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Exploring Prism Graphs with Fractional Domination Parameters

G. Uma[1], **S. Amutha**[2,*], **N. Anbazhagan**[1], **B. Koushick**[1]

[1]Department of Mathematics, Alagappa University, karaikudi-630 003, Tamilnadu, India

[2]Ramanujan Centre for Higher Mathematics, Alagappa University, karaikudi-630 003, Tamilnadu, India

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Abstract In this paper, we consider the prism graph of the cartesian product $C_m \times P_2$ as Z_p. Our primary objectives are to explore the concept of the fractional domination number in the prism graphs by determining the bounds and relations of fractional domination number and other parameters in prism graphs. This makes significant improvements in resource allocation. To attain our objectives, we compute the bounds based on the definitions of the fractional domination number and other parameters. When comparing these parameters in prism graphs, we investigate how the fractional domination number relates to other parameters, such as the fractional mixed domination number, domination number, independence domination number, and vertex independence number of prism graphs. Our main findings include the domination chain of the prism graph, and also insights into how the bounds of the fractional domination number and the fractional mixed domination number change when the vertex or edge is added or removed from the prism graphs which is crucial for analyzing how changes in prism graphs affect the bounds of fractional domination-related parameters.

Keywords Prism Graph, Domination Number, Fractional Domination Number, Fractional mixed Domination Number

1 Introduction

In this paper, we consider finite, undirected, simple graphs. Several researchers have investigated the domination number and its related characteristics, resulting in numerous papers on domination and its variants, [1]. These variants and related outcomes are discussed in several well-known works, [2, 3]. Fractional domination is a variant with numerous developments in its theory and applications. A prism graph is the Cartesian product $C_m \times P_n$, where C_m is a cycle with m vertices and P_n is a path with n vertices [4, 5]. Results related to domination and prism graphs are presented in [6, 7]. Connected prism graphs are equitably $\triangle(G)$ -choosable, leading to equitable list-coloring [8]. The spectrum-based results of generalized prism graphs were analyzed to understand their structural traits and characteristics [9]. The Turan number of an odd prism was calculated using Simonovits theory and Yuan's stability [10]. The fractional domination number of fractal graphs is discussed in [11], while reflexive edge sturdiness on generalized prism graphs is measured in [12]. Consider the prism graph of the Cartesian product $C_m \times P_2$ where C_m is a cycle with m vertices and P_2 is a path with 2 vertices. Let Z_p be a prism graph of this type consisting of $2p$ vertices and $3p$ edges for every $p \geq 3$, as shown in Fig. 1 [13].

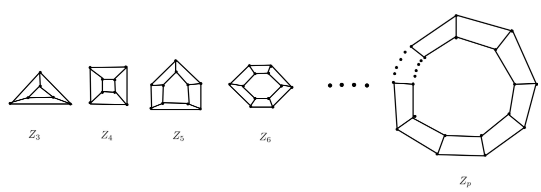

Figure 1. Prism graph Z_p

The vertex(edge) independence number $\beta(Z_p)(\beta_1(Z_p))$ of Z_p is the maximum cardinality of any collection of vertices(edges) in Z_p such that no two vertices(edges) in the set are adjacent(incident). The vertex(edge) covering number $\alpha(Z_p)(\alpha_1(Z_p))$ of Z_p is the minimum cardinality of any collection of vertices(edges) such that every vertex(edge) in Z_p is incident with at least one vertex(edge) from the set. When considering edge covers, it is assumed that they do not contain isolated vertices. The independent domination number

$i(Z_p)$ of Z_p is the minimum cardinality of the largest independent sets in Z_p. A dominating set in Z_p is a collection of vertices such that each vertex in Z_p is either in the dominating set itself or adjacent to a vertex in the dominating set. The domination number $\gamma(Z_p)$ of Z_p is the minimum cardinality of all such dominating sets[14]. For any $v \in V(Z_p)$, the open neighborhood of v consists of all vertices adjacent to v, denoted as $M(v)$, and the closed neighborhood $M[v] = M(v) \cup \{v\}$. Similarly, for any edge $e_1 \in E(Z_p)$, the open neighborhood $M(e_1)$ includes all edges adjacent to e_1, and the closed neighborhood $M[e_1] = M(e_1) \cup \{e_1\}$. If $w_1 \in V(Z_p) \cup E(Z_p)$, the open neighborhood $M^*(w_1)$ consists of all $w_2 \in V(Z_p) \cup E(Z_p)$ adjacent to w_1, and the closed neighborhood $M^*[w_1] = M^*(w_1) \cup \{w_1\}$. A function $f : V(Z_p) \to [0, 1]$ is termed a fractional dominating function(FDF) of Z_p if $f(M[v]) = \sum_{v_1 \in M[v]} f(v_1) \geq 1$ for every $v \in V(Z_p)$. The minimum weight among all other FDFs of Z_p is known as the fractional domination number(FDN), denoted by $\gamma_f(Z_p)$ [15]. The fractional domination chain is $ir_f(Z_p) \leq \gamma_f(Z_p) \leq i_f(Z_p) \leq \beta_f(Z_p) \leq \Gamma_f(Z_p) \leq IR_f(Z_p)$ [16]. A fractional mixed dominating function(FMDF) $f : V(Z_p) \cup E(Z_p) \to [0, 1]$ satisfies $f(M^*[w_1]) = \sum_{w_2 \in M^*[w_1]} f(w_2) \geq 1$ for all $w_1 \in V(Z_p) \cup E(Z_p)$. Among all FMDFs of Z_p, the function with the minimum weight is known as the fractional mixed domination number(FMDN), represented as $\gamma^*_{fm}(Z_p)$ [17].

Previous research on prism graphs and fractional domination has inspired our curiosity to explore the effects and bounds of fractional domination-related parameters in prism graphs. In our study, we carefully examine each vertex's neighborhood $M[v]$ to determine the fractional domination number $\gamma_f(Z_p)$. We assign each vertex v a weight $f(v)$ within the interval $[0, 1]$. The objective is to ensure that for every $v \in V(Z_p)$, $\sum_{v_1 \in M[v]} f(v_1) \geq 1$. We then determine $\gamma_f(Z_p)$ by minimizing the total weight of the FDFs. Additionally, we conduct a thorough analysis using the definitions of various parameters, such as the independence domination number, vertex (edge) independence number, and vertex covering number, to determine the bounds of these parameters in prism graphs. Furthermore, we investigate the bounds of the FDN and FMDN by analyzing the changes in bounds when a vertex is deleted or added to the prism graphs. This approach not only enhances our theoretical understanding of fractional domination in prism graphs but also provides practical insights for significant resource allocation.

Consider a scenario during a devastating hurricane where the government needs to distribute financial resources efficiently to affected areas. They plan to establish centers, represented as vertices in a prism graph Z_p, to manage fund distribution. In this scenario, the minimality condition of fractional domination ensures that every allocation center and its surrounding areas receive adequate financial support for disaster relief. Unlike a dominating set that might allocate funds to only a few dominating vertices, FDFs distribute the total sum of money fractionally across all allocation centers. This approach guarantees that every allocation center in the affected community receives optimal financial support, facilitating timely aid and optimizing resource allocation without unnecessary delays. For example, let's consider a prism graph Z_4. Each allocation center, represented as red points in Fig. 2, receives an equal distribution of the total funds. Due to the prism graph's three regular nature, where every vertex has exactly three neighbors, each allocation center in Z_4 receives one –quarter for distribution. This equal distribution ensures that the fractional domination number $\gamma_f(Z_4)$ is 2, indicating sufficient support for every center and its neighbors. The use of fractional domination in disaster relief exemplifies its role in ensuring efficient and equitable distribution of resources. By adopting this strategy, governments can respond more effectively to disasters, providing timely aid to all affected communities while enhancing overall efficiency in relief efforts.

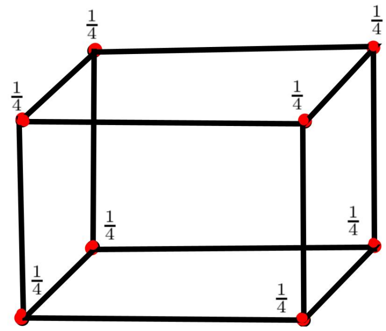

Figure 2. Prism graph Z_4

Practical implementations rely on theoretical insights into the relationship between FDN and other graph parameters. For instance, understanding how FDN interacts with covering numbers in prism graphs is crucial for efficient resource allocation. FDN ensures optimal resource utilization through fractional allocation while covering numbers to guarantee maximum coverage. This theoretical framework not only enhances resource planning but also underscores its application in optimizing resource allocation strategies. Using prism graphs for resource allocation offers distinct advantages and challenges compared to other graph types. The inherent connectivity of prism graphs makes them essential for strategic resource allocation planning. Their regular structure facilitates precise identification of resource requirements and efficient distribution strategies, even in challenging scenarios such as fault tolerance situations. However, as prism graphs scale up in complexity, a practical challenge needs careful consideration.

2 Bounds and effects of fractional domination and related parameters of prism graphs

In this section, we find the bounds of fractional domination number, independent domination number, vertex(edge) independence number, vertex covering number, and fractional mixed domination number of prism graphs and exhibit the relationship among these parameters and provide valuable insights into their structural properties. To proceed with the proof of our main theorem, we first recall an important lemma related to the regular graph of fractional domination number.

Lemma 2.1 [12] *If G is a k-regular graph* ($k \geq 1$) *and has n vertices then* $f(v) = \frac{1}{k+1}$ *for every* $v \in V(G)$ *gives a minimum fractional dominating function and the weight of this function is*

$$\gamma_f(G) = \frac{n}{k+1}. \quad (1)$$

Using the foundational result of fractional domination number of regular graphs, we can now move on to prove our main theorem of bounds of fractional domination number of prism graphs.

Theorem 2.2 *For every prism graph* Z_p *with* $p \geq 3$*, p and the fractional domination number are proportional. i.e, for every* $p \geq 3$,

$$\gamma_f(Z_p) = \frac{p}{2} \quad (2)$$

Proof Let Z_p be the prism graphs, which have $2p$ vertices and $3p$ edges. Since, Z_p is a regular graph of degree 3. Using equation (1), $f(v_i) = \frac{1}{3+1} = \frac{1}{4}$ for every $v_i \in V(Z_p)$. Since this function holds $f(M[v_i]) \geq 1$ for every $v_i \in V(Z_p)$. This function f is the minimum FDF of Z_p and the sum of the weight of the function is $\frac{2p}{4} = \frac{p}{2}$.i.e, for every $p \geq 3$, $\gamma_f(Z_p) = \frac{p}{2}$.

Lemma 2.3 [10] *For a prism graph* Z_p *where* $p \geq 3$*, we have*

$$\gamma(Z_p) = \begin{cases} \frac{P}{2}, & \text{if } p \equiv -4(mod4) \\ \frac{P}{2}+1, & \text{if } p \equiv -2(mod4) \\ \frac{P+1}{2}, & \text{if } p \equiv -1(mod2) \end{cases}$$

We utilize the above lemma to prove the next corollary.

Corollary 2.4 *For every* $p \geq 3$, $\gamma_f(Z_p) \leq \gamma(Z_p)$.

Proof For every $p \geq 3$, comparing the fractional domination number to the domination number of a prism graph Z_p yields the inequality $\gamma_f(Z_p) \leq \gamma(Z_p)$ by equations (2) and Lemma 2.3.
The following theorem gives the bounds for the independent domination number of the prism graphs.

Theorem 2.5 *For every* $p \geq 3$,

$$i(Z_p) = \begin{cases} 2\lfloor \frac{p-2}{4} \rfloor + 4, & \text{if } p \equiv 1(mod4) \\ 2\lfloor \frac{p-2}{4} \rfloor + 2, & \text{otherwise.} \end{cases}$$

Proof As shown in Fig. 3, let the graph vertices of Z_p be $1, 2, 3, ..., 2p$.

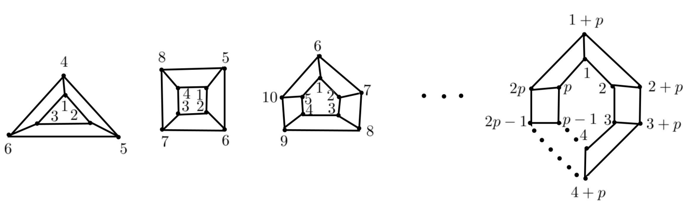

Figure 3. Prism graph of labelled vertices of Z_p

We consider for $p \geq 3$ the three following cases:
Case(i): For $p \equiv 2(mod4)$, we give the set of vertices $M' = M'_1 \cup M'_2 \cup 2p$ is the independent dominating set of Z_p where $M'_1 = \{1 + 4t/t = 0, 1, 2, ..., \lfloor \frac{p-2}{4} \rfloor\}$ and $M'_2 = \{(3 + p) + 4t/t = 0, 1, 2, ..., \lfloor \frac{p-2}{5} \rfloor\}$. It is sufficient to show that there is no proper subset of M' dominating Z_p. So, if v is any vertex in M', $Q = M' - v \subseteq M'$ is independent dominating Z_p and $|Q| \leq 2\lfloor \frac{p-2}{4} \rfloor + 2$. Now, for any $v \in M'$ implies that v is either in M'_1 or in M'_2 or $v = 2p$. According to the structure of the labeling, it is clear that for any choice of $v \in M'$, we have at least 2 vertices of the form $\{2 + 4t, p + 4t/t = 0, 1, 2, ..\lfloor \frac{p-2}{4} \rfloor\}$ that adjacent to v is not independent dominating with any vertex in Q. For $v \in M'_2$, v independent dominates itself and we have 3 vertices of the form: $\{3+4t, (p+2)+4t, (4+p)+4t/t = 0, 1, 2, ..., \lfloor \frac{p-2}{4} \rfloor\}$ not independent dominating with any vertex in Q. Also, if $v = 2p$ then we have at least two vertices of the form $\{2p - 1, 2p - 5\}$ adjacent to v not independent dominating yet with any vertex in Q. So, for any choice of v we have at least 2 vertices that are not independent dominating with any vertex in Q, this shows that there is no proper subset $Q \subseteq M'$ independent dominating of Z_p. Hence, $i(Z_p) \geq |M'| = 2\lfloor \frac{p-2}{4} \rfloor + 2$ implies that $i(Z_p) = 2\lfloor \frac{p-2}{4} \rfloor + 2$.
Case(ii): Suppose $p \equiv 1(mod4)$ and $p \neq 5$, we give the set of vertices $M' = \{1\} \cup \{3\} \cup \{p + 2\} \cup M'_1 \cup M'_2$ is independent dominating set for Z_p where $M_1 = \{6 + 4t/t = 0, 1, 2, 3..., \lfloor \frac{p-6}{4} \rfloor\}$ and $M_2 = \{(4 + p) + 4t/t = 0, 1, 2, 3..., \lfloor \frac{p-6}{3} \rfloor\}$. Then, $|M'| = 1 + 1 + 1 + |M'_1| + |M'_2| = 2\lfloor \frac{p-2}{4} \rfloor + 4$. Hence , $i(Z_p) \leq |M'| = 2\lfloor \frac{p-2}{4} \rfloor + 4$. Now, if $v \in M'_1$ or in M'_2, alike in the case (i) remained when $v = 1$ or 2 or {p+2} $\notin$ Q. This vertex independent dominates itself which is not independently dominated by any vertex in $Q = M'_1 \cup M'_2$. So, we have for all choices of v at least one vertex not independent dominating with any vertex in Q. Therefore, Q is not an independent dominating set of Z_p. Hence, $i(Z_p) \geq |M'| = 2\lfloor \frac{p-2}{4} \rfloor + 4$ implies that $i(Z_p) = 2\lfloor \frac{p-2}{4} \rfloor + 4$.
Case(iii): For $p \equiv 0, 3(mod4)$, we give the independent dominating set for these two cases of Z_p by $M' = M'_1 \cup M'_2$ where the M'_1 and M'_2 are the same sets in case(i). Therefore, $|M'| = |M'_1| + |M'_2| = 2\lfloor \frac{p-2}{4} \rfloor + 2$. Hence, $i(Z_p) \leq 2\lfloor \frac{p-2}{4} \rfloor + 2$. The proof of $i(Z_p) \geq 2\lfloor \frac{p-2}{4} \rfloor + 2$ is similar to that in case(i) when $v \in M'_1$ or $v \in M'_2$ and therefore, $i(Z_p) = 2\lfloor \frac{p-2}{4} \rfloor + 2$.
Case(iv): When $p = 5$, we give the set of vertices $M' = \{1, 3, 7, 9\}$ yield to the cardinality of $i(Z_5)$. There is no Q contained in M'. Hence, $i(Z_5) = |M'| = 4$.
From the above cases, we have $i(Z_p) = \begin{cases} 2\lfloor \frac{p-2}{4} \rfloor + 4, & \text{if } p \equiv 1(mod4) \\ 2\lfloor \frac{p-2}{4} \rfloor + 2, & \text{otherwise.} \end{cases}$.

Corollary 2.6 *For every* $p \geq 3$, $\gamma_f(Z_p) \leq i(Z_p)$.

Proof It follows from the equation (2) and Theorem 2.5.
The theorem that follows establishes bounds for the vertex covering number of the prism graphs.

Theorem 2.7 *For every* $p \geq 3$, $\alpha(Z_p) = \begin{cases} p, & \text{if } p \equiv 0(mod2) \\ p+1, & \text{if } p \equiv 1(mod2). \end{cases}$

Proof
We consider for $p \geq 3$ the two following cases:
Case(i) Suppose $p \equiv 0(mod2)$, the number of vertices is $2p$. We select the set of vertices so that each edge intersects with a minimum of one vertex within the set to compute the vertex covering number. Let $T = M_1 \cup M_2$ where $M_1 = \{2t/t = 1, 2, 3, ..., \frac{p}{2}\}$ and $M_2 = \{1 + 2t + p/t = 0, 1, 2, .., \frac{p-2}{2}\}$ is the minimum vertex covering set consists of p vertices for Z_p. Hence, if $p \equiv 0(mod2)$, then $\alpha(Z_p) = |T| = |M_1| + |M_2| = \frac{p}{2} + \frac{p}{2} = p$.
Case(ii) When $p \equiv 1(mod2)$, similar to Case(i): Let $T = \{M_1 \cup \{p+1\}\}$ where $M_1 = \{2t+1/t = 0, 1, 2, ..., p-1\}$ is the minimum vertex covering set consisting of $p + 1$ vertices for Z_p. Hence, if $p \equiv 1(mod2)$, then $\alpha(Z_p) = |T| = |M_1| + 1 = p + 1$.
From the above cases, we have $\alpha(Z_p) = \begin{cases} p, & \text{if } p \equiv 0(mod2) \\ p+1, & \text{if } p \equiv 1(mod2). \end{cases}$
The next theorem provides bounds for the vertex independence number of the prism graphs.

Theorem 2.8 *For every $p \geq 3$, $\beta(Z_p) = \begin{cases} p, & \text{if } p \equiv 0(mod2) \\ p-1, & \text{if } p \equiv 1(mod2). \end{cases}$*

Proof We consider for $p \geq 3$ the two following cases:
Case(i) Suppose $p \equiv 0(mod2)$, the number of vertices is $2p$. To calculate the vertex independence number, we give the set of vertices such that any two vertices of the set are not adjacent. Let $S = M_1 \cup M_2$ where $M_1 = \{2t+1/t = 0, 1, 2, ..., \lfloor \frac{p-1}{2} \rfloor\}$ and $M_2 = \{p + 2t/t = 1, 2, 3.... \frac{p}{2}\}$ is the maximum vertex independence set consisting of p vertices for Z_p. Hence, if $p \equiv 0(mod2)$, then $\beta(Z_p) = |S| = |M_1| + |M_2| = \frac{p}{2} + \frac{p}{2} = p$.
Case(ii) When $p \equiv 1(mod2)$, similar to Case(i): Let $S = M_1 \cup M_2 \cup 2p$ where $M_1 = \{1, 3, 5, ..., p-2\}$ and $M_2 = \{p + 2t/t = 1, 2, ..., \frac{p-3}{2}\}$ is the maximum vertex independence set consisting of $p - 1$ vertices for Z_p. Hence, if $p \equiv 1(mod2)$, then $\beta(Z_p) = |S| = |M_1| + |M_2| + 1 = \frac{p-1}{2} + \frac{p-3}{2} + 1 = p - 1$.
From the above cases, we have $\beta(Z_p) = \begin{cases} p, & \text{if } p \equiv 0(mod2) \\ p-1, & \text{if } p \equiv 1(mod2). \end{cases}$
The relationship between the vertex independence number and the domination number, as well as the conditions under which equality holds, are explained in the following corollary.

Corollary 2.9 *For every $p \geq 3$ of the prism graph Z_p, $\gamma(Z_p) < \beta(Z_p)$ and $\gamma(Z_p) = \beta(Z_p) \iff p = 3$.*

Proof It follows from the equations Theorem 2.3 and Theorem 2.8.
The next theorem states the bounds for edge independence number of prism graphs.

Theorem 2.10 *For every $p \geq 3$ of the prism graph Z_p, $\beta_1(Z_p) = p$.*

Proof Let the vertices be $\{v_1, v_2, v_3, ...v_{p-1}, v_p, v_{1+p}, v_{2+p}, ...v_{2p-1}, v_{2p}\}$ and edges of the graph as $\{e_1, e_2, e_3, ...e_{3p-2}, e_{3p-1}, e_{3p}\}$. We select the set of edges such that no two of the set are incident to determine the edge independence number. The group of edges $K = \{e_3, e_6, e_9, ..., e_{3p-6}, e_{3p-3}, e_{3p}\}$ which consists of p edges is the maximum cardinalities of all edge independence sets. Therefore, for every $p \geq 3$, $\beta_1(Z_p) = |K| = p$.
Combining all the above results, we establish the domination chain of prism graphs in the next remark.

Remark 2.11 *For every $p \geq 3$, $\gamma_f(Z_p) \leq \gamma(Z_p) = i(Z_p) < \beta(Z_p)$.*

The following results are based on how adding or removing a vertex or edge from the prism graph affects the FDN and FMDN. Knowing these differences is essential in situations where graphs change over time. For dynamic graph analysis, it is crucial to examine how this number varies when vertex or edge is added or removed from the graph. Doing so uncovers deeper features of prism graphs. In the next theorem, we describe how adding a vertex in the prism graphs affects the FDN.

Theorem 2.12 *For every $p \geq 3$ of the prism graph Z_p, $\gamma_f(Z_p + v) = \frac{2p+3}{4}$ where $Z_p + v$ is adding a new vertex v to any other vertex in Z_p.*

Proof Let the vertices of the graph $Z_p(p \geq 3)$ be $\{v_1, v_2, v_3, ...v_{p-1}, v_p, v_{1+p}, v_{2+p}, ...v_{2p-1}, v_{2p}\}$ and then add the vertex v in any of the vertices in Z_p to obtain $Z_p + v$. Let $f : V(Z_p) \longrightarrow [0, 1]$ such that $f(v_i) = \frac{1}{4}$ and $f(v) = \frac{3}{4}$ for every $v_i, v \in V(Z_p + v)$. By adding these function values, the sum of the weight of the function f is $\gamma_f(Z_p + v) = \frac{p}{2} + \frac{3}{4} = \frac{2p+3}{4}$. Since this function holds $f(M[v_i, v]) \geq 1$ for every $v_i, v \in V(Z_p+v)$. Hence, it is a minimum FDF and the weight of the function f is a $\gamma_f(Z_p + v)$. i.e, $\gamma_f(Z_p + v) = \frac{2p+3}{4}$. Now, suppose that $\gamma_f(Z_p + v) \neq \frac{2p+3}{4}$. Then, there is a vertex $v_i, v \in V(Z_p + v)$ such that $f(M[v_i, v]) < 1$. This implies that f is not a minimum FDF and the weight of the function is not a $\gamma_f(Z_p + v)$. This completes the proof.
The next theorem explains the bounds of FDN when deleting any vertex from the vertices in the prism graphs.

Theorem 2.13 *For every $p \geq 3$, $\gamma_f(Z_p - v) = \frac{p+1}{2}$ where $Z_p - v$ is removing a vertex $v \in V(Z_p)$.*

Proof To obtain $Z_p - v$, remove any vertex from Z_p. Suppose we remove $v_1 \in V(Z_p)$, let $f : V(Z_p) \longrightarrow [0, 1]$ such that $f(v_i) = \frac{1}{4}$ $\forall$ $v_i \in V(Z_p - v_1)$ except $\{v_2, v_p, v_{1+p}\}$ and assign the function value to the three vertices in the neighborhood of the removed vertex as $f(v_2) = f(v_p) = f(v_{1+p}) = \frac{1}{2}$. The weight of the function is determined by the sum of these function values f is $\gamma_f(Z_p - v_1) = \frac{p}{2} - 1 + \frac{3}{2} = \frac{p+1}{2}$. Since, this function holds $f(M[v]) \geq 1$ for every $v \in V(Z_p - v_1)$. It is a minimum FDF and the weight of the function f is a $\gamma_f(Z_p - v_1)$. Similarly, the theorem holds for removing any other vertex in Z_p. Hence, $\gamma_f(Z_p - v) = \frac{p+1}{2}$.
In the following theorem, we provide the exact bound of the FDN when adding an edge to the prism graphs.

Theorem 2.14 *For every $p \geq 3$ of the prism graph Z_p, $\gamma_f(Z_p + e) = \frac{2p+3}{4}$ where $Z_p + e$ is adding a new edge $e = v_i v$*

where $v_i \in V(Z_p)$ and v is a new vertex formed by adding an edge.

Proof Let the vertices of the graph $Z_p(p \geq 3)$ be $\{v_1, v_2, v_3, ...v_{p-1}, v_p, v_{1+p}, v_{2+p}, ...v_{2p-1}, v_{2p}\}$ and then add the edge in any of the vertices in Z_p to obtain $Z_p + e$. Let $f : V(Z_p) \longrightarrow [0, 1]$ such that $f(v_i) = \frac{1}{4}$ and $f(v) = \frac{3}{4}$ for every $v_i, v \in V(Z_p + e)$. The total of these function values determine the function's weight f is $\gamma_f(Z_p + e) = \frac{p}{2} + \frac{3}{4} = \frac{2p+3}{4}$. Since this function holds $f(M[v_i, v]) \geq 1$ for every $v_i, v \in V(Z_p + e)$. it is a minimum FDF and the weight of the function f is a $\gamma_f(Z_p + e)$. Hence, $\gamma_f(Z_p + e) = \frac{2p+3}{4}$.
The bound for the FDN of prism graphs when removing an edge is stated in the next theorem.

Theorem 2.15 *For every $p \geq 3$, $\gamma_f(Z_p - e) = \frac{p+1}{2}$ where $Z_p - e$ is removing an edge $e = v_iv_j \in E(Z_p)$ for any $v_i, v_j \in V(Z_p)$*

Proof

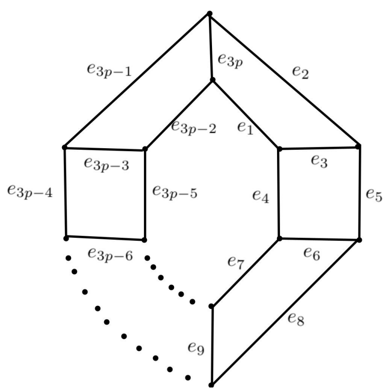

Figure 4. Prism graph Z_p

As shown in Fig. 4, let the graph edges of Z_p be $\{e_1, e_2, e_3, \ldots, e_{3p-2}, e_{3p-1}, e_{3p}\}$. To obtain $Z_p - e$, remove any edge $e \in E(Z_p)$. Suppose we remove $e_1 \in E(Z_p)$, let $f : V(Z_p) \longrightarrow [0, 1]$ so that $f(v_i) = \frac{1}{4}\ \forall\ i \in V(Z_p)$ except $\{v_1, v_2\}$ and assign the function value to the two vertices in the neighborhood of the removed edge as $f(v_1) = f(v_2) = \frac{1}{2}$. By adding the listed function values, the sum of the weight of the function f is $\gamma_f(Z_p - e_1) = \frac{2p-2}{4} + \frac{1}{2} + \frac{1}{2} = \frac{2p-2}{4} + 1 = \frac{p+1}{2}$. Since this function holds $f(M[v]) \geq 1$ for every $v \in V(Z_p)$. It is a minimum FDF and the weight of the function f is a $\gamma_f(Z_p - e_1)$. Similarly, the theorem holds for removing any other edge in Z_p. Hence, $\gamma_f(Z_p - e) = \frac{p+1}{2}$.
Combining the results of the bounds of FDN when removing or adding the vertex or edge in the prism graphs stated in the following remark.

Remark 2.16 *For every $p \geq 3$, $\gamma_f(Z_p) < \gamma_f(Z_p - v) = \gamma_f(Z_p - e) < \gamma_f(Z_p + v) = \gamma_f(Z_p + e)$.*

The FMDF extends the concept of FDF by considering both vertices and edges in the graph. The following theorem establishes the bounds of FMDN of prism graphs.

Theorem 2.17 *For every $p \geq 3$, $\gamma^*_{fm}(Z_p) = \frac{4p}{5}$*

Proof Let $f : V(Z_p) \cup E(Z_p) \to [0, 1]$ so that $f(v) = \frac{1}{5}$ for all $v \in V(Z_p)$ and $f(e_{3i}) = 0$ where $i = 1, 2, 3, ..., p$ except for these edges, all other remaining edges are assigned to be $\frac{1}{5}$. By adding these function values, the sum of the weight of the function f is $\frac{2p}{5} + \frac{2p}{5} = \frac{4p}{5}$. Since this function holds, $f(M^*[w_1]) = \sum_{w_2 \in M^*[w_1]} f(w_2) \geq 1$ for all $w_1 \in V(Z_p) \cup E(Z_p)$. Hence, it is a minimum FMDF and the weight of the function is $\gamma^*_{fm}(Z_p)$. Now, suppose that $\gamma^*_{fm}(Z_p) \neq \frac{4p}{5}$. Then, there is an edge or vertex $w_1 \in V(Z_p) \cup E(Z_p)$ such that $f(M^*[w_1]) < 1$. This implies that f is not a minimum FMDF and the weight of the function f is not a $\gamma^*_{fm}(Z_p)$. Hence, a contradiction. Therefore, for every $p \geq 3$, $\gamma^*_{fm}(Z_p) = \frac{4p}{5}$.
The relation that exists between FDN and FMDN of prism graphs is stated in the following corollary.

Corollary 2.18 *For every $p \geq 3$, $\gamma^*_{fm}(Z_p) > \gamma_f(Z_p)$*

Proof It follows from equation (2) and Theorem 2.17.
The following theorem explains how adding a new vertex to prism graphs affects the FMDN.

Theorem 2.19 *For every $p \geq 3$ of the prism graph Z_p, $\gamma^*_{fm}(Z_p + v) = \frac{4p+4}{5}$ where $Z_p + v$ is adding a new vertex v to any other vertex in Z_p.*

Proof Let the vertices of the graph $Z_p(p \geq 3)$ be $\{v_1, v_2, v_3, ...v_{p-1}, v_p, v_{1+p}, v_{2+p}, ...v_{2p-1}, v_{2p}\}$ and then add the vertex v in any of the vertices in Z_p to obtain $Z_p + v$. Let $f : V(Z_p) \cup E(Z_p) \to [0, 1]$ such that $f(v_1) = \frac{1}{5}$ for all $v_1 \in V(Z_p)$ and $f(e_{3i}) = 0$ where $i = 1, 2, 3, ..., p$ except these edges all other remaining edges are assigned to be $\frac{1}{5}$ and the new vertex v is assigned as $f(v) = \frac{4}{5}$ and the corresponding edges are assigned 0 (or) $f(v) = \frac{1}{5}$ for all $v \in V(Z_p)$ and $f(e_{3i}) = 0$ where $i = 1, 2, 3, ..., p$ except these edges all other remaining edges are assigned to $\frac{1}{5}$ and the new vertex v is assigned as $f(v) = \frac{1}{5}$ and the corresponding edges are assigned $\frac{3}{5}$. In either way, the total weight is calculated by summing up these function values f is $\frac{2p}{5} + \frac{2p}{5} + \frac{4}{5} = \frac{4p+4}{5}$. Since this function holds, $f(M^*[w_1]) = \sum_{w_2 \in M^*[w_1]} f(w_2) \geq 1$ for all $w_1 \in V(Z_p) \cup E(Z_p)$. Hence, it is a minimum FMDF and the weight of the function is $\gamma^*_{fm}(Z_p + v)$. Now, suppose that $\gamma^*_{fm}(Z_p + v) \neq \frac{4p+4}{5}$. Then, there is an edge or vertex $w_1 \in V(Z_p) \cup E(Z_p)$ such that $f(M^*[w_1]) < 1$. This implies that f is not a minimum FMDF and the weight of the function f is not a $\gamma^*_{fm}(Z_p + v)$. Therefore, for every $p \geq 3$, $\gamma^*_{fm}(Z_p + v) = \frac{4p+4}{5}$.
The next corollary illustrates how FMDN change when adding a new edge in the prism graphs.

Corollary 2.20 *For every $p \geq 3$ of the prism graph Z_p, $\gamma^*_{fm}(Z_p + e) = \frac{4p+4}{5}$ where $Z_p + e$ is adding a new edge $e = v_iv$ where $v_i \in V(Z_p)$ and v is a new vertex formed by adding an edge.*

Proof Similar to the Proof of the Theorem 2.19.
The following theorem provides the bounds of FMDN for removing any one of the vertices in the prism graphs.

Theorem 2.21 *For every $p \geq 3$, $\gamma^*_{fm}(Z_p - v) = \frac{4p-1}{5}$ where $Z_p - v$ is removing any vertex $v \in V(Z_p)$.*

Proof To obtain $Z_p - v$, remove any vertex from $V(Z_p)$. Suppose we remove $v_1 \in V(Z_p)$ and the corresponding edges are also removed. Let $f : V(Z_p) \cup E(Z_p) \to [0,1]$ so that $f(v_i) = \frac{1}{5}$ for all $v_i \in V(Z_p - v_1)$ and $f(e_{3i}) = 0$ where $i = 1, 2, 3, ..., p$ except the edges $f(e_{3p-3}) = f(e_3) = \frac{1}{5}$ and all other remaining edges are assigned to be $\frac{1}{5}$. Regarding the possible values that are listed, the sum of the weight of the function f is $\frac{4p-1}{5}$. Since this function holds, $f(M^*[w_1]) = \sum_{w_2 \in M^*[w_1]} f(w_2) \geq 1$ for all $w_1 \in V(Z_p) \cup E(Z_p)$. Hence, it is a minimum FMDF and the weight of the function is $\gamma^*_{fm}(Z_p - v_1)$. Now, suppose that $\gamma^*_{fm}(Z_p - v) \neq \frac{4p-1}{5}$. Then, there is an edge or vertex $w_1 \in V(Z_p) \cup E(Z_p)$ such that $f(M^*[w_1]) < 1$. This implies that f is not a minimum FMDF and the weight of the function f is not a $\gamma^*_{fm}(Z_p - v_1)$. Hence, $\gamma^*_{fm}(Z_p - v_1) = \frac{4p-1}{5}$. Likewise, this applies to every vertex in Z_p. Therefore, for every $p \geq 3$, $\gamma^*_{fm}(Z_p - v) = \frac{4p-1}{5}$.
The next theorem provides the bounds of FMDN when removing any edge in the prism graphs.

Theorem 2.22 *For every* $p \geq 3$, $\gamma^*_{fm}(Z_p - e) =$
$$\begin{cases} \frac{4p}{5}, & \textit{if removable edge is } \{e_{3i}/i = 1, 2, ..., p\} \\ \frac{4p+1}{5}, & \textit{if otherwise} \end{cases}$$
where $Z_p - e$ *is removing a edge* $e = v_i v_j \in E(Z_p)$ *for any* $v_i, v_j \in V(Z_p)$

Proof Similar to the Proof of the Theorem 2.21.
The relation of bounds of the FDN and the FMDN when a vertex or edge is added or removed from the prism graphs stated in the following remark.

Remark 2.23 *For every* $p \geq 3$, $\gamma^*_{fm}(Z_p + v) = \gamma^*_{fm}(Z_p + e) > \gamma^*_{fm}(Z_p - e) \geq \gamma^*_{fm}(Z_p) > \gamma^*_{fm}(Z_p - v) > \gamma_f(Z_p)$.

3 Conclusions

We provided bounds and relations on how fractional domination number correlated with other parameters such as independent domination number, vertex(edge) independence number, vertex covering number, and fractional mixed domination number. Additionally, we stated the result for the bounds for the FDN and the FMDN with the addition or removal of vertex or edge from the prism graphs. This investigation has given a solid basis for future research and implementation of fractional domination in more interesting graph structures. This study strengthened our understanding of fractional domination and related parameters and also advanced the field of prism graphs, which is helpful for the real-world applications of graph theory including resource allocation. In the future, we will try to figure out the bounds and effects of the fractional domination chain in the prism graph on the path of $p > 2$ vertices and state under which conditions the parameter values are equivalent in the prism. Fractional domination parameters in prism can optimize resource allocation in distributed systems by ensuring fair and efficient usage of resources and making them better equipped to handle failures and dynamic changes. Importantly, the specific conditions under which these fractional domination parameters vary will be tailored to the needs and conditions of the real-life scenarios, ensuring optimal allocation across resources. Also, we intend to concentrate on the application-oriented features of prism graphs, utilizing various fractional domination parameters. This technique seeks to connect theoretical notions with practical implementations, improving the real-world applicability and performance of these graph structures across various domains.

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