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Solutions of Fuzzy Fractional Boundary Value Problems: A Novel Approach

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Abstract In this paper, the solution of fractional boundary value problem with fuzzy boundary conditions is investigated. The fuzzy fractional boundary value problem is decomposed into collections of classical fractional boundary value problems. Then Adomian decomposition method is applied to solve these classical fractional boundary value problems. The collection of all solutions to these fractional boundary value problems provide the solution to the fuzzy fractional boundary value problem. The solution to this problem is expressed in terms of a fuzzy collection of crisp real functions. The boundary value problem is satisfied by every real function in the solution set. The degree of membership of each function is determined by the minimum membership degree among its associated fuzzy boundary values. It can be demonstrated that if the corresponding fractional problem has a unique solution, then the fuzzy fractional problem also has one. It showed that when triangular fuzzy numbers are assigned to the boundary values, the resulting solution at a specific time also takes the form of a triangular fuzzy number. An example is provided to explain the suggested method.

Keywords Fuzzy Set, Fuzzy Fractional Boundary Value Problem, Fuzzy Fractional Differential Equations, Adomian Decomposition Method

1 Introduction

Fuzzy set theory is a well-known and effective approach for modeling uncertain problems. Therefore, fuzzy concepts have been used to model a wide range of natural phenomena. In particular, the fuzzy boundary value problem is a significant model in scientific studies. The concept of fuzzy boundary value problem(FBVP) was initially investigated by Lakshmikantham et al. [1] and O'Regan et al. [2]. According to these researchers, the FBVP is converted to a fuzzy integral equation and then solved. Moreover, the existence of the solution to the FBVP is provided by Chen et al. [3, 4]. Agarwal et al. [5] investigated fuzzy solutions to multi-point boundary value problems. Several researchers have explored the uniqueness and existence of the solution to the FBVP [6, 7, 8, 9, 10].

In recent decades, fractional calculus has drawn the attention of many scientists and technologists due to its numerous applications in mathematics, science, rheology, finance, economics, and other domains of engineering. Furthermore, fractional calculus has significant applications in describing memory and hereditary processes. Compared to ordinary derivatives, the notion of fractional-order derivatives provides a more elegant explanation for the aforementioned processes. Due to their importance, researchers have devoted a lot of time and energy to mathematically modeling real-world processes using fractional-order derivatives and integrals instead of classical derivatives and integrals. In this context, numerous books, papers, and monographs have been published that report on a variety of significant findings and applications [11, 12, 13, 14]. As a result, over the last two decades, utilizing fractional differential equations in a fuzzy environment has become essential for generalization in modeling.

Nowadays, the field of fuzzy fractional differential equations (FFDEs) is experiencing significant growth as a novel area of fuzzy mathematics. The FFDEs are a fundamental concept of fuzzy analysis. The solution of fuzzy fractional boundary value problem (FFBVP) has been discussed in this study. Abdel Aal et al. [15] provided a solution for FFBVP with order 2α. The

Fractional Differential Transform Method is utilized to provide an approximative solution for FFBVPs in [16]. Elmfadel et al. [17] explored the existence of solutions to the nonlinear FFBVP. Vinothkumar et al. [18] utilized the finite difference method. Very recently, Jawad Hashim et al. [19] investigated two-point FFBVPs utilizing the homotopy analysis approach. Hasan et al. [20] provided the Hilbert solution for the two-point boundary value problem with fuzzy fractional derivative.

In the above-mentioned studies, it is believed that the derivative is the gH-derivative, which comes in four kinds: (1,1), (1,2), (2,1), and (2,2) derivatives. A drawback of the fuzzy derivative concept is the challenge of determining which kind of derivative to employ when solving FFBVPs, as well as establishing the uniqueness of the solution. In order to overcome this difficulty, Gasilov et al. [21, 22] offered a novel method for solving FBVP. The problem is interpreted as a collection of crisp problems. The authors presented a method that utilizes the properties of linear transformations and also demonstrates that if there is a unique solution to the corresponding crisp problem, there must also be a unique solution to the fuzzy problem. Furthermore, they proved that when triangular fuzzy numbers are assigned to the boundary values, the resulting solution at a specific time also takes the form of a triangular fuzzy number.

Motivated by the approach mentioned above, the objective of our research is to develop the method for solving FFBVP which is extension of the method for solving FBVP proposed by Gasilov et al. [21, 22]. We interpret the problem as a collection of fractional problems.

The organization of this work is structured in the following way: Section 2 serves an introduction to the fundamental ideas of fractional calculus that help us comprehend the work in this study. Section 3 discusses the Adomian decomposition approach. Section 4 proposes a methodology for solving the FFBVP. Section 5 offers a numerical demonstration to showcase and evaluate the capabilities of the proposed method. Section 6 presents the conclusion.

2 Basics Concept

The definitions and notations used in fractional calculus are introduced in this section and will be used throughout the rest of the paper. Gasilov et al. [21, 22] introduced the fundamental idea of fuzzy set theory.

2.1 Definition [13, 23]

The definition of the Riemann-Liouville fractional integral of order $\alpha > 0$ is

$$(\mathcal{J}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds$$

where $0 < t < 1$ and $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$

2.2 Definition [13, 23]

The definition of the Caputo fractional derivative of order $\alpha > 0$ is

$$({}^{C}D^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$$

where $n = [\alpha] + 1$ for $\alpha \notin N_0,\ 0 < t < 1$ and $\Gamma(n-\alpha) = \int_0^{\infty} t^{n-\alpha-1} e^{-t} dt$

2.3 Remarks [24, 12]

1. $\mathcal{J}^{\alpha} \mathcal{J}^{\beta} f = \mathcal{J}^{\alpha+\beta} f,\ \ \alpha, \beta \geq 0$
2. $\mathcal{J}^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}$
3. $\mathcal{J}^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k,\ \ m-1 < \alpha \leq m$

3 Adomian Decomposition Method (ADM)

Another method for solving the boundary value problem (BVP) with fractional order is the ADM. The American mathematician G. Adomian proposed this approach, which has been used to solve various kinds of problems [25, 26, 27]. The benefit of this approach is that it offers a direct methodology for solving the problems, which means it does not require any transformation, linearization, large amounts of computing, or perturbation.

Consider the following BVP with fractional order

$$\begin{cases} {}^{C}D^{\alpha} y(t) + a_1 y'(t) + a_2 y(t) = f(t), & 1 < \alpha \leq 2 \\ y(0) = a \\ y(T) = b \end{cases} \tag{1}$$

Here, ${}^{C}D^{\alpha}$ represents the Caputo fractional derivatives and a_1 and a_2 are constants.

ADM for the boundary value problem with fractional derivative suggested by Jafari and Daftardar [24] is employed for the problem (1).

Applying $\mathcal{J}^{\alpha}$ on both sides of the problem (1), we have

$$\begin{aligned} \mathcal{J}^{\alpha}[{}^{C}D^{\alpha} y(t)] + a_1 \mathcal{J}^{\alpha}[y'(t)] + a_2 \mathcal{J}^{\alpha}[y(t)] &= \mathcal{J}^{\alpha}[f(t)] \\ y(t) = y(0) + \beta t - a_1 \mathcal{J}^{\alpha}[y'(t)] - a_2 \mathcal{J}^{\alpha}[y(t)] &+ \mathcal{J}^{\alpha}[f(t)] \end{aligned} \tag{2}$$

where $y'(0) = \beta$. ADM decomposes the solution $y(t)$ into series form

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{3}$$

Substituting the decomposition series (3) into (2) gives

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(t) = y(0) + \beta t - a_1 \mathcal{J}^{\alpha} \left[\sum_{n=0}^{\infty} y_n'(t) \right] \\ - a_2 \mathcal{J}^{\alpha} \left[\sum_{n=0}^{\infty} y_n(t) \right] + \mathcal{J}^{\alpha} [f(t)] \end{aligned}$$

From this equation, the components $y_k(t),\ k \geq 0$ are determined in the following recursive manner.

$$y_0(t) = y(0) + \beta t + \mathcal{J}^{\alpha}\big[f(t)\big]$$
$$y_{k+1}(t) = -a_1\mathcal{J}^{\alpha}\big[y_k'(t)\big] - a_2\mathcal{J}^{\alpha}\big[y_k(t)\big],\ \ k = 0, 1, ...$$

we approximate the solution $y(t)$ by the truncated series

$$\phi_m = \sum_{k=0}^{m-1} y_k \ \ and \lim_{m\to\infty} \phi_m = y(t) \tag{4}$$

Generally, the solution is very well approximated by the first five or six terms of the series. However, by analyzing more terms, the accuracy level can be raised.

After determining the components $y_n(t), n \geq 0$, the solution is formed using (3) in series form, where the constant $\beta = y'(0)$ is yet to be determined.

An approximate solution is provided by ϕ_n. Consequently, we apply the boundary condition at $t = T$ on $\phi_n\ (n \geq 0)$ in order to find the unknown constant β.

4 Solutions of Fuzzy Fractional Boundary Value Problems

In this section, the FFBVP is explained and the concept of a solution methodology is also explained.

Consider the fractional BVP with fuzzy boundary values.

$$\begin{cases} {}^{C}D^{\alpha}\tilde{y}(t) + a_1\tilde{y}'(t) + a_2\tilde{y}(t) = f(t), \quad 1 < \alpha \leq 2 \\ \tilde{y}(0) = \tilde{A} \\ \tilde{y}(T) = \tilde{B} \end{cases} \tag{5}$$

The boundary values are expressed as follows: $\tilde{A} = a_{cr} + \tilde{a}$ and $\tilde{B} = b_{cr} + \tilde{b}$, where a_{cr} and b_{cr} refer to crisp parts and also $\tilde{a}$ and $\tilde{b}$ refer to uncertainty parts of $\tilde{A}$ and $\tilde{B}$.

The problem (5) is split into the two problems as follows:

1) The corresponding fractional non-homogeneous BVP

$$\begin{cases} {}^{C}D^{\alpha}y(t) + a_1y'(t) + a_2y(t) = f(t) \\ y(0) = a_{cr} \\ y(T) = b_{cr} \end{cases} \tag{6}$$

2) The fractional homogeneous BVP with fuzzy boundary values

$$\begin{cases} {}^{C}D^{\alpha}\tilde{y}(t) + a_1\tilde{y}'(t) + a_2\tilde{y}(t) = 0 \\ \tilde{y}(0) = \tilde{a} \\ \tilde{y}(T) = \tilde{b} \end{cases} \tag{7}$$

Problem (6) has a unique solution $y_{cr}(t)$. ADM [24] is applied to compute this solution which is explained in detail in section 3.

Next, we solve the problem (7) with boundary conditions $\tilde{a} = (\underline{a_0}, 0, \overline{a_0})$ and $\tilde{b} = (\underline{b_0}, 0, \overline{b_0})$.

Consider $y_a(t)$ as a solution to the subsequent problem

$$\begin{cases} {}^{C}D^{\alpha}y(t) + a_1y'(t) + a_2y(t) = 0 \\ y(0) = \underline{a_0} \\ y(T) = \underline{b_0} \end{cases} \tag{8}$$

Moreover, consider $y_b(t)$ as a solution to the subsequent problem

$$\begin{cases} {}^{C}D^{\alpha}y(t) + a_1y'(t) + a_2y(t) = 0 \\ y(0) = \overline{a_0} \\ y(T) = \overline{b_0} \end{cases} \tag{9}$$

Then, the solution of eqn (7) is

$$\tilde{y}_{un}(t) = \Big(min\{y_a(t), 0, y_b(t)\}, 0, max\{y_a(t), 0, y_b(t)\}\Big)$$

$\tilde{y}_{un}(t)$ is a triangular fuzzy number due to its boundary values being triangular fuzzy numbers. As a result, an α-cut of $\tilde{y}_{un}(t)$ can be found as follows.

$$\tilde{y}_{un}(t) = (1-\alpha)\big[\underline{y_{un}}(t), \overline{y_{un}}(t)\big]$$

where

$$\underline{y_{un}}(t) = min\big\{y_a(t), 0, y_b(t)\big\}$$
$$\overline{y_{un}}(t) = max\big\{y_a(t), 0, y_b(t)\big\}$$

Then the solution to the problem (5) can be expressed as:

$$\tilde{y}(t) = y_{cr}(t) + \tilde{y}_{un}(t)$$

4.1 The Algorithm of The Solution

Next, the algorithm for solving the FFBVP (5) is explained.

1. Express the boundary values in the following form: $\tilde{A} = a_{cr} + \tilde{a}$ and $\tilde{B} = b_{cr} + \tilde{b}$.
2. Find the solution y_{cr} for the non-homogeneous fractional BVP problem (6).
3. Find the solutions $y_a(t)$ and $y_b(t)$ to the problems (8) and (9) and then the solution of problem (7) can be defined by
$$\tilde{y}_{un}(t) = \Big(min\big\{y_a(t), 0, y_b(t)\big\}, 0, max\big\{y_a(t), 0, y_b(t)\big\}\Big)$$
4. Finally, the solution to problem (5) can be expressed in the following manner
$$\tilde{y}(t) = y_{cr}(t) + \tilde{y}_{un}(t)$$

5 Numerical Example

Consider the following FFBVP

$$\begin{cases} {}^{C}D^{\alpha}\tilde{y}(t) - 4\tilde{y}'(t) + 4\tilde{y}(t) = 1 - 2t^2, \quad 1 < \alpha \leq 2 \\ \tilde{y}(0) = (2, 3, 4) \\ \tilde{y}(1) = (1, 2, 2.5) \end{cases} \tag{10}$$

where ${}^{C}D^{\alpha}$ represents Caputo fractional derivatives.

The boundary values can be written as follows

$$\tilde{A} = a_{cr} + \tilde{a} = 3 + (-1, 0, 1)$$

$$\tilde{B} = b_{cr} + \tilde{b} = 2 + (-1, 0, 0.5)$$

First, we find a solution to the non-homogeneous fractional BVP

$$\begin{cases} {}^C D^\alpha y(t) - 4y'(t) + 4y(t) = 1 - 2t^2 \\ y(0) = 3 \\ y(1) = 2 \end{cases} \tag{11}$$

To solve problem (11) apply the ADM.
Applying $\mathcal{J}^\alpha$ on both sides of (11), we have

$$\mathcal{J}^\alpha[{}^C D^\alpha y(t)] - 4\mathcal{J}^\alpha[y'(t)] + 4\mathcal{J}^\alpha[y(t)] = \mathcal{J}^\alpha[1 - 2t^2]$$
$$y(t) = 3 + \beta t + 4\mathcal{J}^\alpha[y'(t)] - 4\mathcal{J}^\alpha[y(t)] + \mathcal{J}^\alpha[1 - 2t^2]$$

where $y'(0) = \beta$.
ADM decomposed the solution into infinite series as $y(t) = \sum_{n=0}^{\infty} y_n(t)$.

$$\sum_{n=0}^{\infty} y_n(t) = 3 + \beta t + 4\mathcal{J}^\alpha[\sum_{n=0}^{\infty} y_n'(t)] - 4\mathcal{J}^\alpha[\sum_{n=0}^{\infty} y_n(t)] + \mathcal{J}^\alpha[1 - 2t^2]$$

The following recursive relation is thus obtained.

$$y_0(t) = 3 + \beta t + \mathcal{J}^\alpha[1 - 2t^2]$$
$$y_{k+1}(t) = 4\mathcal{J}^\alpha[y_k'(t)] - 4\mathcal{J}^\alpha[y_k(t)], \quad k = 0, 1, 2, ... \tag{12}$$

According to eqn (12), the components of the solution for problem (11) can be written as follows:

$$y_0(t) = 3 + \beta t + \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+2}}{\Gamma(\alpha+3)}$$
$$y_1(t) = \frac{16t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{16t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{12t^\alpha}{\Gamma(\alpha+1)} - \frac{4t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{4\beta t^\alpha}{\Gamma(\alpha+1)} - \frac{4\beta t^{\alpha+1}}{\Gamma(\alpha+2)}$$
$$y_2(t) = \frac{128t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{16t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{64t^{3\alpha+2}}{\Gamma(3a+3)} + \frac{48t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{48t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{48t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{32t^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{32\beta t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{16\beta t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{16\beta t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$
$$\vdots$$

Then $y_{cr}(t) = y_0(t) + y_1(t) + y_2(t) + ...$
Moreover, we compute $y_{cr}(t)$ for various α values.

For $\alpha = 1.8$,

$$\begin{aligned} y_{cr}(t) = {} & \beta t + 2.386\beta t^{9/5} + 4.305\beta t^{13/5} - 0.8521\beta t^{14/5} \\ & + 6.314\beta t^{17/5} - 2.391\beta t^{18/5} - 4.305\beta t^{22/5} \\ & + 0.2599\beta t^{23/5} + 0.7972\beta t^{27/5} - 0.04152\beta t^{32/5} \\ & - 6.561t^{9/5} - 11.84t^{13/5} - 17.36t^{17/5} + 3.288t^{18/5} \\ & - 0.2242t^{19/5} + 1.965t^{21/5} + 7.893t^{22/5} \\ & - 0.2599t^{23/5} - 1.133t^{26/5} - 0.9965t^{27/5} \\ & + 0.04642t^{28/5} - 0.06093t^{31/5} + 0.08304t^{32/5} \\ & + 0.09309t^{36/5} - 0.005611t^{37/5} - 0.01238t^{41/5} \\ & + 0.0004487t^{46/5} + 3.0 \end{aligned}$$

For $\alpha = 1.9$,

$$\begin{aligned} y_{cr}(t) = {} & \beta t + 3.408\beta t^{14/5} - 1.794\beta t^{19/5} + 2.189\beta t^{19/10} \\ & + 0.1869\beta t^{24/5} - 0.7548\beta t^{29/10} + 4.147\beta t^{37/10} \\ & - 2.647\beta t^{47/10} + 0.4644\beta t^{57/10} - 0.02311\beta t^{67/10} \\ & - 9.373t^{14/5} + 2.467t^{19/5} + 1.04t^{23/5} - 6.02t^{19/10} \\ & - 0.1869t^{24/5} - 0.557t^{28/5} + 0.03222t^{29/5} \\ & - 0.02813t^{33/5} + 0.04072t^{38/5} - 11.41t^{37/10} \\ & - 0.005165t^{43/5} - 0.1935t^{39/10} + 0.0001793t^{48/5} \\ & + 4.853t^{47/10} - 0.5805t^{57/10} + 0.04621t^{67/10} \\ & - 0.003001t^{77/10} + 3.0 \end{aligned}$$

For $\alpha = 2$,

$$\begin{aligned} y_{cr}(t) = {} & \beta t + 2.0\beta t^2 + 2.0\beta t^3 + 1.333\beta t^4 + 0.6667\beta t^5 \\ & - 1.156\beta t^6 + 0.2921\beta t^7 - 0.0254\beta t^8 + 0.0007055\beta t^9 \\ & - 5.5t^2 - 7.333t^3 - 5.667t^4 - 3.067t^5 + 2.978t^6 \\ & - 0.6476t^7 + 0.04762t^8 + 0.006349t^9 - 0.001552t^{10} \\ & + 0.0001026t^{11} - 2.138 \times 10^{-6}t^{12} + 3.0 \end{aligned}$$

We apply the boundary condition at $t = 1$ in order to determine β, then we have

$$y(1) = 2, \ then \begin{cases} \beta = 3.2238 \ \ for \ \ \alpha = 1.8 \\ \beta = 3.0554 \ \ for \ \ \alpha = 1.9 \\ \beta = 2.9752 \ \ for \ \ \alpha = 2 \end{cases}$$

Second, we take into consideration the homogeneous FF-BVP in order to determine the uncertainty.

$$\begin{cases} {}^C D^\alpha \tilde{y}(t) - 4\tilde{y}'(t) + 4\tilde{y}(t) = 0 \\ \tilde{y}(0) = (-1, 0, 1) \\ \tilde{y}(1) = (-1, 0, 0.5) \end{cases} \tag{13}$$

Then, we solve the following fractional BVP

$$\begin{cases} {}^C D^\alpha y(t) - 4y'(t) + 4y(t) = 0 \\ y(0) = \varphi_1 \\ y(1) = \varphi_2 \end{cases} \tag{14}$$

Applying $\mathcal{J}^{\alpha}$ on both sides of the problems (14) and also applying ADM.
For $\varphi_1 = -1\ and\ \varphi_2 = -1$, we have the solution components for problem (14) as follows.

$$\begin{aligned}
y_0(t) &= -1 + \beta_1 t \\
y_1(t) &= \frac{4t^{\alpha}}{\Gamma(\alpha+1)} + \frac{4\beta_1 t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4\beta_1 t^{\alpha+1}}{\Gamma(\alpha+2)} \\
y_2(t) &= \frac{16t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{16t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{32\beta_1 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{16\beta_1 t^{2\alpha-1}}{\Gamma(2\alpha)} \\
&\quad + \frac{16\beta_1 t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\
&\vdots
\end{aligned}$$

Then $y_a(t) = y_0(t) + y_1(t) + y_2(t) + ...$
For $\varphi_1 = 1\ and\ \varphi_2 = 0.5$, we have the solution components for problem (14) as follows.

$$\begin{aligned}
y_0(t) &= 1 + \beta_2 t \\
y_1(t) &= \frac{4\beta_2 t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4\beta_2 t^{\alpha+1}}{\Gamma(\alpha+2)} \\
y_2(t) &= \frac{16t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{16t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{32\beta_2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{16\beta_2 t^{2\alpha-1}}{\Gamma(2\alpha)} \\
&\quad + \frac{16\beta_2 t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\
&\vdots
\end{aligned}$$

Then $y_b(t) = y_0(t) + y_1(t) + y_2(t) + ...$
Moreover, we compute $y_a(t)$ and $y_b(t)$ for various α values.
For $\alpha = 1.8$,

$$\begin{aligned}
y_a(t) &= \beta_1 t + 2.386\beta_1 t^{9/5} + 4.305\beta_1 t^{13/5} - 0.8521\beta_1 t^{14/5} \\
&\quad + 6.314\beta_1 t^{17/5} - 2.391\beta_1 t^{18/5} - 4.305\beta_1 t^{22/5} \\
&\quad + 0.2599\beta_1 t^{23/5} + 0.7972\beta_1 t^{27/5} - 0.04152\beta_1 t^{32/5} \\
&\quad + 2.386t^{9/5} + 4.305t^{13/5} + 6.314t^{17/5} - 1.196t^{18/5} \\
&\quad - 2.87t^{22/5} + 0.2657t^{27/5} - 1.0 \\
y_b(t) &= \beta_2 t + 2.386\beta_2 t^{9/5} + 4.305\beta_2 t^{13/5} - 0.8521\beta_2 t^{14/5} \\
&\quad + 6.314\beta_2 t^{17/5} - 2.391\beta_2 t^{18/5} - 4.305\beta_2 t^{22/5} \\
&\quad + 0.2599\beta_2 t^{23/5} + 0.7972\beta_2 t^{27/5} - 0.04152\beta_2 t^{32/5} \\
&\quad - 2.386t^{9/5} - 4.305t^{13/5} - 6.314t^{17/5} + 1.196t^{18/5} \\
&\quad + 2.87t^{22/5} - 0.2657t^{27/5} + 1.0
\end{aligned}$$

For $\alpha = 1.9$,

$$\begin{aligned}
y_a(t) &= \beta_1 t + 3.408\beta_1 t^{14/5} - 1.794\beta_1 t^{19/5} + 2.189\beta_1 t^{19/10} \\
&\quad + 0.1869\beta_1 t^{24/5} - 0.7548\beta_1 t^{29/10} + 4.147\beta_1 t^{37/10} \\
&\quad - 2.647\beta_1 t^{47/10} + 0.4644\beta_1 t^{57/10} - 0.02311\beta_1 t^{67/10} \\
&\quad + 3.408t^{14/5} - 0.897t^{19/5} + 2.189t^{19/10} + 4.147t^{37/10} \\
&\quad - 1.765t^{47/10} + 0.1548t^{57/10} - 1.0 \\
y_b(t) &= \beta_2 t + 3.408\beta_2 t^{14/5} - 1.794\beta_2 t^{19/5} + 2.189\beta_2 t^{19/10} \\
&\quad + 0.1869\beta_2 t^{24/5} - 0.7548\beta_2 t^{29/10} + 4.147\beta_2 t^{37/10} \\
&\quad - 2.647\beta_2 t^{47/10} + 0.4644\beta_2 t^{57/10} - 0.02311\beta_2 t^{67/10} \\
&\quad - 3.408t^{14/5} + 0.897t^{19/5} - 2.189t^{19/10} - 4.147t^{37/10} \\
&\quad + 1.765t^{47/10} - 0.1548t^{57/10} + 1.0
\end{aligned}$$

For $\alpha = 2$,

$$\begin{aligned}
y_a(t) &= \beta_1 t + 2.0\beta_1 t^2 + 2.0\beta_1 t^3 + 1.333\beta_1 t^4 + 0.6667\beta_1 t^5 \\
&\quad - 1.156\beta_1 t^6 + 0.2921\beta_1 t^7 - 0.0254\beta_1 t^8 \\
&\quad + 0.0007055\beta_1 t^9 + 2.0t^2 + 2.667t^3 + 2.0t^4 + 1.067t^5 \\
&\quad - 0.9778t^6 + 0.1524t^7 - 0.006349t^8 - 1.0 \\
y_b(t) &= \beta_2 t + 2.0\beta_2 t^2 + 2.0\beta_2 t^3 + 1.333\beta_2 t^4 + 0.6667\beta_2 t^5 \\
&\quad - 1.156\beta_2 t^6 + 0.2921\beta_2 t^7 - 0.0254\beta_2 t^8 \\
&\quad + 0.0007055\beta_2 t^9 - 2.0t^2 - 2.667t^3 - 2.0t^4 - 1.067t^5 \\
&\quad + 0.9778t^6 - 0.1524t^7 + 0.006349t^8 + 1.0
\end{aligned}$$

We apply the boundary condition at $t = 1$ in order to determine β_1 and β_2, then we have
$y_a(1) = -1,\ \ and\ \ y_b(1) = 0.5$ then

$$\begin{cases}
\beta_1 = -1.2319\ \ and\ \ \beta_2 = 1.1650\ \ for\ \ \alpha = 1.8 \\
\beta_1 = -1.1717\ \ and\ \ \beta_2 = 1.0908\ \ for\ \ \alpha = 1.9 \\
\beta_1 = -1.1292\ \ and\ \ \beta_2 = 1.0474\ \ for\ \ \alpha = 2
\end{cases}$$

Finally, the solution of problem (10) is obtained in the following form.

$$\tilde{y}(t) = y_{cr}(t) + \tilde{y}_{un}(t)$$

Figure 1 displays $\tilde{y}(t)$ and associated r-cuts for α values of 1.8, 1.9 and 2. When $\alpha = 2$, the result $\tilde{y}(t)$ of the FFBVP matches to the result of the FBVP described in [22]. Here green line represents $r = 1$, blue line represents $r = 0.7$, red line represents $r = 0.3$, and black line represents $r = 0$.

6 Conclusions

In this work, the FFDE with fuzzy boundary values was investigated. The FFBVP has been studied as a collection of fractional BVPs. ADM has been applied to solve these fractional BVPs. The solution of FFBVP was obtained by combining all of the solutions to these fractional BVPs. The solution approach is based on the properties of linear transformations. An example is provided to demonstrate that the proposed method is more efficient than other methods.

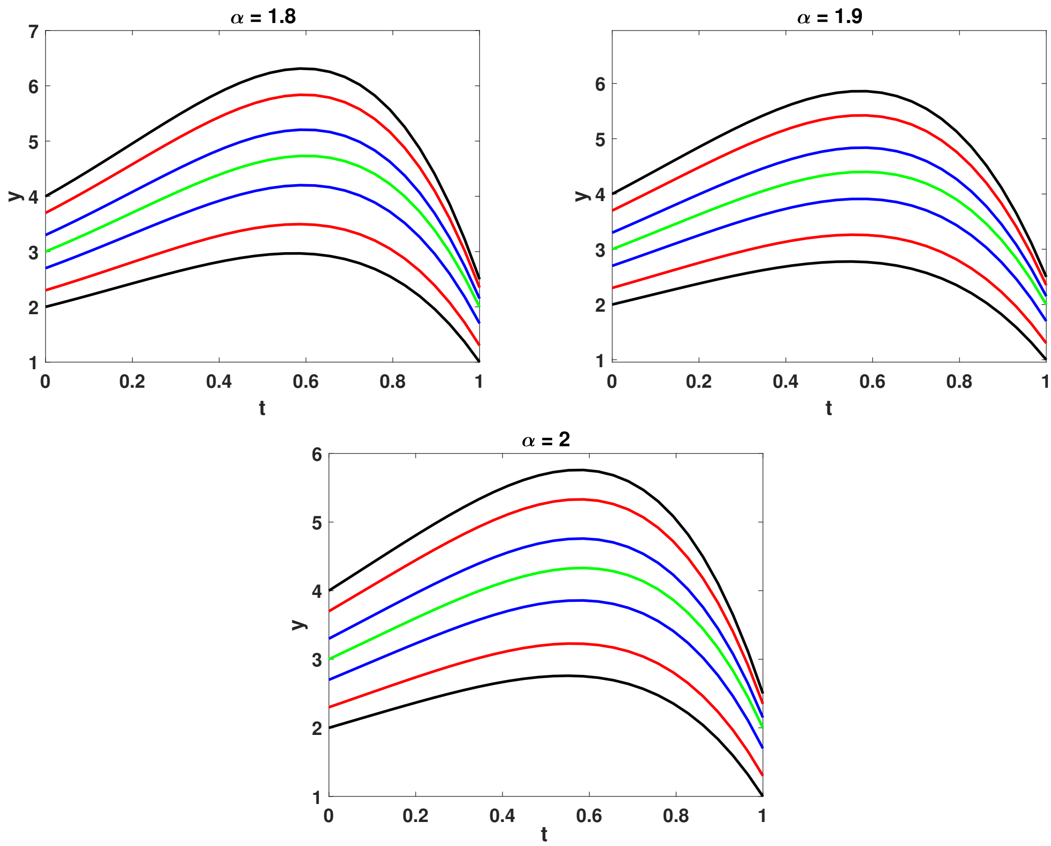

Figure 1. Total solution to the problem (10) and its r-cuts

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