Electronic Reseacrh Archieves, Vol. 30, no. 10, pp. 3494-3731, 2022.

[27] Huang, J., Cen, Z., Xu, A., "An Efficient Numerical Method for a Time-Fractional Telegraph Equation," Math Biosc Eng., Vol. 19, no. 5, pp. 4672-4672, 2022.

[28] Tasbozan, O. and Esen, A., "Collocation Solutions for the Time-Fractional Telegraph Equation using Cubic B-Spline Finite Elements," Seria Matematica-informatica, Vol. 47, no. 2, pp. 131-144, 2019.

[29] Liu, R., "Fractional Difference Approximations for Time Telegraph Equation," Journal of Applied Mathematics and Physics, 2018, doi: 10.42371.jamp.2018.71029.

[30] Van den Nieuwenhof, B., Coyette, J.P., "Modal approaches for the Stochastic Finite Element Analysis of Structures with Material and Geometric Uncertainties," Comput. Methods Appl. Mech. Engrg., Vol. 192, pp. 3705–3729, 2003.

[31] Papadopoulos, V., Papadrakakis, M., "Finite Element Analysis of Cylindrical Panels with Random Initial Imperfections," J. Engrg. Mech. (ASCE), Vol. 130, pp. 867–876, 2004.

[32] Witteveen, J.A.S., Bijl, H., "An Alternative Unsteady Adaptive Stochastic Finite Elements Formulation based on Interpolation at Constant Phase," Comput. Methods Appl. Mech. Engrg., Vol. 198, pp. 578–591, 2008.

[33] Nedoma, J., "The Finite Elements Solution to Elliptic and Parabolic Equations using Simplicial Isoparametric Elements," *R.A.I.R.O.*, Vol. 13, no. 3, pp. 257–289, 1979.

[34] Moës, N., Dolbow, J., Belytschko, T., "A Finite Element Method for Crack Growth without Remeshing," Int. J. Numer. Methods Engrg., Vol. 46, pp. 131–150, 1999.

[35] Zhou, Z., Tan, Z., "Finite Element Approximation of Optimal Control Problem Governed by Space Fractional Equation," J. Sci. Comput., Vol. 78, no. 3, pp. 1840–1861, 2019.

[36] Jiang, Y., Ma, J., "High-Order Finite Element Methods for Time-Fractional Partial Differential Equations," J. Comput. Appl. Math., Vol. 235, pp. 3285–3290, 2011.

[37] Wang, H., Yang, D., Zhu, S., "Accuracy of Finite Element Methods for Boundary-Value Problems of Steady-State Fractional Diffusion Equations," J. Sci. Comput., Vol. 70, pp. 429–499, 2017.

[38] Rannacher, R., Turek, S., "Simple Nonconforming Quadrilateral Stokes Element, Numer.Methods Part D. E., Vol. 8, pp. 97–111, 1992.

[39] Hu, J., Shi, Z.C., "Constrained Quadrilateral Nonconforming Rotated Q1 Element," J. Comput. Math., Vol. 23, pp. 561–586, 2005.