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# g-inverses in Ternary Semiring

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**Abstract** Ternary algebraic systems represent a natural extension of algebraic structures, providing a greater grasp of their features and avenues for further development. Multiplicative semigroups over a field are non-regular, meaning that the regularity equation $axa = a$ is not always solvable. When $x$ exists, it's referred to as a regular. The regularity requirement is a linear condition that solves linear equations, which makes regular rings significant in many areas of mathematics, particularly in matrix theory. The current state of generalized inverses encompasses many different mathematical fields, including semigroups, operator theory, $c^*$-algebras, matrix theory, and semirings. Applications for them can be found in many fields, including robotics, graphics, cryptography, coding theory, Markov chains, linear estimation, differential and difference equations, and graphics. For elements of Ternary semiring, the existence of the generalized inverse is examined. The most general 1- inverse and 1–2 inverse are found for an element over a regular ternary semiring. We looked into the properties and characterization of the g-inverse in ternary Semiring and some fascinating characteristics of the left and right cosets in Partial ordered ternary semiring in this article. Mainly, we investigated the g-inverses using Principal ideals (left and right coset) and found some results in ordered ternary semiring and ternary semiring.

**Keywords** Semirings, g-inverses, Ideal, Ordered Ternary Semiring

## 1 Introduction

Vandiver first proposed the idea of semiring in 1934. Semiring arouse naturally in many branches of Mathematics. Since 1950, the theory of semiring has been developing. Many studies of semirings have been conducted, either in an effort to expand on methods from ring theory or semigroup theory. More information about semiring can be found in [1] by Golan.
In 1932, Lehmer [2] explored ternary algebraic system theory. He studied "triplexes," which are ternary algebraic systems that finally turn out to be ternary groups. A generalization of ternary rings are called ternary semirings, as presented by Dutta *et al* [3]. Ternary semiring is a superset of semiring. Negative integers form a ternary semiring but they do not form a semiring.
An ordered ternary semigroup was introduced by Iampan Aiyared [4], and the theory of ordered Ternary semiring was researched by D. Madhusudanarao *et al*. [5]. In 1920, E.H. Moore established the concept of the generalized inverse of a matrix, defining the unique generalized inverse through the use of matrices projectors. Penrose demonstrated in 1955 that Moore's inverse is the unique matrix that can satisfy four equations. [6]. The notion of regular element was first introduced by Von Neumann in 1936 [7].

In this paper an attempt is made to characterize g-inverses in ternary semiring by using principal ideals( left and right cosets) as a generalization of Hartwig [9] and also we attain some results of partially ordered ternary semiring.

## 2 Preliminaries

In this work, $\mathcal{T}$ represents the Ternary semiring; it is abbreviated as T.S.R. throughout.

**Definition 2.1** *The term "**ternary semiring**" refers to $\mathcal{T} \neq \emptyset$, a binary addition, and a ternary multiplication called juxtaposition if $(\mathcal{T}, +)$ is an commutative semigroup such that the following subsequent holds :*

*(i)* $(stu)vw = s(tuv)w = st(uvw)$,

*(ii)* $(s + t)uv = suv + tuv$,

*(iii)* $s(t + u)v = stv + suv$,

*(iv)* $st(u + v) = stu + stv$, *for all* $s, t, u, v, w$ *in* $\mathcal{T}$.

***Note:1*** *An example of a ternary semiring in nature is* $z^-$, *which has ternary multiplication and regular addition.*

**Definition 2.2** *For* $a \in \mathcal{T}$ *if* $\exists\, b \in \mathcal{T} \ni$ :

$$ababa = a \tag{2.1}$$

$$babab = b \tag{2.2}$$

*If the equation (2.1) is satisfied by* $b \in \mathcal{T}$, *then* $a \in \mathcal{T}$ *is considered regular. Such* $b$ *is called generalizd or 1-inverse of the element* $'a'$.
*If* $b \in \mathcal{T}$ *satisfies the equation (2.2), it is called 2-inverse of* $'a'$, *here* $a$ *is called anti-regular.*

The both equations (2.1, 2.2) satisfied by a single element $b$, represented by $a^+$, is the 1-2 inverse of $'a'$.
In Definition (2.2), When $'b'$ is unit, $'a'$ is referred to as unit regular.

**Definition 2.3** *For any* $t_1, t_2 \in \mathcal{T}$ *and* $i \in I$, *an additive subsemigroup* $I$ *of* $\mathcal{T}$ *is considered a left ( right, lateral) ideal of* $\mathcal{T}$ *if* $t_1t_2i$ *(similarly* $it_1t_2, t_1it_2$) $\in I$. $I$ *is said to be as an ideal* $\mathcal{T}$ *if* $I$ *is both a left, right, and lateral ideal of* $\mathcal{T}$.
*The* $a \in \mathcal{T}$ *generates the principle left ideal, principal right ideal, and principal lateral ideal of* $\mathcal{T}$, *which we refer to as such.*

*(i)* $\mathcal{T}\mathcal{T}a = \{xya \mid x, y \in \mathcal{T}\}$

*(ii)* $a\mathcal{T}\mathcal{T} = \{axy \mid x, y \in \mathcal{T}\}$

*(iii)* $\mathcal{T}a\mathcal{T} = \{xay \mid x \,\&\, y \in \mathcal{T}\}$

**Definition 2.4** *For every* $a \in \mathcal{T}$, $\mathcal{T}$ *is called Boolean T.S.R if* $a^3 = a$.

**Definition 2.5** *An involution* $*$ *of* $\mathcal{T}$ *is an involutory anti-automorphism,*
*i.e.,* $(s^*)^* = s$ , $(s + b)^* = s^* + b^*$, $(sbc)^* = c^*b^*s^*$, $s^* = 0$ *if* $s = 0$ $\forall s, b, cin\ \mathcal{T}$

**Definition 2.6** *When* $a\mathcal{T}b\mathcal{T}a = 0 \Rightarrow a = 0$ *or* $b = 0$, *a Ternary Semiring* $\mathcal{T}$ *is referred to as prime.*

**Definition 2.7** *if* $a\mathcal{T}a\mathcal{T}a = 0 \Rightarrow a = 0$, *then* $\mathcal{T}$ *is called Semiprime.*

**Definition 2.8** *A ternary semiring* $\mathcal{T}$ *is referred to ordered ternary semiring if* $\mathcal{T}$ *admits a compatible relation* $\leq$, *meaning that* $\leq$ *is a partial order on S that satisfies the below contions:*

*(i)* $\mathcal{T}$ *is a partially ordered semigroup with respect to addition.*

*(ii)* $\mathcal{T}$ *is a partially ordered semigroup with respect to multiplication.*

*It is indicated by* $(\mathbb{T}, +, ., \leq)$.

**Definition 2.9** *The ordered Ternary Semiring* $\mathbb{T}$ *is called regular if* $a \leq axaxa$ $\forall a$ *belongs to* $\mathbb{T}$.

**Definition 2.10** *If there is a* $y \in \mathbb{T}$ *such that* $y \leq yayay$, *then* $a$ *of* $\mathbb{T}$ *is called anti-regular.*

In this paper $\mathcal{T}$ denotes the Ternary Semiring with multiplicative cancellation.

## 3 g-inverses in Ternary Semiring

The following results are seen similar as work of hartwig but it appeared for semiring, we have attempted these results for ternary semiring.

**Theorem 3.1** *[9] Suppose* $\mathcal{T}$ *is regular and* $\jmath, b, c \in \mathcal{T}$. *Then the outcomes listed below are valid*

*(i)* $b = y\mathfrak{a}^-\jmath \Leftrightarrow b = b\mathfrak{a}^-\jmath\mathfrak{a}^-\jmath \Leftrightarrow \mathcal{T}eb \subseteq \mathcal{T}\mathfrak{a}^-\jmath$

*(ii)* $c = \jmath a^-x \Leftrightarrow c = \jmath a^-\jmath a^-c \Leftrightarrow ce\mathcal{T} \subseteq \jmath a^-\mathcal{T}$

*(iii)* $b\jmath^2c^-\mathcal{T} = b\jmath\mathcal{T} \Rightarrow \jmath^2c^- = b^-$,
$\mathcal{T}c^-\jmath^2b = \mathcal{T}\jmath b \Rightarrow c^-\jmath^2 = b^-$

*(iv)* $\left.\begin{aligned}\jmath(qps)^-\ \mathcal{T} &= \jmath q^-\mathcal{T}\\ \mathcal{T}q^-\jmath &= \mathcal{T}p\jmath\end{aligned}\right\} \Rightarrow (qps)^- = q^- = \jmath^-$

*(v)* $pwq = \jmath^- \Rightarrow \begin{cases} w = (qap)^-\ \ q\jmath pw\mathcal{T} = q\jmath\mathcal{T} \\ \mathcal{T}q\jmath = \mathcal{T}a^-\jmath \end{cases}$

*(vi)* $bx = \jmath^- \Leftrightarrow (\jmath bx)^3 = \jmath bx$ *and* $\jmath b\mathcal{T} = \jmath bx\jmath\mathcal{T}$

*(vii)* $\mathcal{T}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-} = \mathcal{T}\jmath\mathfrak{c}^{-} \Leftrightarrow \mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathcal{T} = \mathfrak{c}^{-}\jmath\mathcal{T}$

*Proof.* (i)

$$\text{Let } \ b = y\mathfrak{a}^{-}\jmath \tag{3.1}$$

$$\begin{aligned} \Rightarrow b\mathfrak{a}^{-}\jmath\,\mathfrak{a}^{-}\,\jmath &= y\mathfrak{a}^{-}\,\jmath\,\mathfrak{a}^{-}\,\jmath\,\mathfrak{a}^{-}\,\jmath \\ &= y\mathfrak{a}^{-}\,\jmath \\ &= b \\ \therefore b\mathfrak{a}^{-}\jmath\mathfrak{a}^{-}\jmath &= b \\ \text{so that } \ \mathcal{T}eb &\subseteq \mathcal{T}\mathfrak{a}^{-}\jmath \\ \mathcal{T}eb &\subseteq \mathcal{T}\mathfrak{a}^{-}\jmath \\ b &= b\mathfrak{a}^{-}\jmath\mathfrak{a}^{-}\jmath \\ b &= y\mathfrak{a}^{-}\jmath \end{aligned}$$

(ii) In similar manner of (i) we can prove.

(iii)

$$\begin{aligned} b\jmath^2c^{-}\mathcal{T} &= b\jmath\mathcal{T} \\ b\jmath^2c^{-}(c(\jmath^2)^{-}b^{-})^{-}b &= b\jmath b\jmath b \\ b\jmath^2c^{-}b\jmath^2c^{-}b &= b \\ \jmath^2c^{-} &= b^{-} \\ \text{Similarly,} \qquad Tc^{-}\jmath^2b &= T\jmath b \\ b(b^{-}(\jmath^2)^{-}c)^{-}c^{-}\jmath^2b &= b\jmath b\jmath b \\ bc^{-}\jmath^2bc^{-}\jmath^2b &= b \\ c^{-}\jmath^2 &= b^{-} \end{aligned}$$

(iv)

Let $\jmath(qps)^{-}\mathcal{T} = \jmath q^{-}\mathcal{T}$

$\jmath(qps)^{-}(qps\jmath^{-})\jmath = \jmath q^{-}(q\jmath^{-})^{-}\jmath, \mathit{for\ some}\ q^{-}$ in $\jmath\{1\}$

$\jmath(qps)^{-}\jmath(qps)^{-}\jmath = \jmath$

$$\therefore \ (qps)^{-} = \jmath^{-} \tag{3.2}$$

$$\begin{aligned} \mathcal{T}q^{-}\jmath &= \mathcal{T}p\jmath \\ \jmath q^{-}\jmath q^{-}\jmath &= \jmath p\jmath p\jmath \end{aligned}$$

$$q^{-} = \jmath^{-} \tag{3.3}$$

From (3.2) and (3.3) we get

$(qps)^{-} = q^{-} = \jmath^{-}$

(v)

$$\text{Let } \ pwq = \jmath^{-} \Rightarrow \jmath pwq\jmath pwq\jmath = \jmath \tag{3.4}$$

$xa^{-}\jmath pwq\jmath pwq\jmath = xa^{-}\jmath,$ (for certain $x$ in T.S.R)

$\therefore \mathcal{T}q\,\jmath = \mathcal{T}a^{-}\jmath$

$$\text{From(3.4) } q\jmath pwq\jmath pwq\jmath p = q\jmath p \Rightarrow \ \ w = (q\jmath p)^{-} \tag{3.5}$$

and $q\jmath pw\mathcal{T} = q\jmath\mathcal{T}$

(vi) Let $bx = \jmath^{-}$

$$\jmath = \jmath bx\jmath bx\jmath \tag{3.6}$$

$\jmath\, b\, x = \jmath\, b\, x\, \jmath\, b\, x\, \jmath\, b\, x$

$\jmath\, b\, x = (\jmath\, b\, x)^3$ and $\jmath\, b\, \mathcal{T} = \jmath\, b\, x\, \jmath\mathcal{T}$

Conversely, if $\jmath\, b\, \mathcal{T} = \ \jmath\, b\, x\, \jmath\mathcal{T}$

then $\jmath\, b\, x = \jmath\, b\, x\, \jmath\, b\, x\, \jmath\, b\, x$

$\jmath(bx) = \jmath bx\jmath bx\jmath(bx)$

$\jmath = \jmath\, b\, x\, \jmath\, b\, x\, \jmath$

$\therefore \ \ b\, x = \jmath^{-}$

(vii) Let $\mathcal{T}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-} = \mathcal{T}\jmath\mathfrak{c}^{-}$

$$x\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-} = x\jmath\mathfrak{c}^{-} \tag{3.7}$$

$x\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\jmath = x\jmath\mathfrak{c}^{-}\jmath\jmath$

Using cancellation law and Pre multiply "$\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}$"

we have $\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\jmath = \mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\jmath$

$\Rightarrow \mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathcal{T} = \mathfrak{c}^{-}\jmath\mathcal{T}$

Conversly,

$\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathcal{T} = \mathfrak{c}^{-}\jmath\mathcal{T}$

$\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath x = \mathfrak{c}^{-}\jmath x$

$\Rightarrow \jmath\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath x = \jmath\jmath\mathfrak{c}^{-}\jmath x$

$\jmath\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath = \jmath\jmath\mathfrak{c}^{-}\jmath$

Using cancellation law and Post multiply "$\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}$"

$\jmath\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-} = \jmath\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-}$

$\mathcal{T}\jmath\mathfrak{c}^{-}\jmath\mathfrak{c}^{-} = \mathcal{T}\jmath\mathfrak{c}^{-}$

□

**Theorem 3.2** *[8] If $\mathcal{T}$ is Boolean , then $\mathcal{T}$ is regular .*

**Theorem 3.3** *[9] General solution of $lylyl = l$ is*

$$\begin{aligned} &\ell^{-} + i - \ell^{-}l\ell^{-}lil\ell^{-}l\ell^{-} \ (or) \\ &\ell^{-} + (1 - \ell^{-}l\ell^{-}l)i + j(1 - a\ell^{-}l\ell^{-}) \end{aligned} \tag{3.8}$$

here $\ell^{-}$ is any specific generalized inverse of $l$ and $i, j$ are random.

*Proof.* $l = lylyl$

$l = l(\ell^{-} + i - \ell^{-}l\ell^{-}lil\ell^{-}l\ell^{-})l$

$(\ell^{-} + i - \ell^{-}l\ell^{-}lil\ell^{-}l\ell^{-})l$

$l = (l\ell^{-}l + lil - l\ell^{-}l\ell^{-}lil\ell^{-}l\ell^{-}l)$

$(\ell^{-}l + il\ell^{-}l\ell^{-}lil\ell^{-}l\ell^{-}l)$

$l = (l\ell^{-}l + lil - lil)(\ell^{-}l + il - \ell^{-}l\ell^{-}lil)$

$= l$ □

Also $\ell^- l\ell^- l\ell^-$ is also obviously always a 1-2 inverse. Actually, we've

**Lemma 3.1** *[12] The broadest general reflexive inverse of d is*

$$\mathfrak{d}^+ = (\ell^- + i - \ell^- d\ell^- did\ell^- d\ell^-)d(\ell^- + i - \ell^- d\ell^- did\ell^- d\ell^-)d(\ell^- + i - \ell^- d\ell^- did\ell^- d\ell^-) \quad (3.9)$$

*where $\ell^-$ is any generalized inverse and 'i' is random.*

*Proof.* Given (3.9) takes the form $ydydy = y$, where $y$ is a 1-inverse, it's obvious that right hand side of (3.9) is a reflexive inverse.
Conversely, suppose $dydyd = d$ and $ydydy = y$, then it is seen that
$(\ell^- + y - \ell^- d\ell^- dyd\ell^- d\ell^-)d(\ell^- + y - \ell^- d\ell^- dyd\ell^- d\ell^-)d(\ell^- + y - \ell^- d\ell^- dyd\ell^- d\ell^-) = y$ □

Therefore, using equations (3.8) and (3.9), every generalzed inverse and reflexive inverse of $d$ can expressed in terms of a given generalized inverse.

**Theorem 3.4** *[9] Suppose $\mathcal{T}$ is a *-regular T.S.R and $\alpha, d, g$ in T.S.R, If $g = g^3 = g^*$, then*

*(i)* $(\alpha^* g\alpha)^- \alpha^* g = (gg\alpha)^-$

*(ii)* $gd^*(dgd^*)^- = (dgg)^-$

*proof.*

$(i)$*The identity*

$$\alpha^* ggg\alpha[(\alpha^* g\alpha)^-]\alpha^* ggg\alpha[(\alpha^* g\alpha)^-]\alpha^* ggg\alpha = \alpha^* ggg\alpha$$

$$\alpha^* g^* gg\alpha[(\alpha^* g\alpha)^-]\alpha^* ggg\alpha[(\alpha^* g\alpha)^-]\alpha^* ggg\alpha = \alpha^* g^* gg\alpha$$

$$\Rightarrow \quad gg\alpha[(a^* g\alpha)^-]\alpha^* ggg\alpha[(\alpha^* g\alpha)^-]\alpha\alpha^* ggg\alpha = gg\alpha$$

$\therefore \quad [(\alpha^* g\alpha)^-]\alpha^* g$ *is a 1-inverse of* $gg\alpha$.

$(ii)$*Take the identity*

$$dgggd^*(dgd^*)^- dgggd^*(dgd^*)^- dgggd^* = dgggd^*$$

$$dgggd^*(dgd^*)^- dgggd^*(dgd^*)^- dggg^* d^* = dggg^* d^*$$

$$\Rightarrow \quad dgggd^*(dgd^*)^- dgggd^*(dgd^*)^- dgg = dgg$$

$\therefore \quad gd^*(dgd^*)^-$ *is a 1-inverse of* $dgg$

**Lemma 3.2** *[9] If $f \in \mathcal{T}$ is regular, then $f^- ff^- \in f\{1\}$.*

**Theorem 3.5** *[9] If $\mathcal{T}$ is Prime regular T.S.R , then $\imath = f$ iff $\imath^- = \mathfrak{f}^-$ where $\imath, f$ are non zero.*

*Proof.* If $\imath = f$
$\imath = \imath y\imath y\imath$ and $f = fyfyf$
$\Rightarrow y = \imath^-$ and $y = \mathfrak{f}^-$
therefore $y = \imath^- = f^-$
Hence $\imath^- = \mathfrak{f}^-$
converse,

$$f = f\imath^- f\imath^- f = f\imath^- f\imath^- (f\imath^- f\imath^- f)$$

$$f = f(\imath^- f\imath^-)f(\imath^- f\imath^-)f \qquad (\because \ \imath^- f\imath^- \in f\{1\})$$

here, $\imath = \imath\mathfrak{f}^- \imath\mathfrak{f}^- \imath$

$$= \imath(\imath^- f\imath^-)\imath(\imath^- f\imath^-)\imath$$

Since $f = f\imath^- f\imath^- f$

$$f = f[\imath^- + (1 - \imath^- \imath\imath^- \imath)\mathcal{T} + \mathcal{T}(1 - \imath\imath^- \imath\imath^-)]$$

$$f[\imath^- + (1 - \imath^- \imath\imath^- \imath)\mathcal{T} + \mathcal{T}(1 - \imath\imath^- \imath\imath^-)]f$$

$$f = f\imath^- f\imath^- f + f[(1 - \imath^- \imath\imath^- \imath)\mathcal{T} + \mathcal{T}(1 - \imath\imath^- \imath\imath^-)]f[(1 - \imath^- \imath\imath^- \imath)\mathcal{T} + \mathcal{T}(1 - \imath\imath^- \imath\imath^-)]f$$

$$0 = f[(1 - \imath^- \imath\imath^- \imath)\mathcal{T} + \mathcal{T}(1 - \imath\imath^- \imath\imath^-)]$$

$$f[(1 - \imath^- \imath\imath^- \imath)\mathcal{T} + \mathcal{T}(1 - \imath\imath^- \imath\imath^-)]f$$

$$0 = [f(1 - \imath^- \imath\imath^- \imath)\mathcal{T}f + f\mathcal{T}(1 - \imath\imath^- \imath\imath^-)f]$$

$$[(1 - \imath^- \imath\imath^- \imath)\mathcal{T}f + \mathcal{T}(1 - \imath\imath^- \imath\imath^-)f]$$

$$0 = f(1 - \imath^- \imath\imath^- \imath)\mathcal{T}f(1 - \imath^- \imath\imath^- \imath)\mathcal{T}f + f\mathcal{T}(1 - \imath\imath^- \imath\imath^-)f(1 - \imath^- \imath\imath^- \imath)\mathcal{T}f + f(1 - \imath^- \imath\imath^- \imath)\mathcal{T}f\mathcal{T}(1 - \imath\imath^- \imath\imath^-)f + f\mathcal{T}(1 - \imath\imath^- \imath\imath^-)f\mathcal{T}(1 - \imath\imath^- \imath\imath^-)f$$

By Definition (2.6),
$0 = f(1 - \imath^- \imath\imath^- \imath)\mathcal{T}f(1 - \imath^- \imath\imath^- \imath)\mathcal{T}f$
Similarly each terms are become zero.
If $\mathcal{T}$ is prime, then $f \neq 0$
$f(1 - \imath^- \imath\imath^- \imath) = 0 \Rightarrow \ f = f\imath^- \imath\imath^- \imath$
Similarly $(1 - \imath\imath^- \imath\imath^-)f = 0 \Rightarrow \ f = \imath\imath^- \imath\imath^- f$
$\therefore \ \imath\imath^- \imath\imath^- f = f = f\imath^- \imath\imath^- \imath$, and

$$\imath = \imath\imath^- \imath\imath^- \imath$$

$$\imath = \imath\imath^- [\imath\mathfrak{f}^- \imath\mathfrak{f}^- \imath]\imath^- \imath$$

$$= \imath\imath^- \imath[\mathfrak{f}^-]\imath[\mathfrak{f}^-]\imath\imath^- \imath$$

$$= \imath\imath^- \imath(\imath^- f\imath^-)\imath(\imath^- f\imath^-)\imath\imath^- \imath$$

$$= \imath\imath^- \imath\imath^- f(\imath^- \imath\imath^-)f\imath^- \imath\imath^- \imath$$

$$= f(\imath^- \imath\imath^-)f$$

$$= f\imath^- f = f\mathfrak{f}^- f \ \text{(By using Lemma 3.2)}$$

$$\imath = f\imath^- f\imath^- f$$

$\therefore \ \imath$ is equal to $f$.

□

**Theorem 3.6** *[10] Let $\mathcal{T}$ be a semiprime T.S.R and $a \in Reg(\mathcal{T})$. If $\beta \in \mathcal{T}ca \cap a\mathcal{T}a \cap ac\mathcal{T}$ for any $\beta, c \in \mathcal{T}$, then $\beta I(a)\beta I(a)\beta$ is a singleton set.*
*Where $I(a)$ is set of all generalized inverses of $a$.*

*Proof.* Assume that $\exists\ x, y, z\ in\ \mathcal{T}$
such that $\beta = xca = aya = acz$ and let $l^- \in I(a)$.
Then we have that, for any $h \in \mathcal{T}$,

$$\beta I(a)\beta I(a)\beta = \beta(l^- + h - (l^-al^-a)h(al^-al^-))\beta$$
$$(l^- + h - (l^-al^-a)h(al^-al^-))\beta$$
$$= xca(l^- + h - (l^-al^-a)h(al^-al^-))aya$$
$$(l^- + h - (l^-al^-a)h(al^-al^-))acz$$
$$= (xcal^-aya + xcahaya - xcahaya)$$
$$(l^-acz + hacz - l^-al^-ahacz)$$
$$= xcal^-ayal^-acz + xcal^-ayahacz - xcal^-ayal^-al^-ahacz$$
$$+ xcahayal^-acz + xcahayahacz - xcahayal^-al^-ahacz$$
$$- xcahayal^-acz - xcahayahacz + xcahayal^-al^-ahacz$$
$$= xcal^-ayal^-acz$$

This shows that $\beta I(a)\beta I(a)\beta$ is a singleton set. □

**Theorem 3.7** *[9] If $y \in \mathcal{T}$ such that $a - ayaya$ possesses a 1-inverse $z$ if $a$ possess the generalized inverse $(z - yayaz)$*

*Proof.* If $a - ayaya$ is regular, $\exists\ z \in \mathcal{T}$ such that

$$(a - ayaya)z(a - ayaya)z(a - ayaya) = (a - ayaya)$$
$$azaza - azazayaya - azayayaza + azayayazayaya$$
$$-ayayazaza + ayayazazayaya + ayayazayayaza$$
$$-ayayazayayazayaya = (a - ayaya)$$
$$(azaza - azayayaza - ayayazaza + ayayazayayaza)$$
$$(1 - yaya) = a(1 - yaya)$$
$$(azaza - azayayaza - ayayazaza + ayayazayayaza) = a$$
$$a(z - yayaz)a(z - yayaz)a = a$$
$$\therefore\ a \text{ is regular}$$

Conversely,
If $a$ is regular, then $a(z - yayaz)a(z - yayaz)a = a$

$$(azaza - azayayaza - ayayazaza + ayayazayayaza) = a \tag{3.10}$$

$$(azaza - azayayaza - ayayazaza + ayayazayayaza)$$
$$(1 - yaya) = a(1 - yaya)$$
$$azaza - azayayaza - ayayazaza + ayayazayayaza$$
$$-azazayaya + azayayazayaya + ayayazazayaya$$
$$-ayayazayayazayaya) = a - ayaya$$
$$(azaz - azayayaz - ayayazaz + ayayazayayaz)$$
$$(a - ayaya) = (a - ayaya)$$
$$(a - ayaya)z(a - ayaya)z(a - ayaya) = (a - ayaya)$$
$$\therefore a - ayaya \text{ is regular}$$

□

**Theorem 3.8** *[9] Let $a, z, x = a - ayaya$ and $s = (z - yayaz)(1 - ayay) + y$ be elements of $\mathcal{T}$. If $x$ has reflexive inverse $z$ and $a$ has 2-inverse of $y$, then $s$ is 1-2 inverse of $a$.*

*Proof.* Assume

$$[(z - yayaz)(1 - ayay) + y]a[(z - yayaz)(1 - ayay) + y]a$$
$$[(z - yayaz)(1 - ayay) + y] = [(z - yayaz)(1 - ayay) + y] \tag{3.11}$$

L.H.S $= [(z - yayaz)(1 - ayay) + y]a[(z - yayaz)(1 - ayay) + y]a[(z - yayaz)(1 - ayay) + y]$

$= [(z - yayaz)(1 - ayay) + y][a(z - yayaz)$
$(1 - ayay) + ay][a(z - yayaz)(1 - ayay) + ay]$

$= (za - zayaya - yayaza + yayazayaya)(z - yayaz)(a - ayaya)(z - yayaz)(1 - ayay)$
$+(z - yayaz)(1 - ayay)a(z - yayaz)(1 - ayay)ay + (z - yayaz)(1 - ayay)aya(z - yayaz)(1 - ayay)$
$+(z - yayaz)(1 - ayay)ayay + ya(z - yayaz)(1 - ayay)a(z - yayaz)(1 - ayay)$
$+ya(z - yayaz)(1 - ayay)ay + yaya(z - yayaz)(1 - ayay) + y \qquad (\because\ y = yayay)$

$= (zaz - zayayaz - zayayaz + zayayaz - yayazaz + yayazayayaz + yayazayayazyayazayayaz)(az - ayayaz - ayayaz + ayayayayaz)(1 - ayay) + (z - yayaz)(1 - ayay)a(z - yayaz)(ay - ay) + (zaya - zaya - yayazaya + yayazaya)(z - yayaz)(1 - ayay) + (zayay - zayay - yayazayay + yayazayay) + (yazaz - yazayayaz - yazaz + yazayayaz)(1 - ayay) + (yazay - yazay - yazay + yazay) + 0 + y \qquad (\because\ yayay = y)$

$= (zaz - zayayaz - yayazaz + yayazayayaz)(az - azayay - ayayaz + ayayazayay) + y$

$= zazaz - zazazayay - zazayayaz + zazayayazayay - zayayazaz + zayayazazayay + zayayazyayaz - zayayazayayazayay - yayazazaz + yayazazazazayay + yayazazayayaz - yayazazayayazayay + yayazayayazaz - yayazayayazazayay - yayazayayazayayaz + yayazayayazayayazayay + y$

$(\therefore\ zazaz - zazayayaz - zayayazaz + zayayazayayaz = z(a - ayaya)z(a - ayaya)z = z)$

$= (zazaz - zazayayaz - zayayazaz + zayayazayayaz) - (zazaz - zazayayaz - zayayazaz + zayayazayayaz)ayay - yaya(zazaz - zazayayaz -$

$zayayazaz+zayayazayayaz)+yaya(zazaz-zazayayaz-zayayazaz+zayayazayayaz)ayay+y$

$= z - zayay - yayaz + yayazayay + y$

$= (z - yayaz)(1 - ayay) + y$

$\therefore (z - yayaz)(1 - ayay) + y$ is reflexive inverse for $a$. □

**Theorem 3.9** *[11] Suppose $a, b \in \mathcal{T}$ and $a$ has generalized inverse. Then the below are equivalent*

*(i)* $a = axaxa = axaxb = bxaxa,\ x \in \mathcal{T}$

*(ii)* $a = b\hat{b}b\hat{b}b$ *for certain* $\hat{b} \in b\{2\}$

*(iii)* $a = a\hat{b}a\hat{b}a = a\hat{b}a\hat{b}b = b\hat{b}a\hat{b}a$

*Proof.* $(i) \Rightarrow (ii)$ $a = axaxa = axaxb = bxaxa$
set $\hat{b} = xaxax$. Then $\hat{b}b\hat{b}b\hat{b} = (xaxax)b(xaxax)b(xaxax)$
$= x(a)xaxaxax = xaxax = \hat{b}$
Also, $b\hat{b}b\hat{b}b = b(xaxax)b(xaxax)b$
$= bxaxaxaxb$
$= bxaxa = a$
$(ii) \Rightarrow (iii)$ Let's suppose $\hat{b}$ is 2-inverse of $b$ such that $a = b\hat{b}b\hat{b}b$
since $\hat{b}b\hat{b}b\hat{b} = \hat{b}$
$a = a\hat{b}a\hat{b}a = (b\hat{b}b\hat{b}b)\hat{b}(b\hat{b}b\hat{b}b)\hat{b}(b\hat{b}b\hat{b}b)$
$= b\hat{b}b\hat{b}b\hat{b}b\hat{b}b = b\hat{b}b\hat{b}b = a$
Also $a\hat{b}a\hat{b}b = (b\hat{b}b\hat{b}b)\hat{b}(b\hat{b}b\hat{b}b)\hat{b}b$
$= b\hat{b}b\hat{b}b = a$ and
Similarly $b\hat{b}a\hat{b}a = b\hat{b}(b\hat{b}b\hat{b}b)\hat{b}(b\hat{b}b\hat{b}b)$
$= b\hat{b}b\hat{b}b = a$
$\therefore$ $a = a\hat{b}a\hat{b}a = a\hat{b}a\hat{b}b = b\hat{b}a\hat{b}a$
$(iii) \Rightarrow (i)$ $a = a\hat{b}a\hat{b}a$, fix$\hat{b} = xax \Rightarrow a = a(xax)a(xax)a$
$a = axaxa$ and $a = a\hat{b}a\hat{b}b = a(xax)a(xax)b = axaxb$
similarly, we get $a = bxaxa$.

□

**Theorem 3.10** *[14] Suppose $a \in \mathcal{T}$ is regular, $u = aqpaqaa^- + 1 - aqaa^-e$, $w = a^-aqpaqa + 1 - a^-aqae$, $s = paq$, Then the below are equivalent:*
*(i)$ue\mathcal{T} = \mathcal{T}$, (ii)$aqpa\mathcal{T} = aq\mathcal{T}$, (iii)$we\mathcal{T} = \mathcal{T}$*
*If in addition $Tea = Tpa$ these are also equivalent to (iv)$s^2\mathcal{T} = ses$ and $aq\mathcal{T} = ae\mathcal{T}$.*
*here $e$ is the identity element.*

*Proof.* If $a$ is 1-inverse then $aqaa^-\mathcal{T} = aq\mathcal{T}$.
$(i) \Leftrightarrow (ii)$. $ue\mathcal{T} = \mathcal{T} \Rightarrow aqaa^-(aqpaqaa^- + 1 - aqaa^-e)\mathcal{T} = aqaa^-\mathcal{T}$
$\Rightarrow aqpa\mathcal{T} = aq\mathcal{T}$.
conversely, if $aqpax = aqe$——————————(A),
let $y = (axaea^- + 1 - aqaa^-e)$ and form
$uey = (aqpaqaa^- + 1 - aqaa^-e)e(axaea^- + 1 - aqaa^-e)$
$= aqpaxaea^- + 1 - aqaa^-e$
$= 1 + aqaa^-e - aqaa^-e = 1$ (using (A))
$\therefore ue\mathcal{T} = 1$, for some $y \in \mathcal{T}$
Hence $ue\mathcal{T} = \mathcal{T}$.
$(ii) \Leftrightarrow (iii)$. $aqpax = aqe$,
take $z = (a^-axae + 1 - a^-aqae)$.
$wez = (a^-aqpaqa + 1 - a^-aqae)\, e(a^-axae - a^-aqae + 1)$
$= a^-aqpaxae - a^-aqae + 1$
$= a^-aqae - a^-aqae + 1 = 1$
$\therefore we\mathcal{T} = 1$, for some $x \in \mathcal{T}$
Hence $we\mathcal{T} = \mathcal{T}$.
Conversely, if $we\mathcal{T} = \mathcal{T}$ then $aqpa\mathcal{T} = aqav\mathcal{T} = aq\mathcal{T}$.
$(ii) \Leftrightarrow (iv)$. Obviously (ii) $\Rightarrow$ $aq\mathcal{T} = ae\mathcal{T}$.
So $se\mathcal{T} = paqe\mathcal{T} = paqpa\mathcal{T} = paqpaq\mathcal{T} = s^2\mathcal{T}$.
Conversely, given that $aq\mathcal{T} = ae\mathcal{T}$, we've got $paqpaq\mathcal{T} = paq\mathcal{T}$. By (iv) $\exists$ $p'$ such that $1ea = p'pa \Rightarrow a = p'pa$.
Hence, pre multiplication by $p'$ produces $p'paqpaT = p'paq\mathcal{T} \Rightarrow aqpa\mathcal{T} = aq\mathcal{T}$. □

**Remark 3.1** *It follows from symmetry that the following are equivalent:*

*(1) $\mathcal{T}eu = \mathcal{T}$ (2) $\mathcal{T}aqpa = \mathcal{T}qa$ (3) $\mathcal{T}ev = \mathcal{T}$.*

*If also $ae\mathcal{T} = aq\mathcal{T}$, then these also equivalent to*

*(4) $\mathcal{T}t^2 = \mathcal{T}et$ and $\mathcal{T}ea = \mathcal{T}pa$*

**Theorem 3.11** *When $\mathcal{T}$ is a T.S.R and $a \in \mathcal{T}$, the subsequent statements are equivalent*

*(1)* $a \in a_1^- aTaa_2^-$ *for some 1-inverses* $a_1^-, a_2^- \in \mathcal{T}$.

*(2)* $ae\mathcal{T} = a^+a\mathcal{T}$, $a^+e\mathcal{T} = a^+a\mathcal{T}$ *for some reflexive inverse* $a^+$ *in* $\mathcal{T}$.

*(3)* $\mathcal{T}ea = \mathcal{T}aa^+$, $\mathcal{T}ea^+ = \mathcal{T}aa^+$ *for some reflexive inverse* $a^+$ *in* $\mathcal{T}$.

*(4)* $ae\mathcal{T} = a_1^- a\mathcal{T}$ *for some 1-inverse* $a_1^- \in \mathcal{T}$.

*Where 'e' is identity element.*

*Proof.* $(1) \Rightarrow (2)$, Let $a = a_1^- azaa_2^-$, for some $z \in \mathcal{T}$ and set $a^+ = a_1^- aa_2^- aa_2^-$

$$\text{Then } a = a_1^- azaa_2^- = a_1^-(aa_2^- aa_2^- a)zaa_2^- = a^+azaa_2^-$$

$$\therefore a = a^+azaa_2^-$$

Since $a = a_1^- azaa_2^-$
pre multiplying " $a_1^- aa_2^- aa_2^-$ " and post multiply " $a_2^-$ " on

both sides, we get,
$a^+a\mathcal{T} = a^+e\mathcal{T}$
(2) $\Rightarrow$ (3) since $ae\mathcal{T} = a^+a\mathcal{T}$, we have $a = a^+azaa_2^-$
$a = a^+az(aa_2^-aa_2^-a)a_2^- = a^+azaa_2^-(a\,a_1^-\,a\,a_1^-\,a)a_2^-aa_2^-$
$= a^+azaa_2^-aa_1^-\;a\;(\;a_1^-\,aa_2^-aa_2^-)$
$= a^+azaa_2^-aa_1^-aa^+$
$\therefore \mathcal{T}ea = \mathcal{T}aa^+$
(3) $\Rightarrow$ (4) let $\mathcal{T}ea = \mathcal{T}aa^+ \Rightarrow (aa_1^-aa_1^-)ea = (a^+azaa_2^-)aa^+$
$\therefore ae\mathcal{T} = a_1^-a\mathcal{T}$
(4) $\Rightarrow$ (1), $ae\mathcal{T} = a_1^-a\mathcal{T} \Rightarrow a = a_1^-azaa_2^-$ for some $z \in \mathcal{T}$.
Hence $a \in a_1^-a\mathcal{T}aa_2^-$. □

**Lemma 3.3** *If $\mathcal{T}$ is a T.S.R with unity and $a, d$ are regular elements in $\mathcal{T}$. Then $ae\mathcal{T} = de\mathcal{T}$ and $\mathcal{T}ea = \mathcal{T}ed \Leftrightarrow d = uae = ave$ for units $u, v \in \mathcal{T}$.*

# 4 g- inverses in an Ordered Ternary Semiring

**Theorem 4.1** *( [13]) Let $\mathbb{T}$ be an Ordered T.S.R, then the given statements are equivalent*

*(i)* $u \in \mathbb{T}$ *is regular*

*(ii)* $u \in u\mathbb{T}u$

*(iii)* $u \in u^2T \cap \mathbb{T}u^2$

*Proof.* $(i) \Rightarrow (ii)$
Suppose $\mathbb{T}$ is regular, then for $u \in \mathbb{T}$, $\exists\; x \in \mathbb{T} \;\ni\; u \leq uxuxu$
$therefore\;\; u \in u\mathbb{T}u$
$(ii) \Rightarrow (iii)$
Since $u \in u\mathbb{T}u$, there exist $x, y \in \mathbb{T}$
$u \leq u(yxu)u$
$\leq uyxu^2$
$\therefore u \in \mathbb{T}u^2$
Similarly, $u \leq u(uxy)u$
$\leq u^2xyu$
$u \in u^2\mathbb{T}$
$u \in u^2 \cap \mathbb{T}u^2$
$(iii) \Rightarrow (i)$
Since $u \in u^2\mathbb{T}$ and $u \in \mathbb{T}u^2$
$u \in u^2\mathbb{T} \Rightarrow u \leq uu(yuu)$ and $u \in \mathbb{T}u^2 \Rightarrow u \leq (uuy)uu$
$u \leq uuyuu$
$\leq uuyu(uuyuu)$
$\leq u(uyu)u(uyu)u$
Hence $u$ is regular. □

**Theorem 4.2** *A partially ordered T.S.R $\mathbb{T}$ and $a \in \mathbb{T}$ stands regular $\Leftrightarrow \exists\; t, t_1, s_1, s_2 \in \mathbb{T} \;\ni\; a \leq (at_1t)(zaz)(tt_1a)$.*

*Proof.* Given that $a$ is regular, $z \in \mathbb{T}$ such that $a \leq azaza$
for $z \in \mathbb{T}$, $\exists\, t \in \mathbb{T}$ such that $z \leq ztztz$.
Now $a \leq azaza$
$\leq a(ztztz)a(ztztz)a$
$\leq at_1t(zaz)tt_1a$, where $t_1 = ztz$
$\therefore a \leq (at_1t)(zaz)(tt_1a)$
converse is obvious. □

**Theorem 4.3** *A partially ordered ternary semiring $\mathbb{T}$ and $a \in \mathbb{T}$ stands for regular if $a \leq aaayaaa \;\;\forall\; a \in \mathbb{T}$ for certain $y \in \mathbb{T}$.*

*Proof.* Assuming $a$ is regular, $\mathbb{T}$ is a right, left, and lateral regular.
i.e, n the case where $a \in \mathbb{T}$, $a \leq aaass$,
$a \leq saaas$
and $a \leq ssaaa$ for certain $s \in \mathbb{T}$.
$\because\;\; a \leq axaxa$
$\leq (aaass)xax(ssaaa)$
$\leq aaa(ssxaxss)aaa$
$\leq aaayaaa$ for some $y = ssxaxss \in \mathbb{T}$
$\therefore\;\; a \leq aaayaaa$ for $a \in \mathbb{T}$
Conversely, assume that $\mathbb{T}$ satisfies the condition $a \leq aaayaaa$ for certain $y \in \mathbb{T}$.
Take into $a \leq aaayaaa$
$\leq a(aayaa)a$
$\leq ata$ where $t = aayaa \in \mathbb{T}$
$\leq at(aaayaaa)$
$\leq ata(aayaa)a$
$\leq atata \quad (\because\;\; t = aayaa \in \mathbb{T})$
$\therefore\;\; a \leq atata$
so $a$ is regular □

**Theorem 4.4** *Suppose $\mathbb{T}$ is an ordered T.S.R and $a \in \mathbb{T}$. Then below statements are equivalent:*

*(1)* $a \in a_1^-a\mathbb{T}aa_2^-$ *for some 1-inverses* $a_1^-, a_2^- \in \mathbb{T}$.

*(2)* $ae\mathbb{T} \leq \mathfrak{a}^+a\mathbb{T}$, $\mathfrak{a}^+e \leq \mathfrak{a}^+a\mathbb{T}$ *for some reflexive inverse* $\mathfrak{a}^+ \in \mathbb{T}$.

*(3)* $\mathbb{T}ea \leq \mathbb{T}a\mathfrak{a}^+$, $\mathbb{T}e\mathfrak{a}^+ \leq \mathbb{T}a\mathfrak{a}^+$ *for some reflexive inverse* $\mathfrak{a}^+ \in \mathbb{T}$.

*(4)* $ae\mathbb{T} \leq a_1^-a\mathbb{T}$ *for some 1-inverse* $a_1^- \in \mathbb{T}$.

*Where 'e' is identity element.*

*Proof.* (1) $\Rightarrow$ (2), Let $a \leq a_1^-azaa_2^-$, for some $z \in \mathbb{T}$ and set $\mathfrak{a}^+ = a_1^-aa_2^-aa_2^-$

Then $a \leq a_1^- azaa_2^- \leq a_1^-(aa_2^- aa_2^- a)zaa_2^- \leq \mathfrak{a}^+ azaa_2^-$

$$\therefore a \leq \mathfrak{a}^+ azaa_2^-$$

Since $a \leq a_1^- azaa_2^-$
pre multiplying " $a_1^- aa_2^- aa_2^-$ " and post multiply " $a_2^-$ " on both sides, we get,
$\mathfrak{a}^+ a\mathbb{T} \leq \mathfrak{a}^+ e\mathbb{T}$

(2) $\Rightarrow$ (3) since $ae\mathbb{T} \leq \mathfrak{a}^+ a\mathbb{T}$, we have $a \leq \mathfrak{a}^+ azaa_2^-$
$a \leq \mathfrak{a}^+ az(aa_2^- aa_2^- a)a_2^- \leq \mathfrak{a}^+ azaa_2^- \ ( aa_1^- aa_1^- a \ ) \ a_2^- aa_2^-$
$\leq \mathfrak{a}^+ azaa_2^- aa_1^- a(a_1^- aa_2^- aa_2^-)$
$\leq \mathfrak{a}^+ azaa_2^- aa_1^- a\mathfrak{a}^+$
$\therefore \mathbb{T}ea \leq \mathbb{T}a\mathfrak{a}^+$
(3) $\Rightarrow$ (4) let $\mathbb{T}ea \leq \mathbb{T}a\mathfrak{a}^+ \Rightarrow (aa_1^- aa_1^-)ea \leq (\mathfrak{a}^+ azaa_2^-)a\mathfrak{a}^+$
$\therefore ae\mathbb{T} \leq a_1^- a\mathbb{T}$
(4) $\Rightarrow$ (1), $ae\mathbb{T} \leq a_1^- a\mathbb{T} \Rightarrow a \leq a_1^- azaa_2^-$ for some $z \in \mathbb{T}$.
Hence $a \in a_1^- a\mathbb{T}aa_2^-$. □

**Theorem 4.5** *Suppose $a \in \mathbb{T}$ is regular, $u = aqpaqaa^- + 1 - aqaa^- e$, $w = a^- aqpaqa - a^- aqae + 1$ and $s = paq$, then the subsequent are equivalent:*
*(1)$ue\mathbb{T} \geq \mathbb{T}$, (2)$aqpa\mathbb{T} \geq aq\mathbb{T}$, (3)$we\mathbb{T} \geq \mathbb{T}$*
.

*Proof.* Since $a$ has 1-inverses $q, a^-$ then $aqaa^-\mathbb{T} \geq aq\mathbb{T}$.
(1) $\Leftrightarrow$ (2). $ue\mathbb{T} \geq \mathbb{T}$ implies that $aqaa^-(aqpaqaa^- + 1 - aqaa^- e)\mathbb{T} \geq aqaa^-\mathbb{T}$
$aqpa\mathbb{T} \geq aq\mathbb{T}$.
conversely, if $aqpax \geq aqe$————————————(A),
let $y = (1 + axaea^- - aqaa^- e)$ and form
$uey \geq (1 + aqpaqaa^- - aqaa^- e) \ \ e(1 + axaea^- - aqaa^- e)$
$\geq 1 + aqpaxaa^- e - aqaa^- e$
$\geq 1 + aqaa^- e - aqaa^- e \geq 1$ (using (A))
$\therefore ue\mathbb{T} \geq 1$, for some $y \in \mathbb{T}$
Hence $ue\mathbb{T} \geq \mathbb{T}$.

(2) $\Leftrightarrow$ (3). If $aqpax \geq aqe$,
take $z = (a^- axae + 1 - a^- aqae)$.
$\Rightarrow wez \geq (a^- aqpaqa + 1 - a^- aqae)e(a^- axae + 1 - a^- aqea)$
$\geq a^- aqpaxea + 1 - a^- aqea$
$\geq a^- aqae + 1 - a^- aqae = 1$
$\therefore we\mathbb{T} \geq 1$, for some $x \in \mathbb{T}$
Hence $we\mathbb{T} \geq \mathbb{T}$.
Conversely, if $we\mathbb{T} = \mathbb{T}$ then $aqpa\mathbb{T} \geq aqav\mathbb{T} \geq aq\mathbb{T}$. □

**Theorem 4.6** *Suppose $a, c \in \mathbb{T}$ and $a$ stands for regular, then the items listed below are comparable.*

*(i)* $a \leq c$

*(ii)* *for some $u, v \in a\{1\}$ such that $aua \leq auc$ and $ava \leq cva$*

*(iii)* *for some $a^- \in a\{1\}$ such that $a \leq azaa^- c$ and $a \leq ctaa^- a$*

*Proof.* $(i) \Rightarrow (ii)$ Obvious.
$(ii) \Rightarrow (iii)$ let $u = zaa^-, \ \ v = taa^-$ for certain $z, t \in a\{1\}$
$a \leq auaua \leq au(auc)$
$a \leq a(zaa^-)a(zaa^-)c$
$a \leq azaa^- c$
similarly, $a \leq avava \leq av(cva)$
$\leq a(taa^-)c(taa^-)a$
$\leq ataa^- a$
$\therefore \ \ a \leq ataa^- a.$
$(iii) \Rightarrow (i) \quad a \leq azaa^- c$
pre multiply $(azaa^-)$ on both sides
$(azaa^-)a \leq (azaa^-)azaa^- c$
$azaa^- a \leq azaa^- c$
$\therefore \ \ a \leq c$
In a similar manner, we can show $a \leq ctaa^- a \Rightarrow a \leq c$ □

## 5 Conclusion

The study of generalized inverses in a ternary semiring is continued in this publication. Mainly, we focused on finding g-inverses using principal ideals(cosets) in ordered and ternary semiring.

**conflict of interest:** The writers affirm that they do not have any conflict of interest.

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