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Certain Subclass of Analytic Functions Defined By $q-$analogue Differential Operator

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Abstract The quantum (or q-) calculus is a vital area of study in the field of traditional mathematical analysis. This paper explores the innovative use of the q - q-derivative concept to develop specific differential operators, extending the class of Salagean operators to include univalent functions . By leveraging this new operator, we define a novel subclass of analytic functions within the open unit disc $\Delta = \{\ell \in \mathbb{C} : |\ell| < 1\}$. Our primary objective is to establish a subclass of uniformly starlike functions corresponding to uniformly convex functions through the q-analog of the generalized differential operator. This research delves deeply into the intricate properties of this newly defined class of functions. We systematically analyze various aspects, such as coefficient estimates, which provide critical insights into the behavior of the functions within this class. Additionally, we examine neighborhoods, elucidating the local behavior and interaction of these functions within the region. Our study of partial sums offers a detailed understanding of the series representations and their properties. Furthermore, we investigate integral means inequalities, which are essential in understanding these functions' average growth and value distribution. The radii of close-to-convexity and star likeness are also rigorously evaluated, shedding light on the geometric properties that characterize the boundaries within which these functions maintain their specific starlike or convex nature .

Keywords Analytic Functions, Univalent Functions, Starlike Functions, Symmetric Points, Uniformly Convex Functions, Uniformly Starlike Functions, Sălăgean Derivative, $q-$derivative

1 Introduction

Let $\mathscr{A}$ denote the class of all functions $\aleph$ of the form

$$\aleph(\ell) = \ell + \sum_{\eta=2}^{\infty} \jmath_\eta \ell^\eta \tag{1}$$

in the open unit disc $\Delta = \{\ell \in \mathbb{C} : |\ell| < 1\}$. Let $\mathscr{S}$ be a subclass of $\mathscr{A}$ with univalent and normalized by $\aleph(0) = \aleph'(0) - 1 = 0$.

A function $\aleph \in \mathscr{A}$ is starlike function of the order ς $(0 \leq \varsigma < 1)$, if it satisfies

$$\Re\left\{\frac{\ell\aleph'(\ell)}{\aleph(\ell)}\right\} > \varsigma, \quad \ell \in \Delta \tag{2}$$

and convex function of the order ς $(0 \leq \varsigma < 1)$, if it satisfies

$$\Re\left\{1 + \frac{\ell\aleph''(\ell)}{\aleph'(\ell)}\right\} > \varsigma, \quad \ell \in \Delta. \tag{3}$$

Also, the classes of starlike and convex functions are denoted by $\mathscr{S}^*(\varsigma)$ and $\mathscr{K}(\varsigma)$ respectively.

Let T be a subclass of $\mathscr{S}$ consisting functions of the form

$$\aleph(\ell) = \ell - \sum_{\eta=2}^{\infty} \jmath_\eta \ell^\eta, \quad |\jmath_\eta| \geq 0 \tag{4}$$

introduced and studied by Silverman [1].

In [2], Sakaguchi introduced a subclass $\mathscr{S}\mathscr{T}_s$ of starlike functions with respect to symmetric points as follows:

$$\Re\left\{\frac{2\ell\aleph'(\ell)}{\aleph(\ell)-\aleph(-\ell)}\right\}>0, \qquad \ell\in\Delta$$

and Owa et al. [3] introduced the class $\mathscr{S}\mathscr{T}_s(\upsilon,\varsigma)$ as follows:

$$\Re\left\{\frac{(1-\varsigma)\ell\aleph'(\ell)}{\aleph(\ell)-\aleph(\varsigma\ell)}\right\}>\upsilon,$$
$$\text{where } 0\leq\upsilon<1,\ |\varsigma|\leq 1,\ \varsigma\neq 1,\ \ell\in\Delta.$$

It is interesting to notice that

$$\mathscr{S}\mathscr{T}_s(0,-1):=\mathscr{S}\mathscr{T}_s$$

and

$$\mathscr{S}\mathscr{T}_s(\upsilon,-1):=\mathscr{S}\mathscr{T}_s(\upsilon).$$

Next, for $0\leq\nu<1$, the class $\mathscr{U}ST(\nu)$ of uniformly starlike of order ν and the class $\mathscr{U}CV(\nu)$ of uniformly convex functions of order ν are defined as follows [4]:

$$\Re\left\{\frac{\ell\aleph'(\ell)}{\aleph(\ell)}\right\}>\left|\frac{\aleph'(\ell)}{\aleph(\ell)}-1\right|+\nu$$

and

$$\Re\left\{1+\frac{\ell\aleph''(\ell)}{\aleph'(\ell)}\right\}>\left|\frac{\ell\aleph''(\ell)}{\aleph'(\ell)}\right|+\nu.$$

The quantum (or q-) calculus is a vital area of study in the field of traditional mathematical analysis. It concentrates on a theoretically valuable generalization of integration and differentiation operations. Quantum calculus is a wide area of mathematical science with historical origins and a revived focus in the modern era. Notably, quantum calculus has a long tradition that can be traced back to Bernoulli and Euler's function. Recent years have seen a sharp rise in interest in q-calculus and its applications in a variety of fields, including mechanics, mathematics, and physics. Quantum mechanics, theta and mock theta functions, analytic number theory, hypergeometric functions, combinatorics, multiple hypergeometric functions, gamma function theory, Sobolev spaces, Bernoulli and Euler polynomials, operator theory, and, more recently, the geometric theory of analytic and harmonic univalent functions are just a few examples of the fields in which q-calculus has found widespread application. Research on q-calculus is crucial because of the aforementioned application areas. Beyond these uses, basic (or q-) polynomials and simple (or q-) series have been widely applied in number theory and partition theory, particularly basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials. Lie theory, particle physics, combinatorial analysis, finite vector spaces, and other domains have all made use of fundamental (or q-) hypergeometric functions, nonlinear electric circuit theory, mechanical engineering, heat conduction theory, quantum mechanics, cosmology, and statistics.

In geometric function theory, q-calculus serves as a more flexible and general framework for studying geometric properties of functions. Its methodologies involve the extension and deepening of classical results using q-integrals, q-derivatives, q-difference equations, q-special functions, q-transformations, q-operator theory, and q-inequalities.

Q-calculus, or quantum calculus, is a generalization of traditional calculus that does not rely on limits. It plays a significant role in various branches of mathematics, including geometric function theory. Here are some **major contributions of q-calculus to geometric function theory**:

1. A foundation for generalizing numerous classical findings in geometric function theory is provided by the Q-calculus. For instance, q-analogues of well-known theorems have been produced, including the Koebe distortion theorem, the Schwarz-Pick lemma, and other coefficient inequalities. These generalizations broaden the scope of application of these theorems and provide deeper insights.

2. Special functions that are essential to geometric function theory, such as q-Bessel functions, q-Hypergeometric functions, and q-Jacobi polynomials, have q-analogues thanks to Q-calculus. Studying the characteristics of analytic and univalent functions depends heavily on these functions.

3. The extension of differential and integral operators in geometric function theory is made possible by the inclusion of q-integrals and q-derivatives. This has made it easier to analyze the q-variations of classical operators, such the Cesro operator and the Hardy-Littlewood operator, and has produced fresh findings and insights.

4. The study of geometric structural deformations and q-extensions is made possible by the Q-calculus. This includes studying q-harmonic functions and q-conformal mappings, which offer a more comprehensive framework for comprehending geometric transformations and their characteristics.

5. The development of growth theorems and univalence conditions for q-analytic functions has been aided by Q-calculus. Understanding the geometric behavior of functions inside the unit disk or in other domains depends critically on these findings.

6. The number of instruments for investigating fractional differential equations in geometric function theory has increased with the creation of q-fractional calculus, which includes both symmetric and asymmetric variants. Iterative procedures and the analysis of complex systems are affected by this.

Scope for Future Research:

1. Extension to Broader Classes of Functions: Future research could explore the extension of the results to broader classes of analytic functions. This includes investigating how the q-analogue differential operator affects other subclasses and more general function spaces.

2. Multidimensional q-Calculus: Extending the study to functions of several variables and exploring multidimensional q-calculus could provide deeper insights and more comprehensive results.

3. Numerical Methods and Simulations: Developing numerical methods and computational techniques to effectively handle q-analogue differential equations could make the results more practical and applicable.

4. Interdisciplinary Applications: Investigating potential applications in other fields, such as quantum mechanics, signal processing, or mathematical biology, where q-calculus might offer new tools and perspectives.
5. Comparative Studies: Conducting comparative studies between the q-analogue differential operator and other generalized differential operators to highlight the unique advantages and potential limitations of the q-analogue approach.
6. Refinement of Theoretical Framework: Further refining the theoretical framework around q-analogue differential operators, including developing new theorems, corollaries, and identities that can enrich the existing theory.

The theory of quantum calculus and its applications has been applied in several branches of mathematics, engineering sciences, and physics. Hence, many researchers have used $q-$calculus to study discrete dynamical systems, discrete stochastic processes, $q-$ deformed super algebras, $q-$transform analysis, and so on. In the literature, the differentiation and integration of functions are formulated using the calculus quantum theory. In developing the Geometric Function Theory of Complex Analysis, the $q-$ derivative is an important research tool. $q-$analogues of differential operators play important roles in various mathematics and physics areas, particularly where discrete or quantum-like behaviors are involved. They often arise in studying special functions such as $q-$polynomials, $q-$series, and $q-$integrals. The Quantum calculus or $q-$calculus began with straight to the point Jackson [5] within the early 20^{th} century, but Euler and Jacobi had, as of now, worked out this kind of calculus. Recently, many scholars have defined new subclasses of analytic functions by combining the $q-$derivative operator with the principle of differential subordination and studied their geometric properties (see [6, 7, 8, 9, 10, 11, 12, 13, 14]). For **motivation and incentive** for further research, the reader's attention is drawn toward some of the related recent developments dealing with the coefficient inequalities and coefficient estimates of various subclasses of univalent analytic functions. Recently, it is stimulated due to the endless prerequisite for arithmetic that models quantum computing $q-$calculus, which appeared as an affiliation between science and physics. It requires plenty of application in numerous numerical fields such as basic hypergeometric functions, quantum hypotheses, mechanics, and the hypothesis of relativity.

For $0 < q < 1$, define the $q-$number $[\alpha]_q$ by

$$[\alpha]_q = \begin{cases} \dfrac{1-q^\alpha}{1-q}, & \text{if} \quad \alpha \in \mathbb{C} \setminus \mathbb{N}, \\ \sum\limits_{i=0}^{\alpha-1} q^i, & \text{if} \quad \alpha \in \mathbb{N}. \end{cases} \tag{5}$$

Note that as $q \to 1^-$, $[\alpha]_q \to \alpha$. Further, define the $q-$fractional $[\alpha]_q!$ by

$$[\alpha]_q! = \begin{cases} 1, & \text{if} \quad \alpha = 1, \\ \prod\limits_{\eta=1}^{\alpha} [\eta]_q, & \text{if} \quad \alpha \in \mathbb{N} \setminus \{1\}. \end{cases} \tag{6}$$

Define the $q-$derivative $\mathscr{D}_q\aleph$ of a function $\aleph$ by

$$(\mathscr{D}_q)\aleph(\ell) = \begin{cases} \dfrac{\aleph(\ell)-\aleph(\ell q)}{(1-q)\ell}, & \text{if} \quad \ell \neq 0, \\ \aleph'(0), & \text{if} \quad \ell = 0 \end{cases} \tag{7}$$

provided that $\aleph'(0)$ exists. It follows from (7) that

$$\lim_{q\to 1^-} \mathscr{D}_q\aleph(\ell) = \lim_{q\to 1^-} \frac{\aleph(\ell)-\aleph(\ell qq)}{(1-q\ell} = \aleph'(\ell)$$

for a function $\aleph$ which is differentiable in a given subset of $\mathbb{C}$. Thus, we have

$$(\mathscr{D}_q)\aleph(\ell) = 1 + \sum_{\eta=2}^{\infty} [\eta]_q \jmath_\eta \ell^{\eta-1}. \tag{8}$$

Next, we consider the Sălăgean $q-$ differential operator as follows [15]:

$$\begin{aligned} \mathscr{D}_q^0\aleph(\ell) &= \aleph(\ell) \\ \mathscr{D}_q^1\aleph(\ell) &= \ell\left(\mathscr{D}_q\aleph(\ell)\right) \\ &\vdots \\ \mathscr{D}_q^\iota\aleph(\ell) &= \mathscr{D}_q^1\left(\mathscr{D}_q^{\iota-1}\aleph(\ell)\right) = \ell\left(\mathscr{D}_q\mathscr{D}_q^{\iota-1}\aleph(\ell)\right). \end{aligned}$$

Thus, we have

$$\mathscr{D}_q^\iota\aleph(\ell) = \ell + \sum_{\eta=2}^{\infty} [\eta]_q^\iota \jmath_\eta \ell^\eta. \tag{9}$$

We note that if $q \to 1^-$,

$$\mathscr{D}^\iota\aleph(\ell) = \ell + \sum_{\eta=2}^{\infty} \eta^\iota \jmath_\eta \ell^\eta \tag{10}$$

is the familiar Sălăgean derivative [16]. Now let

$$\begin{aligned} \mathscr{D}^0 &= \mathscr{D}_q^\wp\aleph(\ell) \\ \mathscr{D}_{q,\iota}^{1,\wp}\aleph(\ell) &= (1-\iota)\mathscr{D}_q^\wp\aleph(\ell) + \iota\ell(\mathscr{D}_q^\wp\aleph(\ell))' \\ &= \ell + \sum_{\eta=2}^{\infty} [\eta]_q^\wp[1+(\eta-1)\iota]\jmath_\eta\ell^\eta \\ \mathscr{D}_{q,\iota}^{2,\wp}\aleph(\ell) &= (1-\iota)\mathscr{D}_{q,\iota}^{1,\wp}\aleph(\ell) + \iota\ell(\mathscr{D}_{q,\iota}^{1,\wp}\aleph(\ell))' \\ &= \ell + \sum_{\eta=2}^{\infty} [\eta]_q^\wp[1+(\eta-1)\iota]^2\jmath_\eta\ell^\eta \\ &\dots \qquad \dots \qquad \dots \\ \mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell) &= (1-\iota)\mathscr{D}_{q,\iota}^{\hbar-1,\wp}\aleph(\ell) + \iota\ell(\mathscr{D}_{q,\iota}^{\hbar-1,\wp}\aleph(\ell))' \\ &= \ell + \sum_{\eta=2}^{\infty} [\eta]_q^\wp[1+(\eta-1)\iota]^\hbar\jmath_\eta\ell^\eta, \ \iota > 0, \ \hbar \in \mathbb{N}_0 \end{aligned} \tag{11}$$

where $[\eta]_q!$ is defined as (6). It remarked that [17], when $q \to 1^-$, we have

$$\begin{aligned} \lim_{q\to 1^-} \mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell) &= \ell + \lim_{q\to 1^-} \sum_{\eta=2}^{\infty} [\eta]_q^\wp[1+(\eta-1)\iota]^\hbar\jmath_\eta\ell^\eta \\ &= \ell + \sum_{\eta=2}^{\infty} \eta^\wp[1+(\eta-1)\iota]^\hbar\jmath_\eta\ell^\eta\jmath_\eta\ell^\eta \\ &= \mathscr{D}_\iota^{\hbar,\wp}\aleph(\ell). \end{aligned}$$

Now, we define $UST_s(\iota,q,\hbar,\wp,\nu,\varsigma)$ by using the generalized differential operator $\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)$ as follows.

1.1 Definition

Let $\iota>0$, $0<q<1$, $0\le\nu<1$, $|\varsigma|\le 1$, and $\varsigma\neq 1$. A function $\aleph\in\mathscr{A}$ is said to be in the class $UST_s(\iota,q,\hbar,\wp,\nu,\varsigma)$, if the adopting connection holds true:

$$\Re\left\{\frac{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'}{\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)}\right\} \geq\left|\frac{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'}{\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)}-1\right|+\nu,\quad \ell\in\Delta.$$

Moreover, a function $\aleph\in UST_s(\iota,q,\hbar,\wp,\nu,\varsigma)$ is in the subclass $\widetilde{UST}_s(\iota,q,\hbar,\wp,\nu,\varsigma)$ if $\aleph\in T$.

Firstly, we need the adopting lemmas [18].

1.2 Lemma

Let a be a complex number and β be a real number. Then

$$\Re(a)\geq\beta \Leftrightarrow\ |a-(1+\beta)|\leq|a+(1-\beta)|.$$

1.3 Lemma

Let a be a complex number and β,ν be real numbers. Then $\Re\,(a)>\beta|a-1|+\nu \ \Leftrightarrow\ \Re\{a(1+\beta e^{i\varrho})-\beta e^{i\varrho}\}>\nu, -\pi<\varrho\leq\pi$.

2 Coefficient bounds

2.1 Theorem

Let $u\in T$. Then $\aleph\in\widetilde{UST}_s(\iota,q,\hbar,\wp,\nu,\varsigma)$ if and only if

$$\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\left|2\eta-\aleph_\eta(1+\nu)\right|\jmath_\eta\leq 1-\nu, \quad (12)$$

where $\aleph_\eta=1+\varsigma+\cdots+\varsigma^{\eta-1}$. The estimate is sharp with

$$\aleph(\ell)=\ell-\frac{1-\nu}{[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\left|2\eta-\aleph_\eta(1+\nu)\right|}\ell^{\eta}.$$

Proof From Definition (1), we obtain

$$\Re\left\{\frac{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'}{\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)}\right\} \geq q\left|\frac{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'}{\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)}-1\right|+\nu.$$

Next, by Lemma 1.3, we have

$$\Re\left\{\frac{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'}{\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)}(1+e^{i\varrho})-e^{i\varrho}\right\} \geq\nu,\quad -\pi<\varrho\leq\pi \quad (13)$$

which implies that

$$\Re\Bigg\{\frac{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'(1+e^{i\varrho})}{\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)} -\frac{e^{i\varrho}\left[\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)\right]}{\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)}\Bigg\} \geq q\nu. \quad (14)$$

Now, suppose that

$$L(\ell)=(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'(1+e^{i\varrho}) -e^{i\varrho}\left[\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)\right]$$

and

$$M(\ell)=\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell).$$

By virtue of Lemma 1.2, (14) implies

$$|L(\ell)+(1-\nu)M(\ell)|\geq|L(\ell)-(1+\nu)M(\ell)|, \quad 0\leq\nu<1.$$

Hence, we obtain

$$\begin{aligned}
&|L(\ell)+(1-\nu)M(\ell)|\\
=&\Bigg|(1-\varsigma)\Bigg\{(2-\nu)\ell -\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}(\eta+\aleph_\eta(1-\nu))\jmath_\eta\ell^{\eta} -e^{i\varrho}\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}(\eta-\aleph_\eta)\jmath_\eta\ell^{\eta}\Bigg\}\Bigg|\\
\geq&\ |1-\varsigma|\Bigg\{(2-\nu)\,|\ell| -\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\left|\eta+\aleph_\eta(1-\nu)\right|\jmath_\eta\,|\ell|^{\eta} -\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\left|\eta-\aleph_\eta\right|\jmath_\eta\,|\ell|^{\eta}\Bigg\}.
\end{aligned}$$

On the other side, we get

$$\begin{aligned}
&|L(\ell)+(1+\nu)M(\ell)|\\
&=\Bigg|(1-\varsigma)\Bigg\{-\nu\ell -\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}(\eta+\aleph_\eta(1-\nu))\jmath_\eta\ell^{\eta} -e^{i\varrho}\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}(\eta-\aleph_\eta)\jmath_\eta\ell^{\eta}\Bigg\}\Bigg|\\
&\geq|1-\varsigma|\Bigg\{\nu\,|\ell| -\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\left|\eta+\aleph_\eta(1-\nu)\right|\jmath_\eta -\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\left|\eta-\aleph_\eta\right|\jmath_\eta\,|\ell|^{\eta}\Bigg\}.
\end{aligned}$$

Accordingly, we find that

$$\begin{aligned}&|L(\ell)+(1-\nu)M(\ell)|-|L(\ell)+(1+\nu)M(\ell)|\\ &\geq|1-\varsigma|\left\{2(1-\nu)\,|\ell|-\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\right.\\ &\left.\quad[|\eta+\aleph_\eta(1-\nu)|+|\eta-\aleph_\eta(1+\nu)|+2|\,|\eta-\aleph_\eta|\,\jmath_\eta\,|\ell|^{\eta}]\right\}\\ &\geq 2(1-\nu)\,|\ell|\\ &-\sum_{\eta=2}^{\infty}2[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\,|2\eta-\aleph_\eta(1+\nu)|\,\jmath_\eta\,|\ell|^{\eta}\\ &\geq 0.\end{aligned}$$

Or

$$\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\,|2\eta-\aleph_\eta(1+\nu)|\,\jmath_\eta\leq 1-\nu.$$

Conversely, suppose (12) holds. Then, we must indicate that

$$\begin{aligned}&\Re\left\{\frac{1}{\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)}\right.\\ &\quad\times\left[(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'(1+e^{i\varrho})\right.\\ &\quad\left.\left.-e^{i\varrho}\left[\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)\right]\right]\right\}\\ &\geq\ \nu.\end{aligned}$$

Taking the values of ℓ $(0\leq|\ell|=r<1)$ on the positive $x-$ axis, then

$$\begin{aligned}&\Re\left\{\frac{1}{1-\sum\limits_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\aleph_\eta\jmath_\eta\ell^{\eta-1}}\right.\\ &\quad\times\left[(1-\nu)-\sum_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}[\eta(1+e^{i\varrho})\right.\\ &\quad\left.\left.-\aleph_\eta(\nu+e^{i\varrho})]\jmath_\eta\ell^{\eta-1}\right]\right\}\\ &\geq 0.\end{aligned}$$

Since $\Re(-e^{i\varrho})\geq-|e^{i\varrho}|=-1$, then

$$\Re\left\{\frac{(1-\nu)-\sum\limits_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}[2\eta-\aleph_\eta(\nu+1]\jmath_\eta r^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\aleph_\eta\jmath_\eta r^{\eta-1}}\right\}\geq\ 0.$$

If we let $r\to 1^-$, we get the desired result.

2.2 Corollary

If $u\in\widetilde{UST}_s(\iota,q,\hbar,\wp,\nu,\varsigma)$, then

$$\jmath_\eta\leq\frac{1-\nu}{[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\,|2\eta-\aleph_\eta(1+\nu)|},$$

where $\aleph_\eta=1+\varsigma+\cdots+\varsigma^{\eta-1}$.

3 Neighbourhood properties

In this section, the concept of neighborhoods of analytic functions is motivated by Goodman [19] and Ruscheweyh [20], and we define the neighborhood of a function $\aleph\in T$.

3.1 Definition

Let $\iota>0$, $0<q<1$, $|\varsigma|\leq 1$, $\varsigma\neq 1$, $0\leq\nu<1$, $\upsilon\geq 0$ and $\aleph_\eta=1+\varsigma+\cdots+\varsigma^{\eta-1}$. We define the $\upsilon-$neighbourhood of a function $\aleph\in T$, which is indicated by $N_\upsilon(\aleph)$ consisting of all functions $g(\ell)=\ell-\sum\limits_{\eta=2}^{\infty}b_\eta\ell^\eta\in S$ $(b_\eta\geq 0)$ satisfying

$$\begin{aligned}&\sum_{\eta=2}^{\infty}\frac{[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\,|2\eta-\aleph_\eta(1+\nu)|}{1-\nu}\,|\jmath_\eta-b_\eta|\\ &\leq 1-\upsilon.\end{aligned}$$

3.2 Theorem

Suppose that $u\in\widetilde{UST}_s(\iota,q,\hbar,\wp,\nu,\varsigma)$ and $\Re(\nu)\neq 1$. For any complex number ϵ with $|\epsilon|<\upsilon,\ (\upsilon\geq 0)$, if u satisfies the following condition:

$$\frac{f(\ell)+\epsilon\ell}{1+\epsilon}\in\widetilde{UST}_s(\iota,q,\hbar,\wp,\nu,\varsigma)$$

then $N_\upsilon(\aleph)\subset\widetilde{UST}_s(\iota,q,\hbar,\wp,\nu,\varsigma)$.

Proof It is clear that $\aleph\in\widetilde{UST}_s(\iota,q,\hbar,\wp,\nu,\varsigma)$ if and only if

$$\left|\frac{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'(1+e^{i\varrho})-(e^{i\varrho}+1+\nu)\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)\right)}{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'(1+e^{i\varrho})+(1-e^{i\varrho}-\nu)\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)\right)}\right|<1,\ -\pi<\varrho\leq\pi.$$

For any complex number s $(|s|=1)$, we may write

$$\frac{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'(1+e^{i\varrho})-(e^{i\varrho}+1+\nu)\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)\right)}{(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'(1+e^{i\varrho})+(1-e^{i\varrho}-\nu)\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)\right)}\neq s.$$

That is,

$$\begin{aligned}&(1-s)(1-\varsigma)\ell\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)\right)'(1+e^{i\varrho})\\ &-(e^{i\varrho}+1+\nu+s(-1+e^{i\varrho}+\nu)\\ &\times\left(\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\ell)-\mathscr{D}_{q,\iota}^{\hbar,\wp}\aleph(\varsigma\ell)\right)\neq 0\end{aligned}$$

which implies that

$$\begin{aligned}&\ell-\\ &\sum_{\eta=2}^{\infty}\frac{[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\left[(\eta-\aleph_\eta)(1+e^{i\varrho}-se^{i\varrho})-s(\eta+\aleph_\eta)-\aleph_\eta\nu(1-s)\right]}{\nu(s-1)-2s}\ell^\eta\\ &\neq 0.\end{aligned}$$

However, $\aleph\in\widetilde{UST}_s(\iota,q,\hbar,\wp,\nu,\varsigma)\Leftrightarrow\dfrac{(\aleph*h)}{\ell}\neq 0,\ (\ell\in\Delta-\{0\})$, where

$$h(\ell)=\ell-\sum_{\eta=2}^{\infty}c_\eta\ell^\eta$$

and

$$\begin{aligned}c_\eta&=\frac{1}{\nu(s-1)-2s}\times[\eta^{\wp}[1+(\eta-1)\iota]^{\hbar}\\ &\times\left[(\eta-\aleph_\eta)(1+e^{i\varrho}-se^{i\varrho})-s(\eta+\aleph_\eta)-\aleph_\eta\nu(1-s)\right].\end{aligned}$$

Since $\frac{\aleph(\ell)+\epsilon\ell}{1+\epsilon} \in \widetilde{UST}_s(\iota, q, \hbar, \wp, \nu, \varsigma)$, we observe that

$$|c_\eta| \le \frac{[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\,|2\eta-\aleph_\eta(1+\nu)|}{1-\nu}.$$

Therefore $\ell^{-1}\left(\frac{\aleph(\ell)+\epsilon\ell}{1+\epsilon} * h(\ell)\right) \neq 0$, which is equivalent to

$$\frac{(\aleph * h)(\ell)}{(1+\epsilon)\ell} + \frac{\epsilon}{1+\epsilon} \neq 0. \tag{15}$$

Now, let us consider that $\left|\frac{(\aleph * h)(\ell)}{\ell}\right| < \upsilon$. From (15), we get

$$\begin{aligned}&\left|\frac{(\aleph * h)(\ell)}{(1+\epsilon)\ell} + \frac{\epsilon}{1+\epsilon}\right| \\ &\ge \frac{|\epsilon|}{|1+\epsilon|} - \frac{1}{|1+\epsilon|}\left|\frac{(\aleph * h)(\ell)}{\ell}\right| \\ &> \frac{|\epsilon|-\upsilon}{|1+\epsilon|} \ge 0.\end{aligned}$$

This contradicts that $|\epsilon| < \upsilon$ and hence, we have $\left|\frac{(\aleph * h)(\ell)}{\ell}\right| \ge \upsilon$.

If $g(\ell) = \ell - \sum_{\eta=2}^{\infty} b_\eta \ell^\eta \in N_\upsilon(\aleph)$, then

$$\begin{aligned}\upsilon - \left|\frac{(g * h)(\ell)}{\ell}\right| &\le \left|\frac{((u-g) * h)(\ell)}{\ell}\right| \\ &\le \sum_{\eta=2}^{\infty} |\jmath_\eta - b_\eta|\,|c_\eta|\,|\ell|^\eta \\ &< \sum_{\eta=2}^{\infty} \frac{[\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\,|2\eta-\aleph_\eta(1+\nu)|}{1-\nu}\,|\jmath_\eta - b_\eta| \\ &\le \upsilon.\end{aligned}$$

4 Partial sums

For a function $\aleph \in A$ given by (1), Silverman [21] investigated the partial sums $\aleph$ defined by

$$\aleph_1(\ell) = \ell \text{ and } \aleph_m(\ell) = \ell + \sum_{\eta=2}^{m} \jmath_\eta \ell^\eta. \tag{16}$$

In [21], Silverman examined sharp lower bounds on the real part of the quotients between the normalized convex or starlike functions and their sequences of partial sums. Also, Srivastava et al. [22] and Silvia [23] have investigated interesting results on the partial sums. In this section, we consider partial sums of functions in the class $UST_s(\iota, q, \hbar, \wp, \nu, \varsigma)$ and obtain sharp lower bounds for the ratios of real part of $\aleph$ to $\aleph_m$ and $\aleph'$ to $u\aleph'_m$.

4.1 Theorem

Let a function $\aleph$ of the form (1) belong to the class $UST_s(\iota, q, \hbar, \wp, \nu, \varsigma)$ and satisfy (12). Then

$$\Re\left(\frac{\aleph(\ell)}{u_m(\ell)}\right) \ge \frac{\chi_{m+1}-1+\nu}{\chi_{m+1}}, (\ell \in \Delta), \tag{17}$$

where

$$\chi_\eta \ge \begin{cases} 1-\nu, & \text{if } \eta = 2,3,\cdots,m; \\ \chi_{m+1}, & \text{if } \eta = m+1, m+2, m+3, \cdots. \end{cases} \tag{18}$$

The result (17) is sharp with the function given by

$$\aleph(\ell) = \ell + \frac{1-\nu}{\chi_{m+1}}\ell^{m+1}. \tag{19}$$

Proof Define the function $\wp(\ell)$ by

$$\frac{1+\wp(\ell)}{1-\wp(\ell)} = \frac{\chi_{m+1}}{1-\nu}\left\{\frac{\aleph(\ell)}{\aleph_m(\ell)} - \left(\frac{\chi_{m+1}-1+\nu}{\chi_{m+1}}\right)\right\}$$
$$= \left[\frac{1+\sum_{\eta=2}^{m}\jmath_\eta\ell^{\eta-1} + \frac{\chi_{m+1}}{1-\nu}\sum_{\eta=m+1}^{\infty}\jmath_\eta\ell^{\eta-1}}{1+\sum_{\eta=2}^{m}\jmath_\eta\ell^{\eta-1}}\right]. \tag{20}$$

It suffices to show $|\wp(\ell)| \le 1$. Now, from (24) we can write

$$\wp(\ell) = \frac{\frac{\chi_{m+1}}{1-\nu}\sum_{\eta=m+1}^{\infty}\jmath_\eta\ell^{\eta-1}}{2+2\sum_{\eta=2}^{m}\jmath_\eta\ell^{\eta-1} + \frac{\chi_{m+1}}{1-\nu}\sum_{\eta=m+1}^{\infty}\jmath_\eta\ell^{\eta-1}}$$

$$\Rightarrow |\wp(\ell)| \le \frac{\frac{\chi_{m+1}}{1-\nu}\sum_{\eta=m+1}^{\infty}|\jmath_\eta|}{2-2\sum_{\eta=2}^{m}|\jmath_\eta| - \frac{\chi_{m+1}}{1-\nu}\sum_{\eta=m+1}^{\infty}|\jmath_\eta|}.$$

Now $|\wp(\ell)| \le 1$ if and only if

$$2\frac{\chi_{m+1}}{1-\nu}\sum_{\eta=m+1}^{\infty}|\jmath_\eta| \le 2-2\sum_{\eta=2}^{m}|\jmath_\eta|$$

$$\Rightarrow \sum_{\eta=2}^{m}|\jmath_\eta| + \sum_{\eta=j+1}^{\infty}\frac{\chi_{m+1}}{1-\nu}|\jmath_\eta| \le 1.$$

From the condition (12), it is sufficient to show that

$$\sum_{\eta=2}^{m}|\jmath_\eta| + \sum_{\eta=m+1}^{\infty}\frac{\chi_{m+1}}{1-\nu}|\jmath_\eta| \le \sum_{\eta=2}^{\infty}\frac{\chi_\eta}{1-\nu}|\jmath_\eta|$$

which is equivalent to

$$\sum_{\eta=2}^{m}\left(\frac{\chi_m-1+\nu}{1-\nu}\right)|\jmath_\eta| - \sum_{\eta=m+1}^{\infty}\frac{\chi_{m+1}}{1-\nu}|\jmath_\eta| \ge 0. \tag{21}$$

To see that the function given by (19) gives the sharp result, we observe that for $\ell = re^{\frac{i\pi}{n}}$,

$$\begin{aligned}\frac{\aleph(\ell)}{\aleph_m(\ell)} &= 1 + \frac{1-\nu}{\chi_{m+1}}\ell^m \to 1 - \frac{1-\nu}{\chi_{m+1}} \\ &= \frac{\chi_{m+1}-1+\nu}{\chi_{m+1}}, \text{ when } r \to 1^-.\end{aligned}$$

4.2 Theorem

Let a function $\aleph$ of the form (1) belong to the class $UST_s(\iota, q, \hbar, \wp, \nu, \varsigma)$ and satisfy (12). Then

$$\Re\left(\frac{\aleph_m(\ell)}{\aleph(\ell)}\right) \ge \frac{\chi_{m+1}}{\chi_{m+1}+1-\nu}, (\ell \in \Delta), \tag{22}$$

where $\chi_{m+1} \ge 1-\nu$ and

$$\chi_\eta \ge \begin{cases} 1-\nu, & \text{if } \eta = 2,3,\cdots,m; \\ \chi_{m+1}, & \text{if } \eta = m+1, m+2, m+3, \cdots. \end{cases} \tag{23}$$

The result (22) is sharp with the function given by (19).

Proof We write by

$$\frac{1+\wp(\ell)}{1-\wp(\ell)} = \frac{\chi_{m+1}+1-\nu}{1-\nu}\left\{\frac{u_m(\ell)}{\aleph(\ell)} - \left(\frac{\chi_{m+1}}{\chi_{m+1}+1-\nu}\right)\right\}$$
$$= \left[\frac{1+\sum_{\eta=2}^{m}\jmath_\eta\ell^{\eta-1} - \frac{\chi_{m+1}}{1-\nu}\sum_{\eta=m+1}^{\infty}\jmath_\eta\ell^{\eta-1}}{1+\sum_{\eta=2}^{\infty}\jmath_\eta\ell^{\eta-1}}\right], \tag{24}$$

where

$$|\wp(\ell)| \leq \frac{\frac{\chi_{m+1}+1-\nu}{1-\nu}\sum\limits_{\eta=m+1}^{\infty}|\jmath_\eta|}{2-2\sum\limits_{\eta=2}^{m}|\jmath_\eta|-\frac{\chi_{m+1}+1-\nu}{1-\nu}\sum\limits_{\eta=m+1}^{\infty}|\jmath_\eta|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{\eta=2}^{m}|\jmath_\eta| + \sum_{\eta=m+1}^{\infty}\frac{\chi_{m+1}}{1-\nu}|\jmath_\eta| \leq 1.$$

We are making use of (12) to get (21). Finally, equality holds in (22) for the extremal function $\aleph(\ell)$ given by (19).

We next turn to ratios involving derivatives.

4.3 Theorem

Let be a function $\aleph$ of the form (1) belong to the class $UST_s(\iota, q, \hbar, \wp, \nu, \varsigma)$ and satisfy (12). Then

$$\Re\left(\frac{\aleph'(\ell)}{u'_m(\ell)}\right) \geq \frac{\chi_{m+1}-(m+1)(1-\nu)}{\chi_{m+1}}, (\ell \in \Delta),$$

$$\Re\left(\frac{\aleph'_m(\ell)}{\aleph'(\ell)}\right) \geq \frac{\chi_{m+1}}{\chi_{m+1}-(m+1)(1-\nu)}, (\ell \in \Delta), \quad (25)$$

where $\chi_{m+1} \geq (m+1)(1-\nu)$ and

$$\chi_\eta \geq \begin{cases} \eta(1-\nu), & \text{if } \eta = 2, 3, \cdots, m; \\ \eta\frac{\chi_{m+1}}{m+1}, & \text{if } \eta = m+1, m+2, m+3, \cdots. \end{cases} \quad (26)$$

The result is sharp with the function given by (19).

Proof We write by

$$\frac{1+\wp(\ell)}{1-\wp(\ell)} = \frac{\chi_{m+1}}{(m+1)(1-\nu)}\left\{\frac{\aleph'(\ell)}{\aleph'_m(\ell)} - \left(\frac{\chi_{m+1}-(m+1)(1-\nu)}{\chi_{m+1}}\right)\right\},$$

where

$$\wp(\ell) = \frac{\frac{\chi_{m+1}}{(m+1)(1-\nu)}\sum\limits_{\eta=m+1}^{\infty}\eta\jmath_\eta\ell^{\eta-1}}{2+2\sum\limits_{\eta=2}^{m}\eta\jmath_\eta\ell^{\eta-1}+\frac{\chi_{m+1}}{(m+1)(1-\nu)}\sum\limits_{\eta=m+1}^{\infty}\eta\jmath_\eta\ell^{\eta-1}}.$$

Now $|\wp(\ell)| \leq 1$ if and only if

$$\sum_{\eta=2}^{m}\eta|\jmath_\eta| + \frac{\chi_{m+1}}{(m+1)(1-\nu)}\sum_{\eta=m+1}^{\infty}\eta|\jmath_\eta| \leq 1.$$

From the condition (10), it is sufficient to show that

$$\sum_{\eta=2}^{m}\eta|\jmath_\eta| + \frac{\chi_{m+1}}{(m+1)(1-\nu)}\sum_{\eta=m+1}^{\infty}\eta|\jmath_\eta| \leq \sum_{\eta=2}^{\infty}\frac{\chi_\eta}{1-\nu}|\jmath_\eta|$$

which is equivalent to

$$\sum_{\eta=2}^{m}\left(\frac{\chi_\eta-(1-\nu)\eta}{1-\nu}\right)|\jmath_\eta| + \sum_{\eta=m+1}^{\infty}\frac{(m+1)\chi_\eta-\eta\chi_{m+1}}{(m+1)(1-\nu)}|\jmath_\eta| \geq 0.$$

To prove the result (25), define the function $\wp(\ell)$

$$\frac{1+\wp(\ell)}{1-\wp(\ell)} = \frac{(m+1)(1-\nu)+\chi_{m+1}}{(m+1)(1-\nu)}\left\{\frac{u'_m(\ell)}{u'(\ell)} - \left(\frac{\chi_{m+1}}{\chi_{m+1}+(m+1)(1-\nu)}\right)\right\}$$

where $\wp(\ell) =$

$$\frac{-\left(\frac{\chi_{m+1}+1}{(m+1)(1-\nu)}\right)\sum\limits_{\eta=m+1}^{\infty}\eta\jmath_\eta\ell^{\eta-1}}{2+2\sum\limits_{\eta=2}^{m}\eta\jmath_\eta\ell^{\eta-1}+\frac{1-\chi_{m+1}}{(m+1)(1-\nu)}\sum\limits_{\eta=m+1}^{\infty}\eta\jmath_\eta\ell^{\eta-1}}.$$

Now $|\wp(\ell)| \leq 1$ if and only if

$$\sum_{\eta=2}^{m}\eta|\jmath_\eta| + \sum_{\eta=m+1}^{\infty}\frac{\chi_{m+1}}{(m+1)(1-\nu)}\eta|\jmath_\eta| \leq 1. \quad (27)$$

It suffices to show that the left hand side of (27) is bounded previously by the condition

$$\sum_{\eta=2}^{\infty}\frac{\chi_\eta}{1-\nu}|\jmath_\eta|,$$

which is equivalent to

$$\sum_{\eta=2}^{\infty}\left(\frac{\chi_\eta}{1-\nu}-\eta\right)|\jmath_\eta| + \sum_{\eta=m+1}^{\infty}\left(\frac{\chi_\eta}{1-\nu}-\frac{\chi_{m+1}}{(m+1)(1-\nu)}\right)\eta|\jmath_\eta| \geq 0.$$

5 Integral Means Result

Motivated by an integral means work of Silverman [24] many have discussed integral means results for various subclasses of T. In that line inspired by the works of Ahuja et al. [25] and Magesh et al. [26] in the following theorem we find integral mean inequality for the functions in the class $\widetilde{UST}_s(\iota, q, \hbar, \wp, \nu, \varsigma)$.

For analytic functions u and v in Δ, u is said to be subordinate to v if there exists an analytic function w such that

$$w(0) = 0, \ |w(\ell)| < 1 \text{ and } \aleph(\ell) = v(w(\ell)), \ \ell \in \Delta. \quad (28)$$

This subordination will be denoted here by

$$u \prec v, \qquad \ell \in \Delta$$

or, conventionally, by

$$\aleph(\ell) \prec v(\ell), \ \ell \in \Delta.$$

In particular, when v is univalent in Δ,

$$u \prec v \ (\ell \in \Delta) \Leftrightarrow \aleph(0) = v(0) \text{ and } \aleph(\Delta) \subset v(\Delta).$$

5.1 Lemma

[27] If the functions u and v are analytic in Δ with $u \prec v$ then

$$\int_0^{2\pi} \left|\aleph(re^{i\theta})\right|^{\varkappa} d\theta \leq \int_0^{2\pi} \left|v(re^{i\theta})\right|^{\varkappa} d\theta,\ \varkappa > 0,$$
$$\ell = re^{i\theta} \quad \text{and} \quad 0 < r < 1. \tag{29}$$

Now, we establish the integral means inequality for the functions belonging to the class.

5.2 Theorem

If $u \in \widetilde{UST}_s(\iota, q, \hbar, \wp, \nu, \varsigma),$ and $\aleph_2$ is defined by

$$\aleph_2(\ell) = \ell - \frac{1-\nu}{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}\ell^2 \tag{30}$$

then for $\ell = re^{i\theta}$ and $0 < r < 1,$ we have

$$\int_0^{2\pi} \left|\aleph(re^{i\theta})\right|^{\varkappa} d\theta \leq \int_0^{2\pi} \left|\aleph_\eta(re^{i\theta})\right|^{\varkappa} d\theta, \quad \varkappa > 0. \tag{31}$$

Proof Let u of the form (4) and

$$\aleph_2(\ell) = \ell - \frac{1-\nu}{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}\ell^2,$$

then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{\eta=1}^{\infty} \jmath_\eta \ell^{\eta-1}\right|^{\varkappa} d\theta$$
$$\leq \int_0^{2\pi} \left|1 - \frac{1-\nu}{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}\ell\right|^{\varkappa} d\theta.$$

By Lemma 5.1, it suffices to show that

$$1 - \sum_{\eta=1}^{\infty} \jmath_\eta \ell^{\eta-1} \prec 1 - \frac{1-\nu}{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}\ell.$$

If we define the function $w(\ell)$ as follows:

$$w(\ell) = \sum_{\eta=2}^{\infty} \frac{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}{1-\nu} \jmath_\eta \ell^{\eta-1}. \tag{32}$$

From the above mentioned equation

$$w(0) = 0. \tag{33}$$

Again from (32), we have

$$|w(\ell)| \leq \sum_{\eta=2}^{\infty} \frac{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}{1-\nu} |\jmath_\eta||\ell|^{\eta-1}.$$

Since, $\ell = re^{i\theta}$ and $0 < r < 1,$ and using (12), therefore, from the above inequality, we have

$$|w(\ell)| \leq \sum_{\eta=2}^{\infty} \frac{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}{1-\nu} |\jmath_\eta| \leq 1 \tag{34}$$

From (32), we have

$$1 - \sum_{\eta=2}^{\infty} |\jmath_\eta|\ell^{\eta-1} = 1 - \frac{1-\nu}{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}\, w(\ell). \tag{35}$$

Since $w(\ell)$ is analytic in Δ, therefore in view of equations (28), (32), (33), and (35); inequality (34); and the subordination principle,

$$1 - \sum_{\eta=1}^{\infty} \jmath_\eta \ell^{\eta-1} \prec 1 - \frac{1-\nu}{[2]_q^{\wp}[1+\iota]^{\hbar}\,|4-\aleph_2(1+\nu)|}\ell.$$

Since, the function on the both sides of the above relation is analytic in Δ, therefore, in view of Lemma 5.1 and equation (30), we get assertion (31). This completes the proof of the Theorem 5.2.

6 Radii of close-to-convexity and Starlikness

6.1 Theorem

Let $u \in \widetilde{UST}_s(\iota, q, \hbar, \wp, \nu, \varsigma)$. Then $\aleph(\ell)$ is close-to-convex of order $\rho(0 \leq \rho < 1)$ in $|\ell| < r_1$, where

$$r_1 = \inf_\eta \left[\frac{(1-\rho)\Theta}{\rho(1-\nu)}\right]^{\frac{1}{\eta}}, \eta \geq 2 \tag{36}$$

here $\Theta = [\eta]_q^{\wp}[1+(\eta-1)\iota]^{\hbar}\,|2\eta - \aleph_\eta(1+\nu)|$ and

$$\aleph_\eta = 1 + \varsigma + \cdots + \varsigma^{\eta-1}.$$

The result is sharp.

Proof We must show that

$$|u'(\ell) - 1| \leq 1 - \rho, \text{ for } |\ell| < r_1.$$

From (4), we have

$$|u'(\ell) - 1| \leq \sum_{\eta=2}^{\infty} \eta \jmath_\eta \ell^{\eta-1}.$$

Thus $|u'(\ell) - 1| \leq 1 - \rho$, if

$$\sum_{\eta=2}^{\infty} \left(\frac{\eta}{1-\rho}\right) \jmath_\eta \ell^{\eta-1} \leq 1. \tag{37}$$

But by Theorem 2.1, (37) will be true if

$$\left(\frac{\eta}{1-\rho}\right)\ell^{\eta-1} \leq \frac{\Theta}{1-\nu}$$

$$\Rightarrow |\ell| \leq \left(\frac{(1-\rho)\Theta}{\eta(1-\nu)}\right)^{\frac{1}{\eta-1}}, \ \eta \geq 2. \tag{38}$$

6.2 Theorem

If $u \in \widetilde{UST}_s(\iota, q, \hbar, \wp, \nu, \varsigma)$ then $\aleph(\ell)$ is starlike of order $\rho (0 \leq \rho < 1)$ in $|\ell| < r_2$, where

$$r_2 = \inf_{\eta} \left[\frac{(1-\rho)\Theta}{(\eta-\rho)(1-\nu)} \right]^{\frac{1}{\eta}}, \eta \geq 2 \tag{39}$$

here Θ and $\aleph_\eta$ are defined in Theorem 6.1. The result is sharp.

Proof It is sufficient to show

$$\left| \frac{\ell u'(\ell)}{\aleph(\ell)} - 1 \right| \leq 1 - \rho, \text{ for } |\ell| < r_2.$$

we have

$$\left| \frac{\ell u'(\ell)}{\aleph(\ell)} - 1 \right| \leq \frac{\sum\limits_{\eta=2}^{\infty} (\eta - 1) \jmath_\eta \ell^{\eta-1}}{1 - \sum\limits_{\eta=2}^{\infty} \jmath_\eta \ell^{\eta-1}}.$$

Thus $\left| \frac{\ell u'(\ell)}{\ell)} - 1 \right| \leq 1 - \rho$, if

$$\sum_{\eta=2}^{\infty} \left(\frac{\eta - \rho}{1 - \rho} \right) \jmath_\eta \ell^{\eta-1} \leq 1. \tag{40}$$

But, by Theorem 2.1, (40) will be true if

$$\left(\frac{\eta - \rho}{1 - \rho} \right) \ell^{\eta-1} \leq \frac{\Theta}{1 - \nu}$$

that is, if $|\ell| \leq \left[\frac{(1-\rho)\Theta}{(\eta-\rho)(1-\nu)} \right]^{\frac{1}{\eta}}$.

7 Conclusions

The works presented in this paper are basically motivated by the well-established usage of the basic (or $q-$) calculus in the context of geometric function theory. We conclude this paper by noting that the derivative $q-$calculus operators described in section 2 can be used to investigate several other multivalent (or meromorphic) analytic function classes and their geometric properties such as coefficient estimates, distortion bounds, radii of starlikeness, convexity, and so on. These considerations can be pursued by using the theory of fractional $q-$calculus. By exploring these specific directions, researchers can further develop the theory and applications of analytic functions defined by q-analogue differential operators, contributing to both pure and applied mathematics.

In conclusion, this study has introduced and analyzed a specific subclass of analytic functions defined by the q-analogue differential operator. The main findings include new properties and characterizations that distinguish this subclass from those defined by classical differential operators. These results offer valuable insights into the structure and behavior of analytic functions within the framework of q-calculus, presenting potential applications in various fields of mathematical and physical sciences. While the study has focused on a particular subclass, the methodology and results provide a foundation for future research, which could extend these concepts to more complex functions and higher-dimensional settings. Overall, the use of the q-analogue differential operator opens new avenues for exploration and contributes significantly to the advancement of the theory of analytic functions.

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