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On One Property of a Conditional Full Angle

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Abstract One of the main directions of modern differential geometry and topology is the so-called geometry "in large", which is a field that studies geometric objects as a whole. As one of the first results of this field, it is possible to recognize the equality of the polyhedron consisting of two closed, congruent sides with the same constituent sides, proved by O. Cauchy in 1813. The problems of restoring polyhedron and surfaces according to the given geometric characteristics are also included in the problems of geometry "in large". As a geometric characteristic, it is possible to obtain external curvature, internal curvature, and area of a spherical image, area of a cylindrical image of a polyhedron and surface, or any other value related to a geometric object. Geometric methods related to the theory of polyhedra are widely used not only for polyhedra, but also in the general theory of surfaces. Some surfaces are formed from the limit states of polyhedras. An analogy of some properties of polyhedra can be made for surfaces. Therefore, studying the properties of polygons helps to study such properties of surfaces. This article studied the properties of polyhedron isometric on sections. In particular, the concept of a conditional full angle given at the vertex, which is important for the restoration of isometric polygons in terms of their external curvature, is included. It is known that the concept of isometry on sections depends to given direction. The concept of a conditional full angle is generalized for trihedral angles that have edges perpendicular to this direction (a special case) and do not have a support plane perpendicular to this direction. The monotonicity of this conditional full angle for a trihedral angle is proved for the general case.

Keywords Defect, Conditional Full Angle, Invariant, Conditional Curvature, Monotonicity, Isometry on Sections

1 Introduction

One of the first results of geometry "in large" was proved by Cauchy in 1813 [1]. D.Hilbert announced 23 famous problems at the second international congress of mathematicians held in Paris in 1900 [2]. The third of these problems is devoted to the problem of dividing two polyhedrons of the same volume into a finite number of equal parts. D.Hilbert's student Dehn [3] partially solved this problem with additional conditions and introduced an invariant depending on the polyhedron. By 1965, Sydler completely solved Hilbert's Problem 3, that is, he showed that for two polyhedrons of the same size to be divisible into a finite number of equal parts, it is necessary and sufficient that they have equal Dehn invariant [4]. The problem of restoring the surface, in particular, the convex polyhedron according to the external curvature, was solved for the first time by A.D.Alexandrov [5]. He took as the external curvature of the polyhedra the sum of the areas of the spherical image at the vertex of the polyhedra. One of the recovery problems for regular surfaces is the Dirichlet problem. In the works of A.D.Alexandrov [6], A.V.Pogorelov [7], the Dirichlet problem for the Monge-Ampere equation was solved only in the convex and one-connected domain of the plane. A.Artykbaev solved this problem in a non-convex and multiconnected domain in Galilean space [8]. In the works of A.S.Sharipov, F.F.Topvoldiyev [9], [10] introduced the concept of defect for the polygon. Using this defect, the concept of a conditional full angle was introduced for convex polyhedral angles, none of the edges of which are perpendicular to the given vector, and the invariant of this quantity with respect to a linear substitution with a determinant equal to one, but neither orthogonal nor symmetric, was shown. Also, A.S.Sharipov, M.Keunimjaev [11] introduced the concept of conditional curvature of convex polyhedron isometric on sections, and using this conditional curvature, they restored a polygon with no perpendicular edge and supporting plane relative to the given direction. Several properties of a conditional full angle are presented, and with the help of these properties, the problem of restoring a convex polyhedron according to a conditional curvature is solved. It has been shown [12] that the Gaussian and mean curvatures of curved surfaces in three-dimensional Galilean space are invariant quantities. Problems of the immersions and embeddings of manifolds in Euclidean and other spaces are some of the central issues both in differential geometry as well as in topology [13]. In isotropic space some invariants of surfaces are considered by A. Artykbaev and Sh. Ismoilov [14]. In [15] the conditional full angle of the polyhedral angle with edge perpendicular to the given direction is estimated.

For restoring a surface using conditional curvature, the conditional curvature must be a positive definite and monotonic function. When we go to the limit in the transition from a polygon to surface, occurs the edges perpendicular to the given direction.

The purpose of this article is to show the monotonicity of a conditional full angle for a trihedral angle in all possible cases (with and without an edge perpendicular to a given direction, but without a supporting plane perpendicular to this direction).

2 The property of a conditional full angle

We are given a three-dimensional Cartesian coordinate system $OXYZ$, in which let the line formed by the intersection of planes $X = 1$ and $Z = 0$ denote by l . Since the line divides the plane on which it lies into two half-planes, let us consider the part $Z > 0$ of the plane $X = 1$ as the “upper” part of the line l, and the part $Z < 0$ as the “lower” part of the line l.

Let there be a circular cone whose vertex is at the coordinate origin and whose axis lies on the axis OZ. We cut the cone with plane $X = 1$, a hyperbola is formed in the section. We designate the branch of this hyperbola located in the “upper” part of the line l by γ_2, and the branch located in the “lower” part of the line l by γ_1. Let’s transfer the planes to the generators of the cone lying in the plane $X = 0$ and denote these planes by π_1 and π_2, accordingly.

When we intersect these planes with the plane $X = 1$, the section produces two intersecting lines, which are the asymptotes of the curves γ_1 and γ_2, and which we denote by l_1 and l_2, accordingly. We choose the direction of the vector $\overrightarrow{e}$ in the direction of the axis OX and from now on we write this vector in the form $\overrightarrow{e}\,(OX)$. We denote the part of the cone located in space $z \leq 0$ by S.

The trihedral angle XOY, whose vertex is at the origin of the coordinates and one-valued projecting on the plane XOY and consists of rays $a,\ b,\ c$, is inscribed inside the cone, so that this trihedral angle does not have a supporting plane perpendicular to the vector $\overrightarrow{e}\,(OX)$. We cut the trihedral angle with the plane $Z = \sigma$ ($\sigma < 0$), in the section a triangle is formed. Let’s denote this triangle by τ_3.

We denote the part of the trihedral angle between $0 \geq Z \geq \sigma$ by $S|_3$, and its conditional full angle by $\omega\left(S|_3\right)$. We denote the end of τ_3 that intersects with edges a is by A_{τ_3}, the end of τ_3 that intersects with edges b is by B_{τ_3}, and the end of τ_3 that intersects with edges c is by C_{τ_3}. We take an arbitrary point on the surface or inside the trihedral angle and denote it by O^*. Connect point O^* with all points of τ_3 and formed a new trihedral angle. We denote this trihedral angle by $S|_3^*$, and its conditional full angle by $\omega\left(S|_3^*\right)$.

The formula for finding the conditional full angle of a trihedral angle with a one-valued projecting onto the plane XOY and with edge perpendicular to the vector $\overrightarrow{e}\,(OX)$, but without a supporting plane perpendicular to this vector differs from the formula for finding a conditional full angle of a trihedral angle with a one-valued projecting onto the plane XOY and without edge perpendicular to the vector $\overrightarrow{e}\,(OX)$, but without a supporting plane perpendicular to this vector. If the trihedral angle $S|_3$ does not have an edge perpendicular to the vector $\overrightarrow{e}\,(OX)$, we denote it by S_3, and its conditional full angle by $\omega\left(S_3\right)$.

If one of the edges of the trihedral angle $S|_3$ is perpendicular to the vector $\overrightarrow{e}\,(OX)$, we will denote it by $\widetilde{S_3}$, and its conditional full angle by $\omega\left(\widetilde{S_3}\right)$. Then the conditional full angles of S_3 and $\widetilde{S_3}$ are accordingly the following found by the formulas [9], [16]

$$\omega\left(S_3\right) = AB + AC - BC. \tag{1}$$

$$\omega\left(\widetilde{S_3}\right) = AB + AB\cdot\cos\varphi = AB\cdot\left(1+\cos\varphi\right). \tag{2}$$

(in Figure 1, accordingly a) and b)).

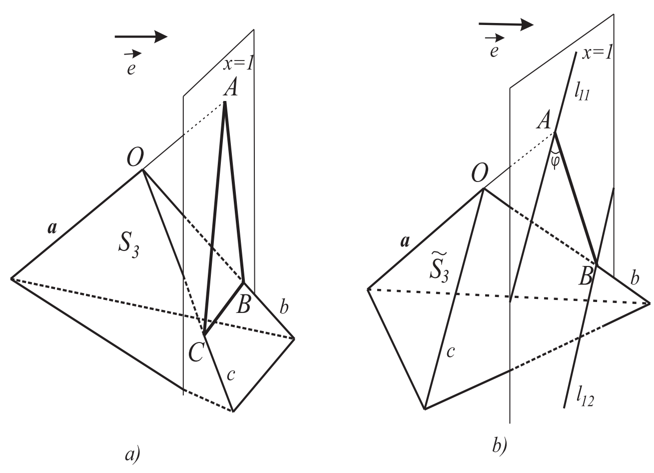

Figure 1. Conditional full angle of trihedral angle: a) no edge perpendicular to vector $\overrightarrow{e}$, and b) with edge c perpendicular to vector $\overrightarrow{e}$.

Therefore, for these two types of trihedral angles, we separately prove one of the properties of conditional full angle.

We denote the edge of $S|_3^*$ connected to points A_{τ_3} by a^*, the edge connected to points B_{τ_3} by b^*, and the edge connected to points C_{τ_3} by c^*. We denote the conditional full angle of $\widetilde{S_3^*}$

by $\omega\left(\widetilde{S_3^*}\right)$. We denote the plane passing through the edges a^* and b^* of $\widetilde{S_3^*}$ by $\pi\left(a^*, b^*\right)$.

I. Let one edge of $S|_3$ be the intersection of plane $X = 0$ and plane π_1, that is, let this edge be perpendicular to vector $\overrightarrow{e}\,(OX)$. Then this edge is parallel to the asymptote l_1. Without limiting the generality, we assume that this edge is edge c. Above we denoted such $S|_3$ by $\widetilde{S_3}$. Let edge a of $\widetilde{S_3}$ intersect plane $X = -1$, and edge b intersect plane $X = 1$. We continue the edge a to the opposite side and denote the resulting ray with $\overline{a}$.

We denote the point of intersection of $\overline{a}$ with the plane $X = 1$ by A, and the point of intersection of the edge b with the plane $X = 1$ by B, where the point A lies on the curve γ_2, and the point B lies on the curve γ_1. We make segment AB in plane $X = 1$. We denote the plane passing through edges b and c of $\widetilde{S_3}$ by $\pi\left(b, c\right)$, the plane passing through edges a and c by $\pi\left(a, c\right)$, and the plane passing through edges a and b by $\pi\left(a, b\right)$. We cut the planes $\pi\left(a, c\right)$ and $\pi\left(b, c\right)$ with the plane $X = 1$ and denote the lines formed in the section by l_{11} and l_{12}, accordingly, and the distance between these lines by h. We denote by ψ the non-intersecting angle between the line l_{11} and the line l, where $0 < \psi < 180°$. We cut the plane $\pi\left(a, b\right)$ with the plane $X = 0$ and denote the line formed in the section by l'_2. We denote the angle lying on the left side of the section AB between the line l_{11} and the section AB by φ, which is $0 < \varphi < \psi$. Also, the angle between the edge c and the line l'_2 is equal to φ.

It is known that the conditional full angle of $\widetilde{S_3}$ is calculated by formula (2).

It can be seen that if it is $\varphi = 90°$, then it will be $\omega\left(\widetilde{S_3}\right) = AB = h$.

We take an arbitrary point on the surface or inside $\widetilde{S_3}$ and denote it by O^*. We connect point O^* with all points of τ_3 and denote the resulting trihedral angle by $\widetilde{S_3^*}$.

Theorem 1. *For trihedral angles $\widetilde{S_3}$ and $\widetilde{S_3^*}$ the following relation is appropriate:*

$$\omega\left(\widetilde{S_3}\right) \geq \omega\left(\widetilde{S_3^*}\right).$$

Proof of Theorem 1. We prove the theorem 1 for the following possible cases:

1. If points O^* overlap with point O, then it will be $\omega\left(\widetilde{S_3}\right) = \omega\left(\widetilde{S_3^*}\right)$.

2. If point O^* lies on plane $Z = \sigma$, then will be $\omega\left(\widetilde{S_3^*}\right) = 0 < \omega\left(\widetilde{S_3}\right)$.

3. Let the point O^* lie in the part between $\sigma < Z < 0$ of the plane $X = 0$, then the following four cases will occur:

a) Let point O^* lie on edge c.

Then we move the vertex O^* of $\widetilde{S_3^*}$ parallel to the point O. The edge of $\widetilde{S_3^*}$ lies on the edge c of $\widetilde{S_3}$. We denote the edge of $\widetilde{S'_3}$ lying on the side $X < 0$ of the plane $X = 0$ by a^*, and the edge lying on the side $X > 0$ of the plane $X = 0$ by b^*.

We cut the plane $\pi\left(a^*, b^*\right)$ with plane $X = 0$ and denote the line formed in the section by l''_2. Continuing the edge a^* of $\widetilde{S_3^*}$ to the opposite side, we denote the resulting ray by $\overline{a^*}$. We cut the ray $\overline{a^*}$ and the edge b^* with the plane $X = 1$ and denote the intersection points by A^* and B^*, accordingly, and made the section A^*B^* in the plane $X = 1$. In this case, point A^* overlaps with point A, and point B^* lies above point B on the line l_{12} and "below" the line l, it will be $\angle BAB^* = \Delta\varphi$.

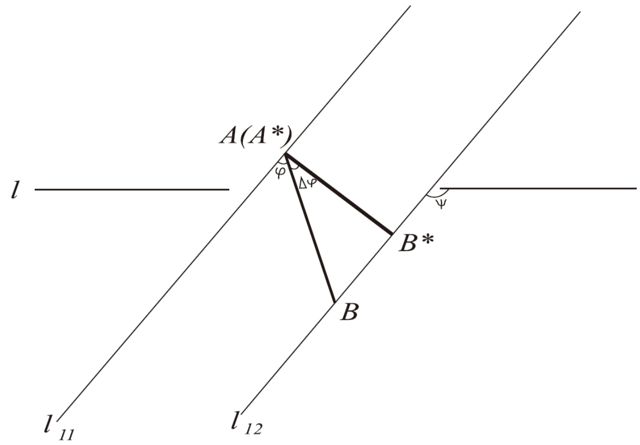

Figure 2. Conditional full angle of trihedral angle for case I.3.a). When angle φ takes a fixed value.

Since the direction of measurement of the angle is from the line l_{11} to the section A^*B^*, it is $\Delta\varphi > 0$. In that case, the angle between the line l_{11} and the section A^*B^* on the left side of the section A^*B^* is equal to $\varphi + \Delta\varphi$, and we denote this angle by φ^*. The conditional full angle of $\widetilde{S_3^*}$ is as following formula:

$$\omega\left(\widetilde{S_3^*}\right) = A^*B^* + A^*B^* \cdot \cos\varphi' = A^*B^* \cdot \left(1 + \cos\varphi^*\right). \quad (3)$$

Where $\varphi^* = \varphi + \Delta\varphi$, from $\Delta\varphi > 0$ it is $\varphi^* > \varphi$, and $\cos\varphi^* < \cos\varphi$ is appropriate. From this we will have $(1 + \cos\varphi) > (1 + \cos\varphi^*)$ (Figure 2).

When $\varphi^* \leq 90°$, as the value of $\Delta\varphi_i$ increases, the segment $A_i^*B_i^*$ moves from the inclined position to the perpendicular position relative to the parallel lines l_{11} and l_{12} and becomes $AB > A^*B^* > A_1^*B_1^* > A_2^*B_2^* > ... > h$. Where, the values $\Delta\varphi_i$ are increasing values of $\Delta\varphi$ and $i = 1, 2, ...$ (Figure 3).

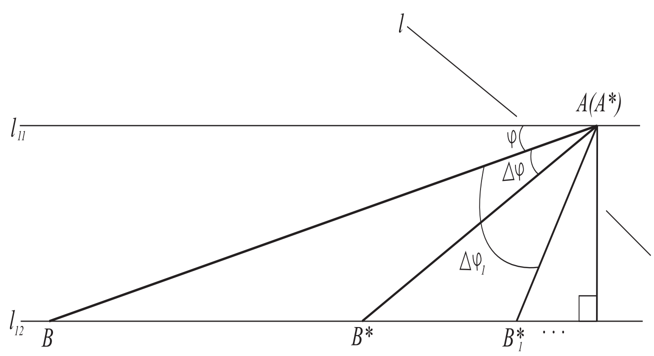

Figure 3. Conditional full angle of trihedral angle for case I.3.a). When $\varphi \leq \varphi^* \leq 90°$.

From this we take into account $(1 + \cos\varphi) > (1 + \cos\varphi^*)$

and $AB > A^*B^* > h$ when it is $\varphi^* \leq 90°$, it follows that relation $AB \cdot (1 + \cos\varphi) > A^*B^* \cdot (1 + \cos\varphi^*)$ is fulfilled.

In the interval $90° < \varphi^* < \psi$ will be:

$$\omega\left(\widetilde{S_3^*}\right) = A^*B^* + A^*B^* \cdot \cos\varphi^* = M^*B^*. \tag{4}$$

And for increasing values $\Delta\varphi_i$ of $\Delta\varphi$, expression (4) can be written as follows:

$$\omega\left(\widetilde{S_3^*}\right)_i = A^*B_i^* + A^*B_i^* \cdot \cos(\varphi + \Delta\varphi_i) =$$
$$= A^*B_i^* + A^*B_i^* \cdot \cos\varphi'_i = M_i^*B_i^*. \tag{5}$$

where $\varphi_i^* = \varphi + \Delta\varphi_i$.

That is, as the angle φ^* increases, firstly, the size of the angle $\angle M_i^*N_iB_i^*$, which separates the segment $M_i^*B_i^*$ from the segment $A^*B_i^*$, decreases, and secondly, the segment $M_i^*B_i^*$ moves closer to the perpendicular situation compared to the segment $M_i^*N_i$ (Figure 4). $BQ = BP$ in Figure 4, formula (1) can be rewritten as follows:

$$\omega\left(\widetilde{S_3}\right) = AB + AB \cdot \cos\varphi =$$
$$= AB \cdot (1 + \cos\varphi) = AB + BQ = AP. \tag{6}$$

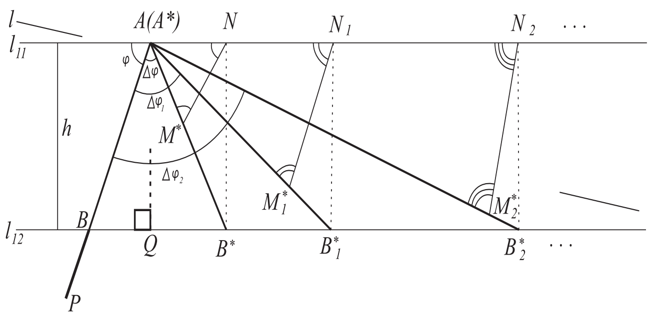

Figure 4. Conditional full angle of trihedral angle for case I.3.a). When $90° < \varphi^* < \psi$.

Therefore it will be $AP > h > M^*B^* > M_1^*B_1^* > M_2^*B_2^* > ....$ So it will be $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$.

In this case, the position of lines l_{11} and l_{12} and the distance between them will not change. Only the length of segment AB and its position relative to these lines will change.

b) Let point O^* lie between l'_2 and $\sigma < Z < 0$. We project point O^* onto axis OZ and denote this point by $Z(O^*)$. We denote the distance between points O and $z(O^*)$ by ρ, which is $0 < \rho < |\sigma|$. In this case, we move ends of O^* of $\widetilde{S_3^*}$ parallel to points O. Since the planes $\pi(a', b')$ and $\pi(a, b)$ are common, the lines l'_2 and l''_2 overlap. We cut the planes $\pi(a', c')$ and $\pi(b', c')$ with the plane $X = 1$, formed parallel lines are in the section. We denote these parallel lines by l'_{11} and l'_{12}, accordingly, and the distance between them by h'. It will be $0 < h' < h$ (Figure 5).

In this case, it is also $\varphi^* = \varphi + \Delta\varphi$, and the points A and A^* overlap. The situation of segment A^*B^* will not change. As the value of ρ increases, the value of $\Delta\varphi$ increases, the parallel lines l'_{11} and l'_{12} change their position with respect to the segment A^*B^*, and the proof of $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$ is

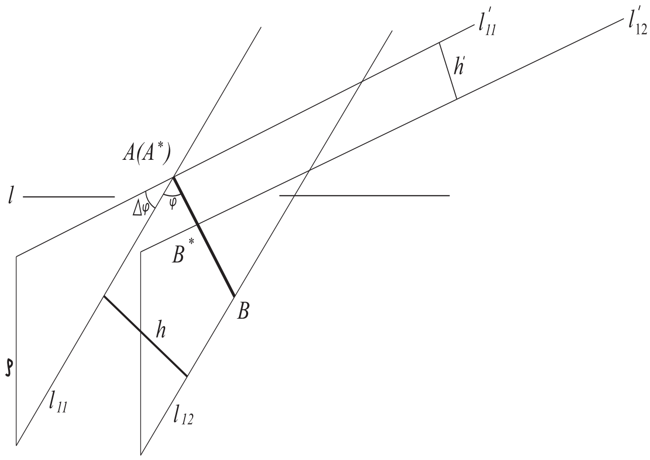

Figure 5. Conditional full angle of trihedral angle for case I.3.b).

similar to case a). At the same time, as the value of ρ increases, the distance between the parallel lines l'_{11} and l'_{12} decreases. This further strengthens the relation $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$ for case.

c) Point O^* lies inside $\widetilde{S_3}$ on plane $X = 0$, but not on axis Oz.

In this case, we move the vertex O^* of $\widetilde{S'}_3$ parallel to the point O. In that case, the position and length of the segment A^*B^* is the same as in case a) ; The position of the parallel lines l'_{11} and l'_{12} changes as in case b) and will be relation $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$.

d) Point O^* lies in the part between $\sigma < Z < 0$ of axis OZ. In the plane $X = 1$, we denote the lines passing through the points A and B and parallel to the axis Oz by l_A and l_B, accordingly. We move the vertex O^* of $\widetilde{S_3^*}$ parallel to the point O. Then the ends of the segment A^*B^* are on lines l_A and l_B, accordingly, and between these lines, segment A^*B^* is closer to the perpendicular situation than segment AB (Figure 6). That's $\varphi^* > \varphi$. So, in this case, $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$ attitude is also appropriate.

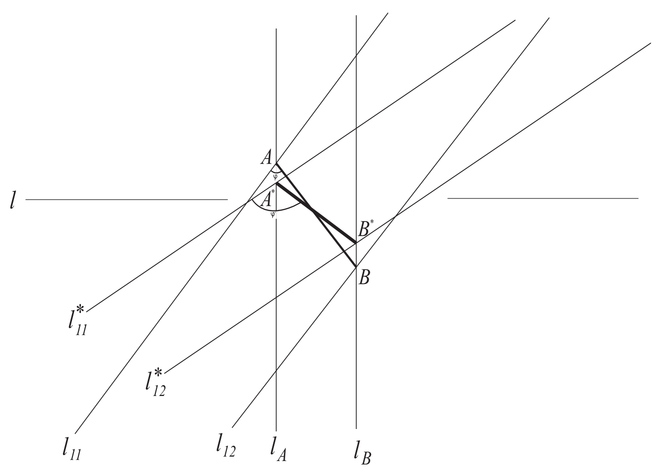

Figure 6. Conditional full angle of trihedral angle for case I.3.d).

4. Let point O^* lie between $\sigma < Z < 0$ of edges a. Then the edge c' loses its perpendicularity to the vector $\overrightarrow{e}(OX)$.

We move the vertex O^* of $\widetilde{S_3^*}$ parallel to the point O. If we continue edge c', it crosses with plane $X = 1$ and we denote the intersection point by C^*. Since edge a' lies on edge a, point A^* overlaps with point A .

Also, since plane $\pi(a', b')$ coincides with plane $\pi(a, b)$, point C^* lies on line l_{11}. Since plane $\pi(a', c')$ coincides with planes $\pi(a, c)$, point B^* lies in segment AB, that is, the situation of segment A^*B^* is the same as the situation of segment AB. As a result, point A^*, B^*, C^* on plane $X = 1$ are formed . Connecting these points, we make triangle $A^*B^*C^*$.

The defect of the sides of this triangle is called the conditional full angle of $\widetilde{S_3^*}$ [9] and it is written in the following form using formula (2.1):

$$\omega\left(\widetilde{S_3^*}\right) = A^*B^* + A^*C^* - B^*C^*. \tag{7}$$

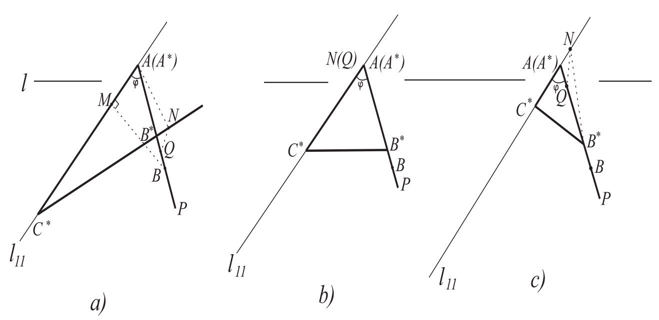

Figure 7. Conditional full angle of trihedral angle for case I.4.

In this case, one of the cases shown in Figure 7 below will be.

In figure 7 a) will be $C^*A^* > C^*B^*$, $C^*A^* = C^*N$, $B^*N = B^*Q$ and formula (7) can be rewritten as follows:

$$\omega\left(\widetilde{S_3^*}\right) = A^*B^* + A^*C^* - B^*C^* =$$
$$= A^*B^* + B^*Q = A^*Q < AP. \tag{8}$$

In figure 7 b) will be $C^*A^* = C^*B^*$ and formula (2.7) can be rewritten as follows:

$$\omega\left(\widetilde{S_3^*}\right) = A^*B^* + A^*C^* - B^*C^* = A^*B^* < AP \tag{9}$$

In figure 7 b) will be $C^*A^* < C^*B^*$, $C^*B^* = C^*N$, $A^*N = A^*Q$ and formula (7) can be rewritten as follows:

$$\omega\left(\widetilde{S_3^*}\right) = A^*B^* + A^*C^* - B^*C^* =$$
$$= A^*B^* - A^*N = A^*B^* - A^*Q = QB^* < AP. \tag{10}$$

It can be seen from these that it is $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$ in all three cases.

5. Let the point O^* lie on the part between the side $\sigma < Z < 0$ between the edges a and c. But do not lie on these edges. In this case, we use the triangle formed in case 3 to prove the theorem 2.1. We denote the edge corresponding to edge c' by c°, the edge corresponding to edge b' by b°, and the edge corresponding to edge a' by a°. Edge c° is not perpendicular to vector $\overrightarrow{e}(OX)$. We denote such a trihedral angle by S_3°. We move vertex O^* of S_3° parallel to point O.

We denote the conditional full angle of S_3° by $\omega(S_3^\circ)$. On the plane $X = 1$ a triangle is formed. It suffices to show that defect of this triangle is smaller than the triangular defect formed in case 3. In this case, we denote the resulting triangle by $A^\circ B^\circ C^\circ$. In this case, points overlap A° with points A^*, and point C° lies on line l_{11}, that is, segment $A^\circ C^\circ$ lies on segment A^*C^*, which will be $A^*C^* > A^\circ C^\circ$. Point B° is located on the right side of segment A^*B^*, in the "bottom" part of line l (Figure 8). In Figure 8, will be $C^\circ B^\circ = C^\circ M$, $MA^\circ = MN$ and $\omega\left(\widetilde{S_3^*}\right) = QB^*$, $\omega(S_3^\circ) = NB^\circ$. It is not difficult to see that it is

$$QB^* > NB^\circ. \tag{11}$$

So, it will be $\omega\left(\widetilde{S_3^*}\right) > \omega(S_3^\circ)$.

From $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$ will be $\omega\left(\widetilde{S_3}\right) > \omega(S_3^\circ)$.

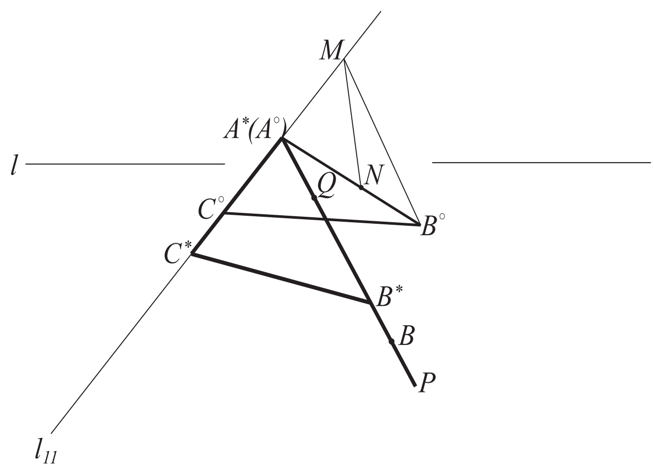

Figure 8. Conditional full angle of trihedral angle for case I.4.

6. Let the point O^* lies on the side $\sigma < Z < 0$ between the edges a and b, on the side $X < 0$ of the plane $X = 0$. In this case, we use the triangle formed in case 3 to prove Theorem 1. We denote the edge corresponding to edge c' by c°, the edge corresponding to edge b' by b°, and the edge corresponding to edge a' by a°. The edge c° is not perpendicular to vector $\overrightarrow{e}(OX)$. We denote such a trihedral angle by S_3°.

We move the vertex O^* of S_3° parallel to the point O. We denote the conditional full angle of S_3° by $\omega(S_3^\circ)$. On the plane $X = 1$ formed triangle. It suffices to show that defect this triangle is smaller than the defect of triangle formed in case 3. In this case, we denote the resulting triangle by $A^\circ B^\circ C^\circ$. In this case, point A° overlaps with point A^*, and point B° lies in segment A^*B^*, that is, the segment $A^\circ B^\circ$ lies in segment A^*B^*, which is $A^*B^* > A^\circ C^\circ$. Point C° is located on the left side of the line l_{11}, in the "bottom" part of the line l (Picture 9).

In Figure 8, will be $C^\circ M = C^\circ B$, $MA^\circ = NA^\circ$ and $\omega\left(\widetilde{S_3^*}\right) = QB^*$, $\omega(S_3^\circ) = NB^\circ$.

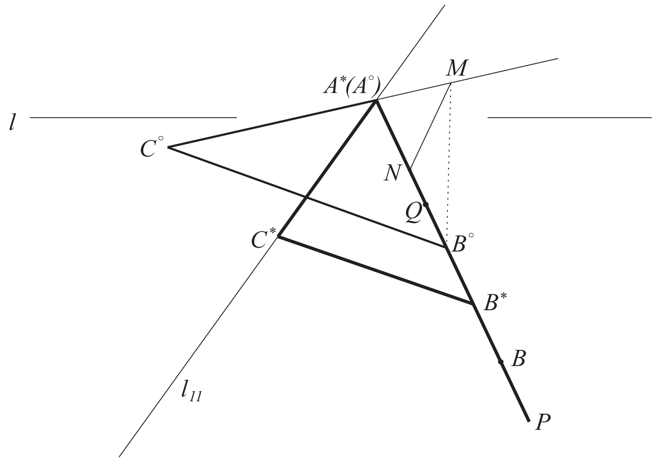

Figure 9. Conditional full angle of trihedral angle for case I.6.

It can be seen that

$$QB^* > NB^\circ. \tag{12}$$

So, it will be $\omega\left(\widetilde{S_3^*}\right) > \omega\left(S_3^\circ\right)$.

It follows that $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$ is $\omega\left(\widetilde{S_3}\right) > \omega\left(S_3^\circ\right)$.

7. If the point O^* lies on the side $X > 0$ of the plane $X = 0$, between the edge or surface $\sigma < Z < 0$ of $\widetilde{S_3}$, then $\omega\left(\widetilde{S_3}\right) > \omega\left(\widetilde{S_3^*}\right)$ is proved as in cases 3, 4, and 5, except that the points A^* and B^* are interchanged.

8. If point O^* lies between $\sigma < Z < 0$ of axis Oz, then point A^* lies on the side of line l of point A in a line passing through point A and parallel to axis OZ. Point B^* lies on the line l side of point B on a line passing through point B and parallel to axis OZ. That is, the section AB approaches the perpendicular situation from the transverse situation to the perpendicular situation between the lines passing through the points A and B, and parallel to the axis OZ, while it is $\varphi^* > \varphi$. So, it will be $\omega\left(\widetilde{S_3}\right) \geq \omega\left(\widetilde{S_3^*}\right)$.

From above 8 cases follows this assertion:

Proposition 1. If point O^* lies on the surface of $\widetilde{S_3}$, then relation $\omega\left(\widetilde{S_3}\right) \geq \omega\left(\widetilde{S_3^*}\right)$ is valid.

9. If the point O^* lies between $\sigma < Z < 0$ inside $\widetilde{S_3}$ and does not lies on the plane $X = 0$, then it is proved to be $\omega\left(\widetilde{S_3}\right) \geq \omega\left(\widetilde{S_3^*}\right)$ using Proposition 1 as shown in [11].

The rest of the cases are proved in the same way as in case 9 above, only the points A and B are replaced, as well as the points A^* and B^* corresponding to these points.

Theorem 1 is proved.

II. Let the triangle $S|_3$ not an edge perpendicular to vector $\overrightarrow{e}\,(OX)$. We denote such a trihedral angle $S|_3$ by S_3. $S|_3^*$ is a trihedral angle and we denote it by $\overline{S_3}$. We denote their conditional full angles by $\omega\left(S_3\right)$ and $\omega\left(\overline{S_3}\right)$, accordingly.

Theorem 2. *For trihedral angles S_3 and $\overline{S_3}$ the following relation is appropriate:*

$$\omega\left(S_3\right) \geq \omega\left(\overline{S_3}\right). \tag{13}$$

Proof of Theorem 2. Before proving the theorem, we introduce the following:

Let the edge a of S_3 intersect with plane $X = -1$, and the edges b and c intersect with plane $X = 1$. We continue the edge a to the opposite side and denote the resulting ray with $\overline{a}$.

We denote the point of intersection of $\overline{a}$ with the plane $X = 1$ by A, and the points of intersection of the edges b and c with the plane $X = 1$ by B and C, accordingly, where the point A lies on the curve γ_2, and the points B and C are lies on the curve γ_1. We connect points $A,\ B,\ C$ and make triangle on plane $X = 1$ and denote triangle by ABC. The defect of this triangle is found by the formula (2.1) and it is called the conditional full angle of S_3 [9].

Let the coordinate of point B_{τ_3} on axis OX be X_B, and the coordinate of point C_{τ_3} on axis OX be X_C. Without limiting generality, we assume $X_C < X_B$. The line formed by the intersection of the plane $X = X_C$ with the plane passing through the edges a and b of S_3 by l''_1, and the line formed by the intersection of the plane $X = X_C$ with the plane passing through the edges b and c we denote the line by l''_2.

Now we turn to the proof of Theorem 2, which we prove for the following possible cases:

1. Let point O^* be the point of intersection of edge b with plane $X = X_C$. We move point O^* parallel to point O. We denote such a trihedral angle by $\overline{S_3}$, and its conditional full angle by $\omega\left(\overline{S_3}\right)$. In this case, edge b^* lies on edge b, so point B^* overlaps with point B. Continuing the edge a^* to the opposite side, intersect it with the plane $X = 0$ and denote the intersection point by A^*.

Since the plane $\pi\left(a', b'\right)$ lies on plane $\pi\left(a, b\right)$, the segment A^*B^* lies in segment AB, it will be $AB > A^*B^*$ (Figure 10). Will be $AC = CN,\quad BN = BP,\quad BM = BQ$ and $\omega\left(S_3\right) = AP,\quad \omega\left(\overline{S_3}\right) = A^*Q$ in Figure 10. Since it is $AP > A^*Q$, relation $\omega\left(S_3\right) > \omega\left(\overline{S_3}\right)$ is appropriate.

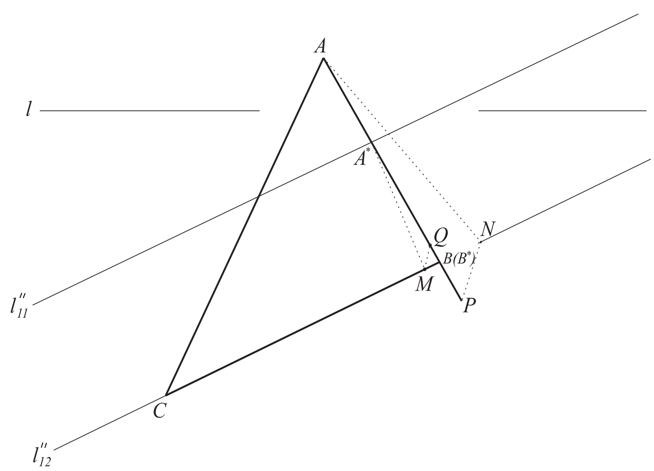

Figure 10. Conditional full angle of trihedral angle for case II.1.

2. Let the point O^* lie on the part of the edge b between the planes $X = 0$ and $X = X_C$. We move point O^* parallel to point O. In this case, edge b^* lies on edge b, so point B^* overlaps with point B. Continuing the edge a to the opposite side, we intersect it with the plane $X = 1$ and denote the intersection point by A^*.

Since plane $\pi(a', b')$ lies in plane $\pi(a, b)$, segment A^*B^* lies in segment AB, because plane $\pi(b', c')$ overlaps with plane $\pi(b, c)$, segment B^*C^* lies in segment BC, where $AB > A^*B^*$ and $BC < B^*C^*$. (Figure 11). Will be $AC = CN$, $BN = BP$, $BM = BQ$ and $\omega(S_3) = AP$, $\omega(\overline{S_3}) = A^*Q$ in Figure 11 . Since it is $AP > A^*Q$ and relation $\omega(S_3) > \omega(\overline{S_3})$ is appropriate.

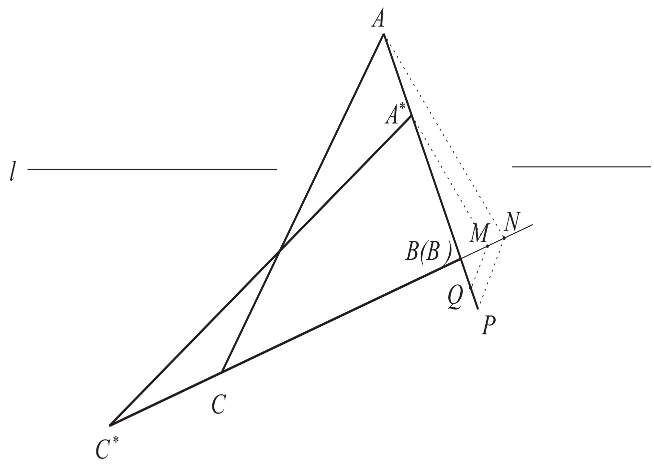

Figure 11. Conditional full angle of trihedral angle for case II.2.

3. Let point O^* lie on the part $\sigma < Z < b$ of $X = X_C$ planes. In this case, $\omega(S_3) > \omega(\overline{S_3})$ is proved for the 1st case of II in the same way as for the 2nd case of I.

4. Let point O^* lie on the part $\sigma < Z < 0$ of edge a. We move point O^* parallel to point O. The edge a' lies on edge a, so point A^* overlap with point A. Since plane $\pi(a', b')$ overlaps plane $\pi(a, b)$, and plane $\pi(a', c')$ overlaps plane $\pi(a, c)$, the segment A^*C^* lies on the segment AC, the segment A^*B^* in the segment AB, where $AC > A^*C^*$ and $AB > A^*B^*$.

Also corresponding to these will be $BC > B^*C^*$ (Figure 12). In Figure 12, will be $AC = CN$, $BN = BP$, $B^*M = B^*Q$ and $\omega(S_3) = AP$, $\omega(\overline{S_3}) = A^*Q$. Since it is $AP > A^*Q$, relation $\omega(S_3) > \omega(\overline{S_3})$ is appropriate.

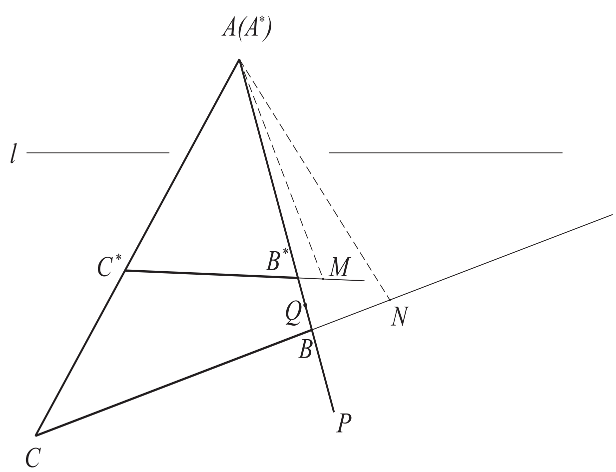

Figure 12. Conditional full angle of trihedral angle for case II.4.

5. Let the point O^* lie on the side $\sigma < Z < 0$ between the edges a and b, on the side $X < X_C$ of the plane $X = X_C$. In this case, the proof of Theorem 2.2 is the same as in Case 4 of I.

6. Let the point O^* lie on the part $\sigma < Z < 0$ between the edges a and c, on the side $X < X_C$ of the plane $X = X_C$. In this case, the proof of Theorem 2 is as in case 5.

From above 6 cases follows this assertion:

Proposition 2. If point O^* is located on the surface of S_3, then relation $\omega(S_3) \geq \omega(\overline{S_3^*})$ is appropriate.

7. Let point O^* lie on the part $\sigma < Z < 0$ of axis Oz. In this case, the proof of Theorem 2.2 is performed as in [11].

8. Let point O^* lie in the part $\sigma < Z < 0$ of S_3, outside the plane $X = X_C$. Then $\omega(S_3) \geq \omega(\overline{S_3^*})$ is proved as shown in [11].

The rest of the cases are proved in the same way as in case 8 above, only the points A and B are replaced, as well as the points A^* and B^* corresponding to these points.

Theorem 2 is proved.

We derive the following theorem from the results of Theorem 1 and Theorem 2:

Theorem 3. *For trihedral angles $S|_3$ and $S|_3^*$ the following relation is appropriate:*

$$\omega(S|_3) \geq \omega(S|_3^*). \tag{14}$$

3 Conclusions

In this paper, the monotonicity of the conditional full angle defined for polyhedral angles is proved. Using the positive definiteness and monotonicity of the conditional full angle, it is possible to prove the monotonicity property, which is important for the conditional extrinsic curvature for solving the problem of recovery by conditional extrinsic curvature. Thus, our result allows us to recover a polyhedral in a certain sense by its conditional extrinsic curvature.

In economics, a production function is a mathematical expression which denotes the physical relations between the output generated by a firm, an industry or an economy and the inputs that have been used. The generalized production function depends on the sign of the Gaussian curvature in the Galilean geometry of the surface. Surfaces isometric on sections (isometric in the sense of Galilean geometry) have the same signs of Gaussian curvature. Therefore, the findings of the work have direct application in economics.

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