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Extending Godunova-Levin Interval-Valued Functions to Stochastic Processes: New Hermite-Hadamard and Jensen-Type Inequalities

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Abstract This paper attempts to broaden the scope of the interval-valued functions by proposing the concept of Godunova-Levin interval-valued functions as stochastic processes. We present a novel framework for interval-valued harmonical (h_1, h_2)-Godunova-Levin stochastic processes. This approach seeks to address the inherent uncertainty and variability in real-world phenomena by establishing a solid mathematical foundation for interval-valued functions in stochastic settings. The fundamental goal of this study is to obtain fresh estimates for interval Hermite-Hadamard and Jensen-type inequalities in the setting of these stochastic processes. We obtain important results using sophisticated stochastic analysis and interval arithmetic approaches, which not only generalize existing inequalities but also provide a deeper understanding of the behavior of interval-valued functions under stochastic effects. The findings of this study have the potential to improve the applicability of interval-valued functions in a variety of stochastic scenarios, including financial modeling, engineering, and decision-making under uncertainty. Furthermore, the theoretical advances discussed here contribute to the larger subject of stochastic processes, bringing up new opportunities for research and application. However, the assumptions underpinning interval-valued functions and stochastic processes may limit the applicability of the presented approaches. Future study could investigate the relaxing of these assumptions and the application of the suggested framework to more complex stochastic systems.

AMS subject classification 2010 : 60G05, 41A55, 32F17, 26D15, 26A51, 26D20.

Keywords Hermite-Hadamard Inequalities, Godunova-Levin Interval-valued Stochastic Process, Harmonical H-convex Stochastic Process

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1 Introduction

1.1 Preliminary

Godunova-Levin interval-valued functions have attracted considerable attention in mathematical research due to their versatility and wide-ranging applications across diverse domains [1]. Originally formulated within deterministic contexts, these functions provide a robust framework for addressing uncertainty and imprecision. However, as contemporary mathematical investigations increasingly intersect with stochastic processes, the significance of studying stochastic processes becomes apparent.

Indeed, the study of stochastic processes holds immense importance across various fields owing to its capability to model and analyze random phenomena. Stochastic processes play a fundamental role in comprehending and forecasting uncertain outcomes, thereby proving indispensable in disciplines such as finance, engineering, biology, and economics ([2], [3], [4] and [5]). Consequently, there emerges a pressing need to extend the applicability of Godunova-Levin functions into the realm of stochastic processes .

One of the key motivations driving this research is the exploration of novel interval Hermite-Hadamard and Jensen-type inequalities within the context of stochastic processes, as demonstrated in the subsequent references [6], [7] and [8]. These inequalities serve as fundamental tools in mathematical analysis, playing pivotal roles in theoretical developments and practical applications across diverse fields [9], [10], [11] and [12]. By establishing estimates for these inequalities within the framework of interval-valued harmonical Godunova-Levin stochastic processes, we aim to enrich both the theoretical foundations and practical utility of interval-valued functions in stochastic contexts.

In this article, we embark on the journey of broadening the horizon of Godunova-Levin interval-valued functions by introducing them as stochastic processes. Our primary objective is to introduce the concept of interval-valued harmonical (h_1, h_2)-Godunova-Levin stochastic processes. By integrating stochasticity into the framework of Godunova-Levin functions, we seek to provide a more comprehensive model for capturing uncertainties inherent in real-world phenomena.

Before delving into the main results of our study, we will first introduce some essential definitions to pave the way forward.

1.2 Definitions

In our previous work (O. Rholam, M. Barmaki and I. Gretete, "Fractional integral inequalities of Hermite-Hadamard type for P-convex and Quasi-Convex stochastic process", the Australian journal of mathematical analysis and applications, vol. 20, no. 10, 2023), we defined the notion of mean square continuity (MS-C), mean-square Differentiability (MS-D), and mean-square integrability (MS-I), as well as the P-convexity and the Quasi-convexityfor a stochastic process, For a more comprehensive understanding of these key concepts and their significance in the context of our current study, we refer readers to [10], [12] and [13] . This papers delves into detailed explanations and analysis of these notions, providing valuable insights for the upcoming discussion .

Theorem 1.1. *(Hermite-Hadamard inequality for Jensen-convex stochastic processes, [6])*
$\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$, which is Jensen-convex and MS-C on $\mathcal{I}$,the following inequality holds for all $(\mu, \nu) \in \mathcal{I}^2$.

$$\mathcal{S}_p\left(\frac{\mu+\nu}{2}, \cdot\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu} \mathcal{S}_p(\chi, \cdot) d\chi \leq \frac{\mathcal{S}_p(\mu, \cdot)+\mathcal{S}_p(\nu, \cdot)}{2}.$$

Over the past decade, there has been significant interest in harmonic convexity and its implications in the study of inequalities. Introduced in 2013 [14], harmonic convexity brought about a series of Hermite-Hadamard inequalities tailored to this specific class of functions. Subsequent research by Noor et al. in 2015 introduced harmonic h-convex functions, along with corresponding Hermite-Hadamard inequalities [15]. Built on these foundations, further advancements extended the h-h inequality to various subclasses of functions, including interval h-convex, interval harmonic h-convex, interval (h_1, h_2)-convex, and interval harmonical (h_1, h_2)-convex functions [16] and [17]. Additionally, researchers such as Ohud Almutairi and Adem Kiliman expanded the scope of the h-h inequality by introducing new concepts like the h-Godunova-Levin function and establishing related inequalities [18]. We aim to introduce and explore the concepts of harmonic convexity and associated inequalities within the framework of stochastic processes. Our goal is to extend these well-established principles from the realm of deterministic functions to the domain of stochastic processes, thereby enriching the understanding and application of convexity-related theories in probabilistic settings.

And to that end , we will revisit some definitions, properties, and notations for stochastic processes that will be helpful throughout this study.

The set of all intervals of real numbers is denoted by $\mathcal{I}_{cb}$, while $\mathcal{I}_{cb}^{+}$ and $\mathcal{I}_{cb}^{-}$ represent the positive and negative interval subsets, respectively

Definition 1.1. *([19])*
The interval $[\mu_1]$, represented by $[\underline{\mu_1}, \overline{\mu_1}]$, consists of all elements χ in the set $\mathcal{I}_{cb}$ such that $\underline{\mu_1} \leq x \leq \overline{\mu_1}$, where $\underline{\mu_1}$ and $\overline{\mu_1}$ belong to $\mathcal{I}_{cb}$. It is characterized as a closed and bounded subset of $\mathbb{R}$. If $\underline{\mu 1}$ equals $\overline{\mu_1}$, $[\mu_1]$ is described as degenerate. It is deemed positive if $\underline{\mu_1} > 0$ or negative if $\overline{\mu_1} < 0$. , "$\subseteq$" is the inclusion relation, defined as follows :

$$[\mu_1] \subseteq [\mu_2] \Longleftrightarrow [\underline{\mu_1}, \overline{\mu_1}] \subseteq [\underline{\mathcal{S}_p}, \overline{\mathcal{S}_p}] \Longleftrightarrow \underline{\mu_2} \leq \underline{\mu_1}, \overline{\mu_1} \leq \overline{\mu_2}.$$

$\forall\alpha \in \mathbb{R}$ and interval $[\mu_1]$, the interval $\alpha[\mu_1]$ is defined as follows :

$$\alpha \cdot [\underline{\mu_1}, \overline{\mu_1}] = \begin{cases} [\alpha\underline{\mu_1}, \alpha\overline{\mu_1}] & , if \quad \alpha > 0 \\ \{0\} & , if \quad \alpha = 0 \\ [\alpha\overline{\mu_1}, \alpha\underline{\mu_1}] & , if \quad \alpha < 0 \end{cases}$$

For $[\mu_1] = [\underline{\mu_1}, \overline{\mu_1}]$ and $[\mu_2] = [\underline{\mu_2}, \overline{\mu_2}]$, we define the following operations by :

$$[\mu_1] + [\mu_2] = [\underline{\mu_1} + \underline{\mu_2}, \overline{\mu_1} + \overline{\mu_2}],$$
$$[\mu_1] - [\mu_2] = [\underline{\mu_1} - \underline{\mu_2}, \overline{\mu_1} - \overline{\mu_2}],$$
$$[\mu_1] \cdot [\mu_2] = [\min\{\underline{\mu_1}\underline{\mu_2}, \underline{\mu_1}\overline{\mu_1}, \overline{\mu_1}\underline{\mu_2}, \overline{\mu_1\mu_2}\}, \max\{\underline{\mu_1}\underline{\mu_2}, \underline{\mu_1}\overline{\mu_1}, \overline{\mu_1}\underline{\mu_2}, \overline{\mu_1\mu_2}\}],$$
$$\frac{[\mu_1]}{[\mu_2]} = \left[\min\{\frac{\underline{\mu_1}}{\underline{\mu_2}}, \frac{\underline{\mu_1}}{\overline{\mu_2}}, \frac{\overline{\mu_1}}{\underline{\mu_2}}, \frac{\overline{\mu_1}}{\overline{\mu_2}}\}, \max\{\frac{\underline{\mu_1}}{\underline{\mu_2}}, \frac{\underline{\mu_1}}{\overline{\mu_2}}, \frac{\overline{\mu_1}}{\underline{\mu_2}}, \frac{\overline{\mu_1}}{\overline{\mu_2}}\}\right],$$

where

$$0 \notin [\underline{\mu_2}, \overline{\mu_2}].$$

The Hausdorff-Pompeiu distance between two intervals $[\underline{\mu_1}, \overline{\mu_1}], [\underline{\mu_2}, \overline{\mu_2}]$ is defined by :

$$\mathcal{D}([\underline{\mu_1}, \overline{\mu_1}], [\underline{\mu_2}, \overline{\mu_2}]) = \max\{|\underline{\mu_1} - \underline{\mu_2}|, |\overline{\mu_1} - \overline{\mu_2}|\}$$

The definition of mean-squar integral of a stochastic process allows us to properly deduce that: $\forall\chi \in [\mu, \nu]$ if :

$$\mathcal{S}_{p1}(\chi, \cdot) \leq \mathcal{S}_{p2}(\chi, \cdot)$$

then:

$$\int_{\mu}^{\nu} \mathcal{S}_{p1}(\chi, \cdot)d\chi \leq \int_{\mu}^{\nu} \mathcal{S}_{p2}(\chi, \cdot)d\chi$$

making it the base for the development of the results that follow via interval calculus for stochastic process.

Definition 1.2.
Let $\mathcal{S}_p : [\mu, \nu] \times \mathcal{E} \to \mathcal{I}_{cb}$ such that $\mathcal{S}_p(\tau, \cdot) = [\underline{\mathcal{S}_p}(\tau, \cdot), \overline{\mathcal{S}_p}(\tau, \cdot)]$ for each $\tau \in [\mu, \nu]$ and $\underline{\mathcal{S}_p}, \overline{\mathcal{S}_p}$ are MS-I on $[\mu, \nu]$. Then :

$$\int_{\mu}^{\nu} \mathcal{S}_p(\tau, \cdot)d\tau = \left[\int_{\mu}^{\nu} \underline{\mathcal{S}_p}(\tau, \cdot)d\tau, \int_{\mu}^{\nu} \overline{\mathcal{S}_p}(\tau, \cdot)d\tau\right].$$

Definition 1.3.
A positive stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is said to be a Godunova-Levin if :

$$\mathcal{S}_p(\tau\mu + (1-\tau)\nu) \leq \frac{\mathcal{S}_p(\mu, \cdot)}{\tau} + \frac{\mathcal{S}_p(\nu, \cdot)}{(1-\tau)}$$

$\forall\mu, \nu \in \mathcal{I}$ and $\tau \in (0, 1)$.

Definition 1.4.
A stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is known as harmonically convex , if :

$$\mathcal{S}_p\left(\frac{\mu\nu}{\tau\mu + (1-\tau)\nu}, \cdot\right) \leq \tau\mathcal{S}_p(\mu, \cdot) + (1-\tau)\mathcal{S}_p(\nu, \cdot)$$

$\forall\mu, \nu \in \mathcal{I}$ and $\tau \in [0, 1]$.

Definition 1.5.
$h : [0, 1] \subseteq \mathcal{I} \to \mathbb{R}$ be a non negative function . A stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is said to be harmonical h-convex if :

$$\mathcal{S}_p\left(\frac{\mu\nu}{\tau\mu + (1-\tau)\nu}, \cdot\right) \leq h(\tau, \cdot)\mathcal{S}_p(\mu, \cdot) + h(1-\tau)\mathcal{S}_p(\nu, \cdot)$$

$\forall\mu, \nu \in \mathcal{I}$ and $\tau \in [0, 1]$.

Definition 1.6.
Let $h : [0,1] \subseteq \mathcal{I} \to \mathbb{R}$ be a positive function with $h \neq 0$. $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is said to be h-Godunova-Levin stochastic process, if :

$$\mathcal{S}_p(\tau\mu + (1-\tau)\nu) \leq \frac{\mathcal{S}_p(\mu,\cdot)}{h(\tau)} + \frac{\mathcal{S}_p(\nu,\cdot)}{h(1-\tau)}$$

for all $\mu, \nu \in \mathcal{I}$ and $\tau \in [0,1]$.

Definition 1.7.
Let $h : [0,1] \subseteq \mathcal{I} \to \mathbb{R}$ be a positive function with $h \neq 0$. $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is a harmonical h-Godunova-Levin stochastic process, if :

$$\mathcal{S}_p\left(\frac{\mu\nu}{\tau\mu + (1-\tau)\nu}, \cdot\right) \leq \frac{\mathcal{S}_p(\mu,\cdot)}{h(\tau)} + \frac{\mathcal{S}_p(\nu,\cdot)}{h(1-\tau)}$$

for all $\mu, \nu \in \mathcal{I}$ and $\tau \in [0,1]$.

Definition 1.8.
Let $h_1, h_2 : [0,1] \subseteq \mathcal{I} \to \mathbb{R}$ be positive functions. $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is a harmonical (h_1, h_2)-Godunova-Levin stochastic process, if :

$$\mathcal{S}_p\left(\frac{\mu\nu}{\tau\mu + (1-\tau)\nu}, \cdot\right) \leq \frac{\mathcal{S}_p(\mu,\cdot)}{h_1(\tau)h_2(1-\tau)} + \frac{\mathcal{S}_p(\nu,\cdot)}{h_1(1-\tau)h_2(\tau)}$$

for all $\mu, \nu \in \mathcal{I}$ and $\tau \in [0,1]$.

Remark 1.1. *In definition 1.8 :*

- *If $h_1(\tau) = h_2(\tau) = 1$, we get the defintion of a harmonical P-convex stochastic process [20].*
- *If $h_1(\tau) = \frac{1}{\tau}$ and $h_2(\tau) = 1$, we obtain the definition of a harmonical convex stochastic process [21].*
- *If $h_1(\tau) = \frac{1}{h(\tau,\cdot)}$ and $h_2(\tau) = 1$, 1.8 becomes the definition of a harmonical h-convex stochastic process [22].*
- *If $h_1(\tau) = (\tau,\cdot)^s$ and $h_2(\tau) = 1$, then definition 1.8 becomes the definition of a harmonical s-Godunova-Levin stochastic process.*

2 Main Results

Definition 2.1.
Let $h_1, h_2 : [0,1] \subseteq \mathcal{I} \to \mathbb{R}$ be positive functions such that $h_1, h_2 \neq 0$.
We define $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathcal{I}cb^+$ as a harmonic (h_1, h_2)-Godunova-Levin ($\mathbb{G}.\mathbb{L}$) interval-valued convex stochastic process if :

$$\frac{\mathcal{S}_p(\mu,\cdot)}{h_1(\tau)h_2(1-\tau)} + \frac{\mathcal{S}_p(\nu,\cdot)}{h_1(1-\tau)h_2(\tau)} \subseteq \mathcal{S}_p\left(\frac{\mu\nu}{\tau\mu + (1-\tau)\nu}, \cdot\right) \tag{1}$$

$\forall \mu, \nu \in \mathcal{I}$ and $\tau \in [0,1]$.
If the previous definition is modified by reversing the inclusion, the stochastic process $\mathcal{S}_p$ is referred to as a harmonical (h_1, h_2)-$\mathbb{G}.\mathbb{L}$ concave interval-valued stochastic process.
The spaces of all harmonical (h_1, h_2)-$\mathbb{G}.\mathbb{L}$ convex and (h_1, h_2)-$\mathbb{G}.\mathbb{L}$ concave interval-valued stochastic processes are denoted by $\mathbb{SPX}\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), S, \mathcal{I}_{cb}^+\right)$ and $\mathbb{SPV}\left(\left(\frac{1}{h1}, \frac{1}{h_2}\right), S, \mathcal{I}_{cb}^+\right)$, respectively.

Remark 2.1.
These spaces represent collections of stochastic processes that satisfy the specified harmonical convexity and concavity properties, where the operators $\mathbb{G}$ and $\mathbb{L}$ are used in conjunction with the intervals. The notations $\mathbb{SPX}$ and $\mathbb{SPV}$ along with the given parameters indicate the specific characteristics and constraints of the stochastic processes within these spaces.

Proposition 2.1.
Let $\mathcal{S}_p : [\mu, \nu] \times \mathcal{E} \to \mathcal{I}cb^+$ be a harmonic interval-valued (h_1, h_2)-$\mathbb{G}.\mathbb{L}$ stochastic process with $\mathcal{S}_p(\tau,\cdot) = [\underline{\mathcal{S}_p}(\tau,\cdot), \overline{\mathcal{S}_p}(\tau,\cdot)]$. Then :

$$\mathcal{S}_p \in \mathbb{SPX}\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mu,\nu], \mathcal{I}_{cb}^+\right) \iff \begin{cases} \underline{\mathcal{S}_p} \in \mathbb{SPX}\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mu,\nu]\ , \mathcal{I}_{cb}^+\right) \\ \overline{\mathcal{S}_p} \in \mathbb{SPV}\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mu,\nu], \mathcal{I}_{cb}^+\right) \end{cases} \tag{2}$$

Proof :
We consider $\mathcal{S}_p$ to be (h_1, h_2)-$\mathbb{G}.\mathbb{L}$ a convex interval valued stochastic process and $\tau \in [0,1]$, thus :

$$\frac{\mathcal{S}_p(\mu,\cdot)}{h_1(\tau)h_2(1-\tau)} + \frac{\mathcal{S}_p(\nu,\cdot)}{h_1(1-\tau)h_2(\tau)} \subseteq \mathcal{S}_p\left(\frac{\mu\nu}{\tau\mu+(1-\tau)\nu},\cdot\right),$$

and so we get :

$$\left[\frac{\underline{\mathcal{S}_p}(\mu,\cdot)}{h_1(\tau)h_2(1-\tau)} + \frac{\underline{\mathcal{S}_p}(\nu,\cdot)}{h_1(1-\tau)h_2(\tau)}, \frac{\overline{\mathcal{S}_p}(\mu,\cdot)}{h_1(\tau)h_2(1-\tau)} + \frac{\overline{\mathcal{S}_p}(\nu,\cdot)}{h_1(1-\tau)h_2(\tau)}\right]$$
$$\subseteq \mathcal{S}_p\left(\frac{\mu\nu}{\tau\mu+(1-\tau)\nu},\cdot\right),$$

which implies that :

$$\frac{\underline{\mathcal{S}_p}(\mu,\cdot)}{h_1(\tau)h_2(1-\tau)} + \frac{\underline{\mathcal{S}_p}(\nu,\cdot)}{h_1(1-\tau)h_2(\tau)} \geq \underline{\mathcal{S}_p}\left(\frac{\mu\nu}{\tau\mu+(1-\tau)\nu},\cdot\right)$$

and

$$\frac{\overline{\mathcal{S}_p}(\mu,\cdot)}{h_1(\tau)h_2(1-\tau)} + \frac{\overline{\mathcal{S}_p}(\nu,\cdot)}{h_1(1-\tau)h_2(\tau)} \leq \overline{\mathcal{S}_p}\left(\frac{\mu\nu}{\tau\mu+(1-\tau)\nu},\cdot\right),$$

we conclude that :
$\underline{\mathcal{S}_p} \in \mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}^+_{cb}\right)$ and $\overline{\mathcal{S}_p} \in \mathbb{SPV}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}^+_{cb}\right)$.
contrariwise, if we suppose that :
$\underline{\mathcal{S}_p} \in \mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}^+_{cb}\right)$ and $\overline{\mathcal{S}_p} \in \mathbb{SPV}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}^+_{cb}\right)$. Based on the above definition and set inclusion, we obtain the following: $\mathcal{S}_p \in \mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}^+_{cb}\right)$.

□

Proposition 2.2.
Any harmonic interval valued (h_1, h_2)-$\mathbb{G}.\mathbb{L}$ stochastic process $\mathcal{S}_p : [\mu,\nu]\times\mathcal{E} \to \mathcal{I}^+_{cb}$, with $\mathcal{S}_p(\tau,\cdot) = [\underline{\mathcal{S}_p}(\tau,\cdot), \overline{\mathcal{S}_p}(\tau,\cdot)]$ verifies the following statement :

$$\mathcal{S}_p \in \mathbb{SPV}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}^+_{cb}\right) \iff \begin{cases}\underline{\mathcal{S}_p} \in \mathbb{SPV}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu]\mathcal{I}^+_{cb}\right)\\ \overline{\mathcal{S}_p} \in \mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}^+_{cb}\right)\end{cases} \tag{3}$$

2.1 Hermite-Hadamard Inequalities

The aim of this section is to establish of several Hermite-Hadamard-type inequalities for interval-valued stochastic processes that satisfy harmonically the (h_1, h_2)-Godunova-Levin condition. We will consistently use to the relation $\Lambda(\chi,\psi) = h_1(\chi)h_2(\psi)$ forall $\chi,\psi \in [0,1]$ throughout the process.

Theorem 2.1.
Let $\mathcal{S}_p : [\mu,\nu]\times\mathcal{E} \to \mathcal{I}^+_{cb}$ be a stochastic process.
If $\mathcal{S}_p \in \mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\ \mathcal{I}^+_{cb}\right)$, where $h_1, h_2 : [0,1] \to \mathbb{R}^+$ are continuous functions, then the following is the case.

$$\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right) \supseteq \frac{\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau$$
$$\supseteq [\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\int_0^1 \frac{d\chi}{\Lambda(\chi,1-\chi)} \tag{4}$$

Proof :
If $\mathcal{S}_p \in \mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}^+_{cb}\right)$, we have :

$$\frac{\mathcal{S}_p(\rho,\cdot)}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]} + \frac{\mathcal{S}_p(\sigma,\cdot)}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]} \subseteq \mathcal{S}_p\left(\frac{2\rho\sigma}{\rho+\sigma},\cdot\right)$$

where :

$$\rho = \frac{\mu\nu}{\chi\mu + (1-\chi)\nu} \quad \text{and} \quad \sigma = \frac{\mu\nu}{(1-\chi)\mu + \chi\nu},$$

and so we get :

$$\frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}\left[\mathcal{S}_p\left(\frac{\mu\nu}{\chi\mu + (1-\chi)\nu},\cdot\right) + \mathcal{S}_p\left(\frac{\mu\nu}{(1-\chi)\mu + \chi\nu},\cdot\right)\right] \subseteq \mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right),$$

we multiply by $\Lambda\left(\frac{1}{2},\frac{1}{2}\right)$ to obtain :

$$\left[\mathcal{S}_p\left(\frac{\mu\nu}{\chi\mu + (1-\chi)\nu},\cdot\right) + \mathcal{S}_p\left(\frac{\mu\nu}{(1-\chi)\mu + \chi\nu},\cdot\right)\right] \subseteq \Lambda\left(\frac{1}{2},\frac{1}{2}\right)\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right).$$

Integrating both sides of the aforementioned inequality over $[0,1]$, we obtain :

$$\int_0^1 \underline{\mathcal{S}_p}\left(\frac{\mu\nu}{\chi\mu + (1-\chi)\nu},\cdot\right)d\chi + \int_0^1 \underline{\mathcal{S}_p}\left(\frac{\mu\nu}{(1-\chi)\mu + \chi\nu},\cdot\right)d\chi$$
$$\geq \Lambda\left(\frac{1}{2},\frac{1}{2}\right)\int_0^1 \underline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)d\chi,$$

and

$$\int_0^1 \overline{\mathcal{S}_p}\left(\frac{\mu\nu}{\chi\mu + (1-\chi)\nu},\cdot\right)d\chi + \int_0^1 \overline{\mathcal{S}_p}\left(\frac{\mu\nu}{(1-\chi)\mu + \chi\nu},\cdot\right)d\chi$$
$$\leq \Lambda\left(\frac{1}{2},\frac{1}{2}\right)\int_0^1 \overline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)d\chi.$$

Wich implies that :

$$\frac{2\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\underline{\mathcal{S}_p}(\tau,\cdot)}{\tau^2}d\tau \geq \Lambda\left(\frac{1}{2},\frac{1}{2}\right)\int_0^1 \underline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)d\chi$$
$$= \Lambda\left(\frac{1}{2},\frac{1}{2}\right)\underline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right).$$

In the way we obtain that :

$$\frac{2\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\overline{\mathcal{S}_p}(\tau,\cdot)}{\tau^2}d\tau \leq \Lambda\left(\frac{1}{2},\frac{1}{2}\right)\int_0^1 \overline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)d\chi$$
$$= \Lambda\left(\frac{1}{2},\frac{1}{2}\right)\overline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right).$$

Thus :

$$\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]\left[\underline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right),\overline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right] \supseteq \frac{2\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau.$$

And by multiplying the inclusion by $\frac{1}{2}$, we get the first result :

$$\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\left[\underline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right),\overline{\mathcal{S}_p}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right] \supseteq \frac{\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau$$

On the other hand we have :

$$\frac{\mathcal{S}_p(\mu,\cdot)}{h_1(\chi)h_2(1-\chi)} + \frac{\mathcal{S}_p(\nu,\cdot)}{h_1(1-\chi)h_2(\chi)} \subseteq \mathcal{S}_p\left(\frac{\mu\nu}{\chi\mu + (1-\chi)\nu},\cdot\right) \quad (*)$$
$$\frac{\mathcal{S}_p(\mu,\cdot)}{h_1(1-\chi)h_2(\chi)} + \frac{\mathcal{S}_p(\nu,\cdot)}{h_1(\chi)h_2(1-\chi)} \subseteq \mathcal{S}_p\left(\frac{\mu\nu}{(1-\chi)\mu + \chi\nu},\cdot\right) \quad (**),$$

we integrate the sum of (*) and (**) over $[0,1]$ to get :

$$[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\int_0^1\frac{1}{h_1(\chi)h_2(1-\chi)}d\chi+[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\int_0^1\frac{1}{h_1(1-\chi)h_2(\chi)}d\chi$$
$$\subseteq\int_0^1\left[\mathcal{S}_p\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)+\mathcal{S}_p\left(\frac{\mu\nu}{(1-\chi)\mu+\chi\nu},\cdot\right)\right]d\chi.$$

For $\chi=\frac{1}{2}$, both integrals are equal, and by multiplying by $\frac{1}{2}$ we get the second inclusion :

$$2\left(\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)\right)\int_0^1\frac{1}{\Lambda(\chi,1-\chi)}d\chi\subseteq\frac{2\mu\nu}{\nu-\mu}\int_\mu^\nu\frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau$$

□

Remark 2.2.

1. *For $h_1(\chi)=h_2(\chi)=1$, Theorem 2.1 is reduced to harmonical interval-valued P-stochastic process*

$$\frac{1}{2}\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\supseteq\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu\frac{\mathcal{S}_p(\tau,\cdot)d\tau}{\tau^2}\supseteq[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)].$$

2. *If $\Lambda(\chi,\psi)=h(\chi)$ Theorem 2.1 expresses harmonical h-Godunova-Levin interval-valued stochastic process*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{2}\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\supseteq\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu\frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau\supseteq[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\int_0^1\frac{d\chi}{h(\chi)}$$

3. *For $\Lambda(\chi,\psi)=\frac{1}{h(\chi)}$ in Theorem 2.1 we obtain a harmonical interval-valued h-convex stochastic process*

$$\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\supseteq\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu\frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau\supseteq[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\int_0^1 h(\chi)d\chi$$

4. *If $\Lambda(\chi,\psi)=\frac{1}{\Lambda(\chi,1-\chi)}$ Theorem 2.1 then we get a harmonic (h_1,h_2)-convex interval-valued stochastic process*

$$\frac{1}{2\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\supseteq\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu\frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau$$
$$\supseteq[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\int_0^1\Lambda(\chi,1-\chi)d\chi.$$

Theorem 2.2.
Let $h_1,h_2:[0,1]\to\mathbb{R}^+$ be continuous functions and $\mathcal{S}_p:[\mu,\nu]\times\mathcal{E}\to\mathcal{I}^+_{cb}$ a stochastic process such that $\mathcal{S}_p\in\mathbb{SPX}\left(\left((\frac{1}{h_1}\frac{1}{h_2}),[\mu,\nu],\mathcal{I}^+_{cb}\right)\right.$, then we have

$$\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\supseteq\mathcal{S}_1\supseteq\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu\frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau\supseteq\mathcal{S}_2$$
$$\supseteq\left\{[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\left[\frac{1}{2}+\frac{1}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}\right]\right\}\int_0^1\frac{d\chi}{\Lambda(\chi,1-\chi)}. \tag{5}$$

where

$$\mathcal{S}_1=\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\left[\mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right)+\mathcal{S}_p\left(\frac{4\mu\nu}{\nu+3\mu},\cdot\right)\right]$$
$$\mathcal{S}_2=\left[\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)+\left(\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}\right)\right]\int_0^1\frac{d\chi}{\Lambda(\chi,1-\chi)}$$

Proof :
Assuming that $\mathcal{S}_p \in \mathbb{SPX}\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mu,\nu], \mathcal{I}_{cb}^{+}\right)$ for $\left[\mu, \frac{2\mu\nu}{\mu+\nu}\right]$, we get

$$\frac{\mathcal{S}_p\left(\frac{\mu\frac{2\mu\nu}{\mu+\nu}}{\chi\mu+(1-\chi)\frac{2\mu\nu}{\mu+\nu}}\right)}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}+\frac{\mathcal{S}_p\left(\frac{\mu\frac{2\mu\nu}{\mu+\nu}}{(1-\chi)\mu+\chi\frac{2\mu\nu}{\mu+\nu}},\cdot\right)}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]} \subseteq \mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right).$$

Thus

$$\frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}\left[\mathcal{S}_p\left(\frac{\mu\frac{2\mu\nu}{\mu+\nu}}{\chi\mu+(1-\chi)\frac{2\mu\nu}{\mu+\nu}},\cdot\right)+\mathcal{S}_p\left(\frac{\mu\frac{2\mu\nu}{\mu+\nu}}{(1-\chi)\mu+\chi\frac{2\mu\nu}{\mu+\nu}},\cdot\right)\right] \subseteq \mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right).$$

By integrating over $[0,1]$, we obtain

$$\frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]} \times \int_0^1 \begin{bmatrix} \underline{\mathcal{S}_p}\left(\frac{\mu\frac{2\mu\nu}{\mu+\nu}}{\chi\mu+(1-\chi)\frac{2\mu\nu}{\mu+\nu}},\cdot\right)d\chi+\underline{\mathcal{S}_p}\left(\frac{\mu\frac{2\mu\nu}{\mu+\nu}}{(1-\chi)\mu+\chi\frac{2\mu\nu}{\mu+\nu}},\cdot\right)d\chi, \\ \overline{\mathcal{S}_p}\left(\frac{\mu\frac{2\mu\nu}{\mu+\nu}}{\chi\mu+(1-\chi)\frac{2\mu\nu}{\mu+\nu}},\cdot\right)d\chi+\overline{\mathcal{S}_p}\left(\frac{\mu\frac{2\mu\nu}{\mu+\nu}}{(1-\chi)\mu+\chi\frac{2\mu\nu}{\mu+\nu}},\cdot\right)d\chi \end{bmatrix}$$
$$\subseteq \mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right).$$

Then

$$\frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]} \times \begin{bmatrix} \frac{2\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\underline{\mathcal{S}_p}(\tau,\cdot)}{\tau^2}d\tau+\frac{2\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\underline{\mathcal{S}_p}(\tau,\cdot)}{\tau^2}d\tau, \\ \frac{2\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\overline{\mathcal{S}_p}(\tau,\cdot)}{\tau^2}d\tau+\frac{2\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\overline{\mathcal{S}_p}(\tau,\cdot)}{\tau^2}d\tau \end{bmatrix}$$
$$\subseteq \mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right)$$
$$=\frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}\left[\frac{4\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\underline{\mathcal{S}_p}(\tau,\cdot)}{\tau^2}d\tau, \frac{4\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\overline{\mathcal{S}_p}(\tau,\cdot)}{\tau^2}d\tau\right] \subseteq \mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right)$$
$$=\frac{4}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}\left[\frac{\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau\right] \subseteq \mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right)$$
$$=\frac{\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau \subseteq \frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right) \quad (*)$$

Likewise, in $\left[\frac{2\mu\nu}{\mu+\nu}, \nu\right]$, we get the following

$$\frac{\mu\nu}{\nu-\mu}\int_{\frac{2\mu\nu}{\mu+\nu}}^{\nu} \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau \subseteq \frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\mathcal{S}_p\left(\frac{4\mu\nu}{\nu+3\mu},\cdot\right), \quad (**)$$

And by adding (*) and (**), we obtain

$$\begin{aligned}
\mathcal{S}_1 &= \frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\left[\mathcal{S}_p\left(\frac{4\mu\nu}{3\mu+\nu},\cdot\right)+\mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right)\right] \\
&\supseteq \left[\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau\right] \\
&= \frac{1}{2}\left[\frac{2\mu\nu}{\nu-\mu}\int_\mu^{\frac{2\mu\nu}{\mu+\nu}} \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau+\frac{2\mu\nu}{\nu-\mu}\int_{\frac{2\mu\nu}{\mu+\nu}}^{\nu} \frac{\mathcal{S}_p(\tau,\cdot)}{\tau^2}d\tau\right] \\
&\supseteq \frac{1}{2}\left[\left[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right]\int_0^1 \frac{d\chi}{\Lambda(\chi,1-\chi)}\right] \\
&+ \frac{1}{2}\left[\left[\mathcal{S}_p(\nu,\cdot)+\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right]\int_0^1 \frac{d\chi}{\Lambda(\chi,1-\chi)}\right] \\
&= \frac{1}{2}\left[\left\{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)+2\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right\}\int_0^1 \frac{d\chi}{\Lambda(\chi,1-\chi)}\right] \\
&= \left[\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}+\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right]\int_0^1 \frac{d\chi}{\Lambda(\chi,1-\chi)} = \mathcal{S}_2
\end{aligned}$$

Then again

$$\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)=\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\mathcal{S}_p\left(\frac{2\frac{4\mu\nu}{\mu+3\nu}\frac{4\mu\nu}{\nu+3\mu}}{\frac{4\mu\nu}{\mu+3\nu}+\frac{4\mu\nu}{\nu+3\mu}},\cdot\right)$$

$$\supseteq\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\left[\frac{\mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right)}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}+\frac{\mathcal{S}_p\left(\frac{4\mu\nu}{\nu+3\mu},\cdot\right)}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}\right]$$

$$\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\left[\mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right)+\mathcal{S}_p\left(\frac{4\mu\nu}{\mu+3\nu},\cdot\right)\right]$$

$$\supseteq\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\times\left[\begin{array}{c}\frac{1}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}\left[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right]\\+\frac{1}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}\left[\mathcal{S}_p(\nu,\cdot)+\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right]\end{array}\right]$$

$$=\frac{1}{4}\left\{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)+2\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right\}$$

$$\supseteq\left[\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}+\mathcal{S}_p\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\right]\int_0^1\frac{d\chi}{\Lambda(\chi,1-\chi)}$$

$$\supseteq\left[\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}+\frac{\mathcal{S}_p(\mu,\cdot)}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}+\frac{\mathcal{S}_p(\nu,\cdot)}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}\right]\int_0^1\frac{d\chi}{\Lambda(\chi,1-\chi)}$$

$$=\left[\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}+\frac{1}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\right]\int_0^1\frac{d\chi}{\Lambda(\chi,1-\chi)}$$

$$=\left\{[\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)]\left[\frac{1}{2}+\frac{1}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}\right]\right\}\int_0^1\frac{d\chi}{\Lambda(\chi,1-\chi)}.$$

□

Theorem 2.3.
Let $\mathcal{S}_{p1},\mathcal{S}_{p2}:[\mu,\nu]\times\mathcal{E}\to\mathcal{I}_{cb}^+$ be two stochastic processes. If $\mathcal{S}_{p1},\mathcal{S}_{p2}$ are $\mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}_{cb}^+\right)$, where $h_1,h_2:[0,1]\to\mathbb{R}^+$ are continuous functions, then

$$\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu\frac{\mathcal{S}_{p1}(\tau,\cdot)\mathcal{S}_{p2}(\tau,\cdot)}{\tau^2}d\tau\supseteq\mathcal{S}_1(\mu,\nu)\int_0^1\frac{1}{\Lambda^2(\chi,1-\chi)}d\chi$$
$$+\mathcal{S}_2(\mu,\nu)\int_0^1\frac{1}{\Lambda(\chi,\chi)\Lambda(1-\chi,1-\chi)}d\chi,\tag{6}$$

where

$$\mathcal{S}_1(\mu,\nu)=\mathcal{S}_{p1}(\mu,\cdot)\mathcal{S}_{p2}(\mu,\cdot)+\mathcal{S}_{p1}(\nu,\cdot)\mathcal{S}_{p2}(\nu,\cdot),$$
$$\mathcal{S}_2(\mu,\nu)=\mathcal{S}_{p1}(\mu,\cdot)\mathcal{S}_{p2}(\nu,\cdot)+\mathcal{S}_{p1}(\nu,\cdot)\mathcal{S}_{p2}(\mu,\cdot).$$

Proof : If $\mathcal{S}_{p1},\mathcal{S}_{p2}\in\mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}_{cb}^+\right)$; then, the following is satisfied

$$\frac{\mathcal{S}_{p1}(\mu,\cdot)}{h_1(\chi)h_2(1-\chi)}+\frac{\mathcal{S}_{p1}(\nu,\cdot)}{h_1(1-\chi)h_2(\chi)}\subseteq\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)$$
$$\frac{\mathcal{S}_{p2}(\mu,\cdot)}{h_1(\chi)h_2(1-\chi)}+\frac{\mathcal{S}_{p2}(\nu,\cdot)}{h_1(1-\chi)h_2(\chi)}\subseteq\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)$$

Thus

$$\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\supseteq$$
$$\frac{\mathcal{S}_{p1}(\mu,\cdot)\mathcal{S}_{p2}(\mu,\cdot)}{\Lambda^2(\chi,1-\chi)}+\frac{\mathcal{S}_{p1}(\mu,\cdot)\mathcal{S}_{p2}(\nu,\cdot)+\mathcal{S}_{p1}(\nu,\cdot)\mathcal{S}_{p2}(\mu,\cdot)}{\Lambda(\chi,\chi)\Lambda(1-\chi,1-\chi)}+\frac{\mathcal{S}_{p1}(\nu,\cdot)\mathcal{S}_{p2}(\nu,\cdot)}{\Lambda^2(1-\chi,\chi)}.$$

We integrate the inclusion above over $[0,1]$,to get :

$$\begin{aligned}
&\int_0^1 \mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)d\chi\\
&=\Big[\int_0^1 \underline{\mathcal{S}_{p1}}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\underline{\mathcal{S}_{p2}}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)d\chi,\\
&\int_0^1 \overline{\mathcal{S}_{p1}}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\overline{\mathcal{S}_{p2}}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)d\chi\Big]\\
&=\left[\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\underline{\mathcal{S}_{p1}}(\tau,\cdot)\underline{\mathcal{S}_{p2}}(\tau,\cdot)}{\tau^2}d\tau,\ \frac{\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\overline{\mathcal{S}_{p1}}(\tau,\cdot)\overline{\mathcal{S}_{p2}}(\tau,\cdot)}{\tau^2}d\tau\right]\\
&=\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\mathcal{S}_{p1}(\tau,\cdot)\mathcal{S}_{p2}(\tau,\cdot)}{\tau^2}d\tau
\end{aligned}$$

$$\supseteq\left[\begin{array}{c}\int_0^1 \frac{[\mathcal{S}_{p1}(\mu,\cdot)\mathcal{S}_{p2}(\mu,\cdot)+\mathcal{S}_{p1}(\nu,\cdot)\mathcal{S}_{p2}(\nu,\cdot)]}{\Lambda^2(\chi,1-\chi)}d\chi\\ +\int_0^1 \frac{[\mathcal{S}_{p1}(\mu,\cdot)\mathcal{S}_{p2}(\nu,\cdot)+\mathcal{S}_{p1}(\nu,\cdot)\mathcal{S}_{p2}(\mu,\cdot)]}{\Lambda(\chi,\chi)\Lambda(1-\chi,1-\chi)}d\chi\end{array}\right],$$

which implies that :

$$\begin{aligned}
&\frac{\mu\nu}{\nu-\mu}\int_s^\nu \frac{\mathcal{S}_{p}(\tau,\cdot)\mathcal{S}_{p2}(\tau,\cdot)}{\tau^2}d\tau\\
&\supseteq \mathcal{S}_1(\mu,\nu)\int_0^1\frac{d\chi}{\Lambda^2(\chi,1-\chi)}+\mathcal{S}_2(\mu,\nu)\int_0^1\frac{d\chi}{\Lambda(\chi,\chi)\Lambda(1-\chi,1-\chi)}
\end{aligned}$$

□

Theorem 2.4.
Let $\mathcal{S}_{p1},\mathcal{S}_{p2}:[\mu,\nu]\times\mathcal{E}\to\mathcal{I}_{cb}^+$ be two stochastic processes. If $\mathcal{S}_{p1},\mathcal{S}_{p2}$ are $\mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}_{cb}^+\right)$, where $h_1,h_2:[0,1]\to\mathbb{R}^+$ are continuous functions, then we have :

$$\begin{aligned}
&\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\mathcal{S}_{p1}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\supseteq\frac{\mu\nu}{\nu-\mu}\int_\mu^\nu\frac{\mathcal{S}_{p1}(\tau,\cdot)\mathcal{S}_{p2}(\tau,\cdot)}{\tau^2}d\tau\\
&+\mathcal{S}_1(\mu,\nu)\int_0^1\frac{1}{\Lambda(\chi,\chi)\Lambda(1-\chi,1-\chi)}d\chi+\mathcal{S}_2(\mu,\nu)\int_0^1\frac{1}{\Lambda^2(\chi,1-\chi)}d\chi. \qquad (7)
\end{aligned}$$

Proof : Using the fact that $\mathcal{S}_{p1},\mathcal{S}_{p2}$ are $\mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}_{cb}^+\right)$ we get :

$$\begin{aligned}
\mathcal{S}_{p1}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)&\supseteq\frac{\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}+\frac{\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)},\\
\mathcal{S}_{p2}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)&\supseteq\frac{\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}+\frac{\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)}{\Lambda\left(\frac{1}{2},\frac{1}{2}\right)}.
\end{aligned}$$

Then :

$$\begin{aligned}
&\mathcal{S}_{p1}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\\
&\supseteq\frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\times\left[\begin{array}{c}\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\\+\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\nu+(1-\chi)\mu},\cdot\right)\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\nu+(1-\chi)\mu},\cdot\right)\end{array}\right]\\
&+\frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\times\left[\begin{array}{c}\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\nu+(1-\chi)\mu},\cdot\right)\\+\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\nu+(1-\chi)\mu},\cdot\right)\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\end{array}\right]
\end{aligned}$$

$$= \frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2} \times \begin{bmatrix} \mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\mu+(1-\chi)\nu},\cdot\right) \\ +\mathcal{S}_{p1}\left(\frac{\mu\nu}{\chi\nu+(1-\chi)\mu},\cdot\right)\mathcal{S}_{p2}\left(\frac{\mu\nu}{\chi\nu+(1-\chi)\mu},\cdot\right) \end{bmatrix}$$

$$+\frac{1}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2} \times \begin{bmatrix} \left(\frac{2}{\Lambda(\chi,\chi)\Lambda(1-\chi,1-\chi)}\right)\mathcal{S}_1(\mu,\nu) \\ +\left(\frac{1}{\Lambda^2(\chi,1-\chi)}+\frac{1}{\Lambda^2(1-\chi,\chi)}\right)\mathcal{S}_2(\mu,\nu) \end{bmatrix}.$$

We integrate the inequality over $[0,1]$ to get :

$$\int_0^1 \mathcal{S}_{p1}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)d\chi$$

$$= \begin{bmatrix} \int_0^1 \underline{\mathcal{S}_{p1}}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)d\chi \\ ,\int_0^1 \overline{\mathcal{S}_{p1}}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)d\chi \end{bmatrix}$$

$$= \mathcal{S}_{p1}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)\mathcal{S}_{p2}\left(\frac{2\mu\nu}{\mu+\nu},\cdot\right)d\chi \supseteq \frac{2}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\frac{2\mu\nu}{\nu-\mu}\int_\mu^\nu \frac{\mathcal{S}_{p1}(\tau,\cdot)\mathcal{S}_{p2}(\tau,\cdot)}{\tau^2}d\tau\right]$$

$$+\frac{2}{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathcal{S}_1(\mu,\nu)\int_0^1 \frac{d\chi}{\Lambda(\chi,\chi)\Lambda(1-\chi,1-\chi)} + \mathcal{S}_2(\mu,\nu)\int_0^1 \frac{d\chi}{\Lambda^2(\chi,1-\chi)}\right].$$

And by multiplying the equality above by $\frac{\left[\Lambda\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}$ we obtain the result.

□

Jensen-Type Inequality

Our aim in this part is to establish a Jensen-type inequality for interval-valued stochastic processes that are harmonically (h_1,h_2)-Godunova-Levin.

Theorem 2.5. *Consider $\varsigma_1,\varsigma_2,\ldots\varsigma_\eta \in \mathcal{I}_{cb}^+$ with $\eta \geq 2$, and let $\mathcal{S}_p$ be a stochastic process of $\mathcal{S}_p \in \mathbb{SPX}\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mu,\nu],\mathcal{I}_{cb}^+\right)$ where h_1,h_2 are super multiplicative positive functions.*
If $\chi_1,\chi_2,\ldots,\chi_\eta \in I \subseteq \mathcal{I}_{cb}^+$, then

$$\mathcal{S}_p\left(\frac{1}{\frac{1}{\mathcal{S}_\eta}\sum_{i=1}^{\eta}\varsigma_i\chi_i},\cdot\right) \supseteq \sum_{i=1}^{\eta}\left[\frac{\mathcal{S}_p(\chi_i,\cdot)}{\Lambda\left(\frac{\varsigma_i}{\mathcal{S}_\eta},\frac{\mathcal{S}_{\eta-1}}{\mathcal{S}_\eta}\right)}\right] \tag{8}$$

where

$$\mathcal{S}_\eta = \sum_{i=1}^{\eta}\varsigma_i\chi_i$$

.

Proof : For $\eta = 2$, the result in our theorem is verified. We suppose (8) holds for $\eta-1$, thus

$$\mathcal{S}_p\left(\frac{1}{\frac{1}{\mathcal{S}_\eta}\sum_{i=1}^{\eta}\varsigma_i\chi_i},\cdot\right) = \mathcal{S}_p\left(\frac{1}{\frac{\varsigma_\eta}{\mathcal{S}_\eta}\chi_\eta+\sum_{i=1}^{\eta-1}\frac{\varsigma_i}{\mathcal{S}_\eta}\chi_i},\cdot\right)$$

$$\supseteq \frac{\mathcal{S}_p(\chi_\eta,\cdot)}{h_1\left(\frac{\varsigma_\eta}{\mathcal{S}_\eta}\right)h_2\left(\frac{\mathcal{S}_{\eta-1}}{\mathcal{S}_\eta}\right)} + \frac{\mathcal{S}_p\left(\sum_{i=1}^{\eta-1}\frac{\varsigma_i}{\mathcal{S}_\eta}\chi_i,\cdot\right)}{h_1\left(\frac{\mathcal{S}_{\eta-1}}{\mathcal{S}_\eta}\right)h_2\left(\frac{\varsigma_\eta}{\mathcal{S}_\eta}\right)}$$

$$\supseteq \frac{\mathcal{S}_p(\chi_\eta,\cdot)}{h_1\left(\frac{\varsigma_\eta}{\mathcal{S}_\eta}\right)h_2\left(\frac{\mathcal{S}_{\eta-1}}{\mathcal{S}_\eta}\right)} + \sum_{i=1}^{\eta-1}\left[\frac{\mathcal{S}_p(\chi_i,\cdot)}{\Lambda\left(\frac{\varsigma_i}{\mathcal{S}_\eta},\frac{\mathcal{S}_{\eta-2}}{\mathcal{S}_{\eta-1}}\right)}\right]\frac{1}{h_1\left(\frac{\mathcal{S}_{\eta-1}}{\mathcal{S}_\eta}\right)h_2\left(\frac{\varsigma_\eta}{\mathcal{S}_\eta}\right)}$$

$$\supseteq \frac{\mathcal{S}_p(\chi_\eta,\cdot)}{h_1\left(\frac{\varsigma_\eta}{\mathcal{S}_\eta}\right)h_2\left(\frac{\mathcal{S}_{\eta-1}}{\mathcal{S}_\eta}\right)} + \sum_{i=1}^{\eta-1}\left[\frac{\mathcal{S}_p(\chi_i,\cdot)}{\Lambda\left(\frac{\varsigma_i}{\mathcal{S}_\eta},\frac{\mathcal{S}_{\eta-2}}{\mathcal{S}_{\eta-1}}\right)}\right]$$

$$\supseteq \sum_{i=1}^{\eta}\left[\frac{\mathcal{S}_p(\chi_i,\cdot)}{\Lambda\left(\frac{\varsigma_i}{\mathcal{S}_\eta},\frac{\mathcal{S}_{\eta-1}}{\mathcal{S}_\eta}\right)}\right]$$

Therefore, the theorem is validated through mathematical induction.

□

3 Conclusions

In this study, we introduced the concept of harmonical interval-valued (h_1, h_2)-Godunova-Levin stochastic processes and established Jensen- and Hermite-Hadamard-type inequalities for them. To validate our main results, we explored several exceptional cases, with the hope that this study can lead to more robust models and methods for dealing with interval uncertainties in stochastic environments, and ultimately providing more accurate predictions and analyses in various applications.

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