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On the Number of Monochromatic Triples Associated with Binary Equations over Coloured Algebraic Groups

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Abstract Schur's Theorem on integer colouring states that colouring integers using finitely many colours yields at least one monochromatic solution to the equation $x + y = z$. An extension of Schur's theorem on integer lattices is explored by Vishal Balaji, Andrew Lott and Alex Rice. Schur tried to prove Fermat's Last Theorem by proving non-existence of the solution to the equation $x^n + y^n = z^n \ (mod\ p)$ for prime p. But he in fact proved that "for every integer $n \geq 1$, there exists p_0 such that for any prime $p \geq p_0$, the congruence $x^n + y^n = z^n \ (mod\ p)$ has a solution" where he failed to prove Fermat's Last Theorem in this route of attack. This demonstrates that Fermat's Last Theorem does not hold in the finite field $\mathbb{Z}_p$ for any sufficiently large prime p. We investigate Schur's Theorem on integer colouring and the corresponding theoretical framework in algebraic groups, and we classify colourings that yield a monochromatic solution to $xy = z$ (not all are equal). We use combinatorial tools like bijective counting and Pigeonhole principle to arrive at Theorem 3.9. Our methods include Principle of Inclusion-Exclusion formula to prove the principal result Theorem 3.20. We have used Python language to implement algorithms developed during the research to showcase Schur triples associated to the group $\mathbb{Z}_7$ and some given colouring maps. We illustrated various groups and its colouring properties. We were able to find bounds using certain parameters involving special subgroups of the group for Schur triples (x, y, xy) such that x, y and xy get the same colour in algebraic groups when coloured using finitely many colours. We also find the connection between proper vertex colouring and group colouring via Cayley graphs of semigroups. Our study throws light on new combinatorial perspectives on colouring problems on finite algebraic groups. The results help to enhance new algorithms related to Cayley graph colourings associated with finite semigroups. The research helps in combinatorial studies equipped by colouring problems involving network theory.

Keywords Algebraic Group Colouring, Schur Colourings, Schur Triples and Cayley Graphs of Semigroups

2020 AMS Subject Classification 05E15, 05C25 and 05C15.

1 Introduction

Schur's theorem, named after the German mathematician Issai Schur [1], is an essential outcome in combinatorics and number theory. The theorem deals with colouring the positive integers with a finite number of colours so that certain arithmetic progressions cannot be monochromatic (all elements having the same colour) [2]. An extension of Schur's theorem on integer lattices is explored by Vishal Balaji, Andrew Lott and Alex Rice [3]. Robertson, Aaron and Doron Zeligberger [4] explore lower bounds 2-colouring of $[1, n]$. More precisely, Schur's theorem states that for any positive integer k, there exists a positive integer $N(k)$ such that if the positive integers are coloured with k colours, there must exist three integers x, y and z all of the same colours, such that $x + y = z$. This can be restated in group theoretic term in the following way. For each k there is a group $\mathcal{G}$ with $N(k)$ number of elements such that if $\mathcal{G}$ is coloured by k colours then there is a monochromatic solution to the equation $xy = z$. Here the group is evidently $\mathbb{Z}_{\lceil ek! \rceil}$, where $\lceil x \rceil$ denotes the least integer greater than x and $e \approx 2.71828$ [5]. In simpler terms, no matter how you colour the positive integers with a finite number of colours, there will al-ways be a monochromatic solution to the equation $x+y=z$ where x, y and z are distinct positive integers. Schur's theorem has important implications in Ramsey theory, which studies the emergence of order in seemingly chaotic structures. The proof of Schur's theorem involves a clever application of the Pigeonhole Principl [6] and is a classic example of how seemingly random configurations can reveal hidden structures in mathematics. In this paper, we investigate the colouring problem associated with Algebraic structures, especially of Group structure. We introduce an analogue of concepts involved in Schur's theorem. Schurs theorem immediately tells us that the monogenic semigroup of natural numbers solves $x + y = z$ monochromatically when it is coloured using finitely many colours [7]. Hence it solves the same for infinite cyclic groups which are isomorphic to the group of integers under addition [8]. This paper studies the set of finite colourings of a finite group which produces a single-colored solution to the equation $x\#y = z$(not all are equal). Where $\#$ is the binary multiplication in the respective finite group. We are exploring weak lower bounds of such colourings through combinatorial tools. Vertex colouring of Cayley graphs is explored by the authors [9] and [10]. Cayley graph colourings, which involve assigning colors to the vertices or edges of Cayley graphs subject to certain constraints, find applications in various areas of physics [11, 12]. These applications often exploit the symmetry properties and structural characteristics of Cayley graphs. We find the interconnection between vertex colouring Cayley graphs of finite semigroups and Schur colourings of the respective semigroup.

2 Preliminaries

Definition 2.1. [1] A t-colouring of set X is a map from X to $\mathbb{Z}_t$. Where $\mathbb{Z}_t = \{0, 1, 2, ..., t-1\}$. Here we have t colours $0, 1, 2, .., t-1$.

Definition 2.2. [1] Let θ be a t-colouring of X. Define the relation $\sim_\theta$ on the set X such that $x \sim_\theta y$ if and only if $\theta(x) = \theta(y)$. Equivalently x is related to y if x and y have the same colour under the t-colouring θ. Then $\sim_\theta$ is an equivalence relation.

Theorem 2.3. [1] *Let $t \geq 2$. For any t-colouring of $\mathbb{Z}$ there exists a single-coloured solution to the equation x+y=z(not necessarily distinct). The triples (x, y, z) are called Schur triples.*

Definition 2.4. [6] For directed graph X with vertex set $V(X)$. A colouring map $\theta : V(X) \longrightarrow C$ to a set of colours C is a proper colouring if for each edge (x, y) of the edge set of X, $\theta(x) \neq \theta(y)$. For simple graph X, chromatic number $\chi(X)$ is the minimum number of colours that are needed to colour vertices of X properly.

Theorem 2.5. [6] *Let n and k positive integers such that $n > k$. If n balls are distributed to k baskets, then there exists at least one basket containing $\left\lceil \frac{n}{k} \right\rceil$ number of balls or more. Where $\lceil x \rceil$ is the least integer bigger than x.*

Definition 2.6. [6] Suppose $n > 0$ and $S_1, ..., S_n$ are finite sets. Then

$$\left| \bigcup_{1 \leq j \leq n} S_j \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \left| \bigcap_{1 \leq t \leq k} S_{i_t} \right|. \tag{2.1}$$

Proposition 2.7. [7] *Let θ be a t-colouring of X. Denote M_x^θ be the equivalence class containing x under the equivalence relation $\sim_\theta$. Also called the colour class containing x. Then*

1. *For each $x \in X$, $(\theta^{-1} \circ \theta)(x) = M_x^\theta$*
2. *For each $y \in \mathbb{Z}_t$, $(\theta \circ \theta^{-1})(y) = y$ if $\theta^{-1}(y) \neq \phi$.*

Proof.

1. Let $x \in X$ and $s = \theta(x)$.
Take $r \in \theta^{-1}(s) \iff \theta(r) = \theta(x) \iff r \in M_x^\theta$.
2. Since $\theta^{-1}(y) = \{x \in X |\ \theta(x) = y\}$. □

Definition 2.8. [7, 8] Let $\mathcal{G}$ be a non-void set with a binary process $\# : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ ($\#(g, h)$ will be denoted simply as 'gh'). For an element $a \in \mathcal{G}$ define $a\mathcal{G} = \{ag : \ g \in \mathcal{G}\}$ and $\mathcal{G}a = \{ga : \ g \in \mathcal{G}\}$

1. $\mathcal{G}$ is semigroup if the operation $\#$ is associative. That is for any $g, h, k \in \mathcal{G}$ we have $g(hk) = (gh)k$.
2. A semigroup $\mathcal{G}$ is a group if for any $a \in \mathcal{G},\ a\mathcal{G} = \mathcal{G}$ and $\mathcal{G}a = \mathcal{G}$.
3. Let H be a non-empty subset of $\mathcal{G}$ then H is called a subgroup of $\mathcal{G}$ if for any $a \in H,\ aH = H = Ha$.
4. The quantity of elements in a group $\mathcal{G}$ with identity e is denoted by $O(\mathcal{G})$. And for an element $g \in \mathcal{G}$ order of g is the minimal positive integer n such that $g^n = e$ and the order of an element g is denoted by $o(g)$.

Definition 2.9. [7] A left-zero semigroup S_L is a semigroup in which $xy = x$ for all $x, y \in S_L$. And right-zero semigroup S_R is a semigroup in which $xy = y$ for all $x, y \in S_R$.

Theorem 2.10. [8] *For finite group $\mathcal{G}$ and a subgroup H of $\mathcal{G}$. We have $O(H)|O(\mathcal{G})$.*

Definition 2.11. [8] Let $\mathcal{G}$ be a group and H be a subgroup of $\mathcal{G}$ then the index of H in $\mathcal{G}$ is the number of the left(right) cosets of H in $\mathcal{G}$ and it is denoted by $[\mathcal{G} : H]$. By Lagrange's theorem For finite groups, we have $[\mathcal{G} : H] = \frac{O(\mathcal{G})}{O(H)}$.

Definition 2.12. [8] A homomorphism ϕ from a group $\mathcal{G}_1$ to a group $\mathcal{G}_2$ is a mapping from $\mathcal{G}_1$ into $\mathcal{G}_2$ that maintains the group operation; that is, $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in \mathcal{G}$.

Definition 2.13 (Kernel of a Homomorphism)**.** [8] The kernel of a homomorphism ϕ from a group $\mathcal{G}$ to a group whose identity element is e is the set of elements in $\mathcal{G}$ that map to e. This set is denoted as $Ker(\phi)$ and is defined by $\{x \in \mathcal{G} : \phi(x) = e\}$.

Theorem 2.14. [8] *Let ϕ be a homomorphism from a group $\mathcal{G}_1$ to a group $\mathcal{G}_2$. Then ϕ transports the identity of $\mathcal{G}_1$ to the identity of $\mathcal{G}_2$.*

Theorem 2.15. [8] *If $\mathcal{G}$ is a finite group with order $p^r m$, where p is a prime and $p \nmid m$, then $\mathcal{G}$ contains a subgroup of order p^r.*

Definition 2.16. [13] Let n and k positive integers such that $n \geq k$. Then the number of ways to partition an n-element set to k-blocks is given by Stirling's numbers which is denoted by $S(n, k)$. A special case is given by

$$S(n, 2) = 2^{n-1} - 1. \tag{2.2}$$

Definition 2.17. [14] Let $\mathcal{X}$ be graph with vertex set V and an edge set E and C be a finite set of colours. A vertex colouring $\alpha : V \longrightarrow C$ is proper if both ends of each edge in E get different colours.

Definition 2.18. [15] Let $\widehat{\mathcal{G}}$ be a Semigroup and $S \subset \widehat{\mathcal{G}}$. Then $Cay(\widehat{\mathcal{G}}, S)$ is called the Cayley graph of $\widehat{\mathcal{G}}$ with connection set S if the vertex set of the graph is equal to $\widehat{\mathcal{G}}$ and the edge set $E(Cay(\widehat{\mathcal{G}}, S)) = \{(x, y) \mid y = sx \text{ for some } s \in S\}$.

3 Main Results

Definition 3.1 (Trivial and Non-trivial colouring of a Group)**.** Let $\mathcal{G}/\sim_\theta$ be the collection of colour classes under the equivalence relation $\sim_\theta$.

1. θ is called trivial if and only if $\mathcal{G}/\sim_\theta = \{\mathcal{G}\}$.
2. θ is called non-trivial if and only if $|\mathcal{G}/\sim_\theta| \geq 2$.
3. θ is said to be Exhaustive if $|\mathcal{G}/\sim_\theta| = t$.

Remark 3.2. For a 2-colouring θ Exhaustiveness and non-triviality are interchangeable.

Proposition 3.3. *Let θ be a t-colouring of a finite group $\mathcal{G}$ with n elements and $n \geq t$. Then θ is Exhaustive if and only if θ is surjective.*

Proof. Assume that θ is surjective. Then for each $s \in \mathbb{Z}_t$ there exists a $q \in \mathcal{G}$ such that $\theta(q) = s$. This implies $|\mathcal{G}/\sim_\theta| \geq t$. Since there are at most t colour classes available, we have $|\mathcal{G}/\sim_\theta| = t$.
Conversely, assume that θ is Exhaustive. That is $|\mathcal{G}/\sim_\theta| = t$. Let $s \in \mathbb{Z}_t$ and M be the colour class such that $\theta(M) = s$. Then M is non-empty. If M is empty then $|\mathcal{G}/\sim_\theta| < t$ which is a contradiction. That is there exists a $y \in M \subseteq \mathcal{G}$ such that $\theta(y) = s$. This shows that θ is surjective. ☐

Definition 3.4 (Schur colouring of a Group)**.** A non-trivial t-colouring θ of a group $\mathcal{G}$ is said to be Schur colouring if there exists a monochromatic solution to the equation xy=z(not all are equal).

Example 3.5. Consider 3-colouring of the group $\mathbb{Z}_5$ with an addition modulo 5 as the group operation. Let B, R and G be Blue, Red and Green respectively.
Let

$$\theta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ B & R & B & B & G \end{pmatrix}$$

and

$$\tilde{\theta} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ B & R & G & G & R \end{pmatrix}$$

be two 3-colourings.
Since 2+3=0 and $\theta(2) = \theta(3) = \theta(0) =$ Blue. That is θ is a Schur colouring. Now for $\tilde{\theta}$ consider all monochromatic sums ,

$$\begin{array}{cc} 1+4=0 & 2+3=0 \\ 1+1=2 & 2+2=4 \\ 3+3=1 & 4+4=3 \\ R+R=B & G+G=B \\ R+R=G & G+G=R \\ G+G=R & R+R=G \end{array}$$

In all cases $\tilde{\theta}$ fails to produce a monochromatic solution to the equation $x + y = z(\text{mod } 5)$. That is $\tilde{\theta}$ is not a Schur colouring.

Proposition 3.6. *Let $\mathcal{G}$ be a finite group with n elements and $n > t$. Then the probability that a t-colouring θ is a Schur colouring is at least $\frac{1}{t}$.*

Proof. denote $P_t(g; s)$= The probability that g gets the colour s when $\mathcal{G}$ is coloured using at most t colours. Then the number of t-colourings that maps g to s is t^{n-1}. And the total number of t-colourings is t^n. Then

$$P_t(g; s) = \frac{t^{n-1}}{t^n} = \frac{1}{t}.$$

Let θ be a t-colouring of the group $\mathcal{G}$. Since $n > t$ by the pigeonhole principle, there are at least two distinct elements g and h in $\mathcal{G}$ such that $\theta(g) = \theta(h) = s \in \mathbb{Z}_t$. Since $\mathcal{G}$ is group $gh = k \in \mathcal{G}$. If k gets the colour s then we have a monochromatic solution to the equation $xy = z$ and θ will be a Schur colouring. Then the probability that a t-colouring θ is a Schur colouring $\geq P_t(gh; s) = P_t(k; s) = \frac{1}{t}$. ☐

Definition 3.7 (Collections of Schur colourings and its counts)**.** Let $\mathcal{G}$ be a finite group with n elements and H be a subgroup of $\mathcal{G}$. Then denote

1. $\Theta_t(\mathcal{G})$:= The set of all Schur t-colourings of $\mathcal{G}$ and $\eta_t(\mathcal{G}) = |\Theta_t(\mathcal{G})|$.
2. $\Theta_t^E(\mathcal{G})$:= The set of all Exhaustive Schur t-colourings of $\mathcal{G}$ and $\eta_t^E(\mathcal{G}) = |\Theta_t^E(\mathcal{G})|$.
3. $\Theta_t^H(\mathcal{G})$:= The set of all Schur t-colourings of $\mathcal{G}$ leaves H monochromatic and $\eta_t^H(\mathcal{G}) = |\Theta_t^H(\mathcal{G})|$.
4. $\Theta_t^{EH}(\mathcal{G})$:= The set of all Exhaustive Schur t-colourings of $\mathcal{G}$ leaves H monochromatic and $\eta_t^{EH}(\mathcal{G}) = |\Theta_t^{EH}(\mathcal{G})|$.

Remark 3.8. It immediately follows that

1. $\Theta_t^E(\mathcal{G}) \subset \Theta_t(\mathcal{G})$ and $\eta_t^E(\mathcal{G}) \leq \eta_t(\mathcal{G})$.
2. $\Theta_t^{EH}(\mathcal{G}) \subset \Theta_t^H(\mathcal{G})$ and $\eta_t^{EH}(\mathcal{G}) \leq \eta_t^H(\mathcal{G})$.

Theorem 3.9. *Let $\mathcal{G}$ be a finite group of odd order. Then*

$$\eta_2(\mathcal{G}) = 2^{n-1} - 1 - \binom{n}{\frac{n-1}{2}}. \tag{3.1}$$

Proof. Let $\theta : \mathcal{G} \longrightarrow \mathbb{Z}_2$ be a non-trivial 2-colouring of the group $\mathcal{G}$. Assume that θ is not a Schur colouring. Since θ is non-trivial there are two subsets A and B such that they form a partition of the group $\mathcal{G}(A \cup B = \mathcal{G}$ and $A \cap B = \phi)$ such that $\theta(A) = 0$ and $\theta(B) = 1$. Without sacrificing generality assume that the identity e of the group $\mathcal{G}$ belongs to A.
We claim that for each $g \in A - \{e\},\ g^2 \in B$ and for each $g \in B,\ g^2 \in A$. If not, there is a $g \in A - \{e\}$ such that $g^2 \in A$. Then $\theta(g) = \theta(g^2) = 0$. Since $g \neq e$ we have $g^2 \neq g$. Then θ is Schur colouring which is a contradiction. Similarly for $g \in B,\ g^2 \notin B$. But $g^2 \in A$.
Now notice that since $\mathcal{G}$ is of odd order we have $O(\mathcal{G}) + 1$ is even. Since for any $g \in \mathcal{G},\ g^{O(\mathcal{G})} = e$ we have $(g^{\frac{O(\mathcal{G})+1}{2}})^2 = g$. That is for $g \in \mathcal{G}$ there is an $h \in \mathcal{G}$ such that $g = h^2$.
Now define a function $\zeta_1 : A - \{e\} \longrightarrow B$ such that $\zeta_1(g) = g^2$. This is possible by the previously justified claim. We will show that ζ_1 is onto. Take $g \in B$ then $g = h^2$ for some $h \in G$. We have $h \in A - \{e\}$. If not, $h \in B$ then θ is Schur colouring which is a contradiction. That is ζ_1 is onto. This implies $|A - \{e\}| \geq |B|$.
Similarly define $\zeta_2 : B \longrightarrow A - \{e\}$ such that $\zeta_2(g) = g^2$. Take $g \in A$ then $g = h^2$ for some $h \in G$. We have $h \in A - \{e\}$. If not, $h \in B$ then θ is Schur colouring which is a contradiction. By similar arguments, it is proved that ζ_2 is onto. Then $|A - \{e\}| \leq |B|$. That is $|A| = |B| + 1$. Then we have non-Schur colourings in bijection with partitions of $\mathcal{G}$ into two blocks A and B such that one contains $\frac{n-1}{2}$ and other one contains $\frac{n-1}{2} + 1$. Then the number of non-Schur colourings is exactly $\binom{n}{\frac{n-1}{2}}$. That is the total number of Schur colourings given by

$$\eta_2(\mathcal{G}) = S(n, 2) - \binom{n}{\frac{n-1}{2}}.$$

Since $S(n, 2) = 2^{n-1} - 1$, By equation 2.2 we have

$$\eta_2(\mathcal{G}) = 2^{n-1} - 1 - \binom{n}{\frac{n-1}{2}}.$$

Hence the proof. □

Theorem 3.10. *Let $\mathcal{G}$ be a finite group with n elements and $S(m, k)$ be the number of partitions of an m-element set into k blocks. Then*

1.
$$\eta_t(\mathcal{G}) \geq (n-1)\, t^{n-1}. \tag{3.2}$$

2.
$$\eta_t^E(\mathcal{G}) \geq S(n-2, t)\, t!\, t \text{ for } n - 2 \geq t. \tag{3.3}$$

Proof. (1) Let e be the identity element of the group $\mathcal{G}$. Then we have $eg = g = ge$ for every element $g \in \mathcal{G}$. If we colour e and g with the same colour then we have a single-coloured solution to the equation $xy = z$ with $y = z = g$ and $x = e$ or $x = z = g$ and $y = e$. We count these t-colourings in the following way. For a fixed $g \in \mathcal{G}$ and e and a fixed colour $s \in \mathbb{Z}_t$ We assign the colour s to the set $\{g, e\}$. Then the number of t-colourings in this fashion is the number of maps from $\mathcal{G} - \{e, g\}$ to $\mathbb{Z}_t$. It is equal to t^{n-2}. So for t colours, it is equal to $t \times t^{n-2} = t^{n-1}$. Now there are $(n-1)$ elements in $\mathcal{G} - \{e\}$. This implies there are at least $(n-1)\, t^{n-1}$ Schur t-colourings.
(2) For a fixed $g \in \mathcal{G}$ and e and a fixed colour $s \in \mathbb{Z}_t$ we assign the colour s to the set $\{g, e\}$. Then the number of t-colourings in this fashion is the number of surjective maps from $\mathcal{G} - \{e, g\}$ to $\mathbb{Z}_t$. Since $n - 2 \geq t$ the collection of such maps is non-empty. The number of surjective maps is given by $S(n-2, t)\, t!$. If we vary colours we get $S(n-2, t)\, t!\, t$. This implies there are at least $S(n-2, t)\, t!$ Exhaustive Schur t-colourings.

□

Theorem 3.11. *Let $\mathcal{G}$ be a finite group and H be a subgroup of $\mathcal{G}$. Then*

1.
$$\eta_t^H(\mathcal{G}) = t^{O(\mathcal{G}) - O(H) + 1}. \tag{3.4}$$

2.
$$\eta_t^{EH}(\mathcal{G}) = S(O(\mathcal{G}) - O(H), t)\, t!\, t \text{ for } O(\mathcal{G}) \geq O(H) + t. \tag{3.5}$$

Proof. (1) Let H be a subgroup of $\mathcal{G}$. Then for a fixed colour $s \in \mathbb{Z}_t$ we have $t^{O(\mathcal{G}) - O(H)}$ Schur t-colourings are possible. If we vary the colours we get $t^{O(\mathcal{G}) - O(H)} \times t = t^{O(\mathcal{G}) - O(H) + 1}$. Since there are at most t colours. Now for any $\theta \in \Theta_t^H(\mathcal{G})$ we have a colour s assigned to H and t colours assigned to $O(\mathcal{G}) - O(H)$.
(2) For a fixed colour $s \in \mathbb{Z}_t$ we have $S(O(\mathcal{G}) - O(H), t)\, t!$ Exhaustive Schur t-colourings are possible. If we vary the colours we get $S(O(\mathcal{G}) - O(H), t)\, t!\, t$ total number of Exhaustive Schur t-colourings. Now for any $\theta \in \Theta_t^H(\mathcal{G})$ we have a colour s assigned to the subgroup H and we have a surjective mapping from $\mathcal{G} - H$ to $\mathbb{Z}_t$. This is possible because $O(\mathcal{G}) \geq O(H) + t$. □

Corollary 3.12. *Let H be a subgroup of finite $\mathcal{G}$ of index $[\mathcal{G} : H] = k$. Then*

1.
$$\eta_t^H(\mathcal{G}) = t^{(k-1)O(H) + 1}. \tag{3.6}$$

2.
$$\eta_t^{EH}(\mathcal{G}) = S((k-1)O(H), O(H))\, t!\, t \text{ if } O(H) \geq \frac{t}{k-1}. \tag{3.7}$$

Proof. The proof is immediate from Lagrange's theorem for finite groups. Since $[\mathcal{G} : H] = k$ we have $O(\mathcal{G}) = k\, O(H)$ and the corollary follows from Theorem 3.11. □

Corollary 3.13. *Let $\mathcal{G}$ be a finite group, p be a prime factor of $O(\mathcal{G})$ and H_p be the Sylow-p subgroup of $\mathcal{G}$. Write $O(\mathcal{G}) = p^n\, m$ where $p \nmid m$. Then*

1.
$$\eta_t^{H_p}(\mathcal{G}) = t^{p^n(m-1)+1}. \tag{3.8}$$

2.
$$\eta_t^{EH_p}(\mathcal{G}) = S(p^n(m-1), t)\, t!\, t \text{ for } p^n(m-1) \geq t. \tag{3.9}$$

Proof. The proof follows immediately from Sylow theorems and Theorem 5. □

Definition 3.14. Let $\mathcal{G}$ be a finite group. Then define

$$\gamma_t(\mathcal{G}) = \left|\{\theta \mid \theta \in \Theta_t^H(\mathcal{G}) \text{ for some } H \leq \mathcal{G}\}\right|. \tag{3.10}$$

That is $\gamma_t(\mathcal{G})$ is the number of t-colourings which leaves at least one subgroup H of $\mathcal{G}$ monochromatic. Then we have $\eta_t(\mathcal{G}) \geq \gamma_t(\mathcal{G})$.

Theorem 3.15. *Let $\mathcal{G}$ be a finite group and $\mathbb{Z}_t$ be the set of colours with a group structure under the operation addition modulo t. Then a t-colouring θ is Schur colouring. Moreover θ belongs to $\Theta_t^H(\mathcal{G})$ for some subgroup H of $\mathcal{G}$ if θ is a non-trivial non-injective homomorphism from $\mathcal{G}$ to $\mathbb{Z}_t$.*

Proof. Let $\theta : \mathcal{G} \longrightarrow \mathbb{Z}_t$. be group homomorphism from $\mathcal{G}$ to $\mathbb{Z}_t$. Then $Ker\,\theta$ is a normal subgroup of $\mathcal{G}$. That is θ colours $Ker\,\theta$ monochromatically with colour 0. We have θ is non-trivial and non-injective. This implies that $Ker\,\theta$ contains at least two elements but not all of $\mathcal{G}$. That is θ is a Schur colouring. □

Definition 3.16. Let $\mathcal{G}$ be a finite group. Then define $\beta_t(\mathcal{G})$ = The number of non-trivial non-injective homomorphisms from $\mathcal{G} \longrightarrow \mathbb{Z}_t$. Then $\gamma_t(\mathcal{G}) \geq \beta_t(\mathcal{G})$.

Theorem 3.17. *For the group $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$ under the operation addition modulo n and $gcd(n, t)$ denotes the greatest common divisor of n and t. Also $\Phi(t)$ is the number of elements less than t which is co-prime to t. Then we have*

$$\beta_t(\mathbb{Z}_n) = \begin{cases} t - \Phi(t) - 1 & \textit{if n is a multiple of t.} \\ gcd(n,t) - 1 & \textit{if n is not a multiple of t.} \end{cases} \tag{3.11}$$

Proof. The function $\theta : \mathbb{Z}_n \longrightarrow \mathbb{Z}_t$ given by $\theta(x) = sx$ for some $s \in \mathbb{Z}_t$ fixed is a homomorphism of groups if and only if $na \equiv 0 \bmod t$. Now the congruence $na \equiv 0 \bmod t$ has $b = gcd(n, t)$ solutions. The solutions are given by $s = \frac{n}{t}r$, where $r = 0, 1, 2, ..., b-1$. Therefore the number of group homomorphisms $\theta : \mathbb{Z}_n \longrightarrow \mathbb{Z}_t$ is $b = gcd(n, t)$. Among these homomorphisms, we sort out all possible non-trivial non-injective group homomorphisms in the following way,
For $\theta(x) = sx \bmod t$, $Ker\,\theta = \{x \in \mathbb{Z}_n \mid sx \equiv 0 \bmod t\}$. The congruence $sx \equiv 0 \bmod t$ has $gcd(s, t)$ solutions. If $gcd(s, t) = 1$ then by the property of group homomorphisms $Ker\,\theta = \{0\}$. We count such homomorphisms and we avoid these from the set of all group homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}_t$.
If

$$\begin{aligned} gcd(s,t) &= 1 \\ \Longrightarrow gcd(\frac{t}{b}k, t) &= 1 \\ \Longrightarrow gcd(\frac{t}{b}k, \frac{t}{b}b) &= \frac{t}{b}gcd(k, b) = 1 \\ \Longrightarrow gcd(k, b) &= \frac{b}{t}. \end{aligned}$$

Since b is the divisor of t, we have $b \leq t$. Then $gcd(k, b)$ is an integer only when $b = gcd(n, t) = t$ if and only if t divides n. That is n has to be a multiple of t. If it happens there are $\Phi(b)$ elements such that $gcd(k, b) = 1$, which is exactly the number of injective homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}_t$. Then there are $b - \Phi(b)$ non-injective group homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}_t$. From this avoid trivial homomorphism. Then we have the desired lower bound. □

t\n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	0	0	0	0	0	0	0	0	0	0
3	0	0	2	0	0	2	0	0	2	0
4	0	1	0	3	0	1	0	3	0	1
5	0	0	0	0	4	0	0	0	0	4
6	0	1	2	1	0	5	0	1	2	1
7	0	0	0	0	0	0	6	0	0	0
8	0	1	0	3	0	1	0	7	0	1
9	0	0	2	0	0	2	0	0	8	0
10	0	1	0	1	4	1	0	1	0	9

Table 1. The $t \times n$ table of size 10 of the function $\beta_t(\mathbb{Z}_n)$.

The positive entries in **Table 1** give us the effective lower bounds of $\beta_t(\mathbb{Z}_n)$ for $1 \leq t \leq 10$ & $1 \leq n \leq 10$.

Corollary 3.18. *For the group $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$ under the operation addition modulo n.*

$$\gamma_t(\mathbb{Z}_n) \geq \begin{cases} gcd(n,t) - \Phi(t) - 1 & \textit{if n is a multiple of t.} \\ gcd(n,t) - 1 & \textit{if n is not a multiple of t.} \end{cases} \tag{3.12}$$

Proof. The proof immediately follows from the fact that $\gamma_t(\mathcal{G}) \geq \beta_t(\mathcal{G})$. □

Definition 3.19. Let θ be a t-colouring of the group $\mathcal{G}$ and H be a proper subgroup of $\mathcal{G}$ and $s \in \mathbb{Z}_t$. Then

1. $\Gamma_\theta(\mathcal{G}) = \{H \leq \mathcal{G} \mid \theta(H) = s \text{ for some } s \in \mathbb{Z}_t\}$ = The set of subgroups of $\mathcal{G}$ which are monochromatically coloured by θ.
2. $\chi_\theta(\mathcal{G}) = \min\{\, [\mathcal{G} : H] \mid H \in \Gamma_\theta(\mathcal{G})\}$.
3. $\tau_{\theta,s}(\mathcal{G})$ = The number of Schur triples (x, y, z), $\theta(x) = \theta(y) = \theta(z) = s$ and $x + y = z$. Clearly $\tau_\theta = \sum_s \tau_{\theta,s}$.

Theorem 3.20. *Let $\mathcal{G}$ be a finite group and θ be a t-colouring such that $|\mathcal{G}| > t$ then the following statements are true.*

1. *If $s_0 \in \theta(\mathcal{G})$ such that $|\theta^{-1}(s_0)|$ is the largest among all fibres of colours then*

$$\tau_\theta(\mathcal{G}) < |\theta(\mathcal{G})||\theta^{-1}(s_0)|^2. \quad (3.13)$$

2. *If $\Gamma_\theta(\mathcal{G}) \neq \phi$ then, there is unique colour $s \in \mathbb{Z}_t$ such that for all $H \in \Gamma_\theta(\mathcal{G})$, $\theta(H) = s$ and*

$$\sqrt{\tau_\theta(\mathcal{G})+1} \geq \sum_{k=1}^{r}(-1)^{k-1} \sum_{1\leq q_1<q_2<\ldots<q_k\leq r} |H_{q_1} \cap H_{q_2} \cap \ldots \cap H_{q_k}| \geq \frac{O(\mathcal{G})}{\chi_\theta(\mathcal{G})}. \quad (3.14)$$

Where $\Gamma_\theta(\mathcal{G}) = \{H_i\}_{i=1}^r$, such that $\theta(H_i) = s$ and s is the unique colour associated with $\Gamma_\theta(\mathcal{G})$. If $\tau_\theta(\mathcal{G}) > 1$ and s_0 is a colour that satisfies the condition in (1) then,

$$|\theta(\mathcal{G})||\theta^{-1}(s_0)|^2 > \tau_\theta(\mathcal{G}) > \frac{O(\mathcal{G})}{\chi_\theta(\mathcal{G})}. \quad (3.15)$$

Proof. (1) Let S_θ be the collection of all Schur triples when $\mathcal{G}$ is coloured by θ. Let us recall the number $\tau_{\theta,s}(\mathcal{G})$ which counts the Schur triples (x, y, xy) such that $\theta(x) = \theta(y) = \theta(xy) = s$. If $x, y \in \theta^{-1}(s)$ then $(x, y, xy) \in S_\theta$ if $xy \in \theta^{-1}(s)$. That is,

$$\tau_{\theta,s}(\mathcal{G}) \leq \left|\{(x, y, xy)|\, x, y \in \theta^{-1}(s)\}\right| = |\theta^{-1}(s)|^2.$$

But by carefully avoiding (e, e, e),

$$\tau_\theta(\mathcal{G}) = \sum_s \tau_{\theta,s}(\mathcal{G}) \leq \sum_{s\in\theta(\mathcal{G})} |\theta^{-1}(s)|^2 - 1$$

which then arrives at,

$$\tau_\theta(\mathcal{G}) < \tau_\theta(\mathcal{G}) + 1 \leq \sum_{s\in\theta(\mathcal{G})} |\theta^{-1}(s)|^2 \leq \sum_{s\in\theta(\mathcal{G})} |\theta^{-1}(s_0)|^2$$

$$= |\theta(\mathcal{G})||\theta^{-1}(s_0)|^2.$$

(2) Suppose that $H, K \in \Gamma_\theta(\mathcal{G})$ and $s, t \in \mathbb{Z}_t$ such that $\theta(H) = s$ and $\theta(K) = t$. But $H \cap K = \{e\}$, e is the identity of $\mathcal{G}$. Then we have $\theta(e) = s = t$. In essence s is unique. Notice that $|S_\theta| = \tau_\theta(\mathcal{G})$. If $H \in \Gamma_\theta(\mathcal{G})$ then the set $\{(x, y, xy) \mid x, y \in H\} \subseteq S_\theta$. The number of elements in the set $\{(x, y, xy) \mid x, y \in H\}$ is exactly equal to $O(H)^2$. But we avoid the triple (e, e, e), e is identity of $\mathcal{G}$. Then $\tau_\theta + 1 \geq O(H)^2$. More generally let $\Gamma_\theta(\mathcal{G}) = \{H_1, ..., H_r\}$, $\theta(H_i) = s$. Then

$$B = \{(x, y, xy)|\, x, y \in \bigcup_{i=1}^r H_i\} \subseteq S_\theta.$$

Now notice that $|B| = \left|\bigcup_{i=1}^r H_i\right|^2 - 1 \leq \tau_\theta$. That is $\left|\bigcup_{i=1}^r H_i\right| \leq \sqrt{\tau_\theta + 1}$. Then by the principle of Inclusion-Exclusion formula we have,

$$\bigcup_{i=1}^r H_i = \sum_{k=1}^r (-1)^{k-1} \sum_{1\leq q_1<q_2<\ldots<q_k\leq r_i} |H_{q_1} \cap H_{q_2} \cap \ldots \cap H_{q_k}|.$$

This proves that,

$$\sqrt{\tau_\theta(\mathcal{G}) + 1} \geq \sum_{k=1}^r (-1)^{k-1} \sum_{1\leq q_1<q_2<\ldots<q_k\leq r} |H_{q_1} \cap H_{q_2} \cap \ldots \cap H_{q_k}|.$$

By well-ordering principle there is a subgroup K such that $\theta(K) = s$ and $[\mathcal{G} : K] = \chi_\theta(\mathcal{G})$. Then by Lagrange's theorem, $O(K) = \frac{O(\mathcal{G})}{\chi_\theta(\mathcal{G})}$. Evidently $O(H) \leq \sqrt{\tau_\theta + 1}$. This implies,

$$\sqrt{\tau_\theta(\mathcal{G}) + 1} \geq \frac{O(\mathcal{G})}{\chi_\theta(\mathcal{G})}.$$

If $\tau_\theta(\mathcal{G}) > 1$ we have $\sqrt{\tau_\theta(\mathcal{G}) + 1} < \tau_\theta(\mathcal{G})$. That is we have,

$$|\theta(\mathcal{G})||\theta^{-1}(s_0)|^2 > \tau_\theta(\mathcal{G}) > \frac{O(\mathcal{G})}{\chi_\theta(\mathcal{G})}.$$

Now the order of the inequality follows from the fact that $K \subseteq \Gamma_\theta(\mathcal{G})$. This justifies the truth of the inequality. □

Corollary 3.21. *Let $\mathcal{G}$ be a group of order p^2 for some prime p. Then exists at least t^{p^2-p+1} number of t-colourings θ such that*

$$\sqrt{\tau_\theta(\mathcal{G}) + 1} \geq p. \quad (3.16)$$

Proof. Let $\mathcal{G}$ be a group order p^2. Then by Lagrange's theorem, the possible orders of the subgroups are $1, p, p^2$. The orders 1 and p^2 correspond to trivial and the entire group respectively.
Now by Sylow's theorem, there is a subgroup of order p. Denote it by H_p. Then again by Lagrange's theorem, we have

$$[\mathcal{G} : H_p] = \frac{p^2}{p} = p.$$

Using Theorem 3.20,

$$\eta_t^{H_p}(\mathcal{G}) = t^{p^2-p+1} \geq 1, \quad t \geq 1.$$

That is there is at least one Schur colouring which leaves H_p monochromatic and this implies $\chi_\theta(\mathcal{G}) = p$. Then by theorem 11, we have

$$\tau_\theta(\mathcal{G}) \geq p.$$

□

Proposition 3.22. *For a finite group $\mathcal{G}$ and $A \subseteq \mathcal{G}$, $P_A(g)$ denotes the number of tuples (h, k) such that $g = hk$, $h, k \in A$ for $g \in \mathcal{G}$. Then,*

$$\tau_\theta(\mathcal{G}) + 1 = \sum_M \sum_{g\in M} P_M(g). \quad (3.17)$$

Where M varies over all the complete set of colour classes under θ.

Proof. Let $\theta^{-1}(s)$ be the colour class corresponding to the colour s. For $g \in \theta^{-1}(s)$, the set $F_g = \{(h,k) \mid h,k \in \theta^{-1}(s), \ g = hk\}$ exhaust all monochromatic factorization of g. Since $\theta(h) = \theta(k) = \theta(g) = s$. $|F_g| = P_{\theta^{-1}(s)}(g)$ counts all Schur triples of the form (h,k,g). Since $F_g \cap F_h = \phi$ for $g \neq h$, we have

$$|\bigcup_{g\in\theta^{-1}(s)} F_g| = \sum_{g\in\theta^{-1}(s)} P_{\theta^{-1}(s)}(g) = \tau_{\theta,s}(\mathcal{G}).$$

which counts all the Schur triples (x,y,z) such that $\theta(x) = \theta(x) = \theta(y) = \theta(z) = s$. Then the total number of Schur triples forbidding (e,e,e) satisfies,

$$\tau_\theta(\mathcal{G}) + 1 = \sum_{s\in\theta(\mathcal{G})} \sum_{g\in\theta^{-1}(s)} P_{\theta^{-1}(s)}(g) = \sum_{M} \sum_{g\in M} P_M(g).$$

Where M varies over the complete set of colour classes under θ. □

Example 3.23. $\mathcal{G} = \mathbb{Z}_7$ and
$\theta_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathfrak{G} & \mathfrak{B} & \mathfrak{R} & \mathfrak{R} & \mathfrak{B} & \mathfrak{B} & \mathfrak{G} \end{pmatrix}$.

Then the colour classes are $M_{\mathfrak{B}} = \{1,4,5\}$, $M_{\mathfrak{G}} = \{0,6\}$, $M_{\mathfrak{R}} = \{2,3\}$. Now for the class $M_{\mathfrak{B}}$, use 3.17 to compute $1+4=5$, $4+1=5$, $1+5=6$, $5+1=6$, $1+1=2$, $4+5=2$, $5+4=2$, $4+4=1$, $5+5=3$ From this $F_1 = \{(4,4)\}$, $F_4 = \phi$, $F_5 = \{(1,4),(4,1)\}$. That is $P_{M_{\mathfrak{B}}}(1) = 1$, $P_{M_{\mathfrak{B}}}(4) = 0$, $P_{M_{\mathfrak{B}}}(5) = 2$. For the class $M_{\mathfrak{R}}$, compute $2+3=5$, $3+2=5$, $3+3=6$, $2+2=4$ but $F_2 = \phi$, $F_3 = \phi$ implies $P_{M_{\mathfrak{R}}}(2) = 0 = P_{M_{\mathfrak{R}}}(3)$. For the class $M_{\mathfrak{G}}$, compute, $0+0=0$, $0+6=6$, $6+0=6$, $6+6=5$. Then $F_6 = \{(6,0),(0,6)\}$, $F_0 = \{(0,0)\}$. Thus we have $P_{M_{\mathfrak{G}}}(0) = 1$, $P_{M_{\mathfrak{G}}}(6) = 2$. Then,

$$\tau_\theta(\mathcal{G})+1 = \sum_M \sum_{g\in M} P_M(g) = P_{M_{\mathfrak{B}}}(1)+P_{M_{\mathfrak{B}}}(4)+P_{M_{\mathfrak{B}}}(5)$$

$$+P_{M_{\mathfrak{R}}}(2) + P_{M_{\mathfrak{R}}}(3) + P_{M_{\mathfrak{G}}}(0) + P_{M_{\mathfrak{G}}}(6).$$

That is $\tau_\theta(\mathcal{G}) + 1 = 1+0+2+0+0+2+1 = 6$ implies $\tau_\theta(\mathcal{G}) = 5$. This can be confirmed by the following Cayley colour table for θ where Schur triples are coloured yellow.

$*$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$
$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$
$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$

3.1 Schur triples over $\mathbb{Z}_7$ using different colourings using Python language

Algorithm

1. **Group Elements**$\longleftarrow$ $\{0,1,2,3,4,5,6\}$
2. **colour**$\longleftarrow$ user input colours(**Group Elements**)
3. **Table**$\longleftarrow$ ϕ
4. For each $x \in$ **Group Elements** do:
   - For each $y \in$ **Group Elements** do:
   - **Result**$\longleftarrow$ $(x+y)$ mod 7
   - **Table**$[x][y]$ $\longleftarrow$**Result**
5. Print coloured Cayley table(**Table**,**colours**)
6. **Triple count**$\longleftarrow$ 0
7. For each $x \in$ **Group Elements** do:
   - For each $y \in$ **Group Elements** do:
   - **Result**$\longleftarrow$**Table**$[x][y]$
   - If **colour**$[x]$=**colour**$[y]$=**colour**[**Result**]
   - **Triple count**$\longleftarrow$**Triple count** + 1
8. **Print**("Number of triples $(x,y,x+y)$ with the same colour:" + **Triple count**).

3.2 Output using Python Programming Language

1. **Input colouring:**

$$\theta_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathfrak{G} & \mathfrak{B} & \mathfrak{R} & \mathfrak{R} & \mathfrak{B} & \mathfrak{B} & \mathfrak{G} \end{pmatrix}$$
$$\theta_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathfrak{G} & \mathfrak{B} & \mathfrak{B} & \mathfrak{R} & \mathfrak{B} & \mathfrak{G} & \mathfrak{G} \end{pmatrix}$$
$$\theta_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ Y & R & \mathfrak{G} & \mathfrak{G} & \mathfrak{B} & \mathfrak{Y} & \mathfrak{B} \end{pmatrix}$$
$$\theta_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathfrak{R} & \mathfrak{G} & \mathfrak{B} & \mathfrak{Y} & \mathfrak{G} & \mathfrak{Y} & \mathfrak{G} \end{pmatrix}$$

2. **Output:** Cayley table for $\mathbb{Z}_7$ as a matrix:

$*$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

3. **Output:**

   (a) Cayley table for θ_1 and the entries coloured yellow corresponds to Schur triples. Schur triples: $(0,6,6),(1,4,5),(4,1,5),(4,4,1),(6,0,6)$. Then $\tau_{\theta_1} = 5$.

$*$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$
$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$
$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{B}$

(b) Cayley colour table for θ_2. Schur triples: $(0,5,5), (0,6,6), (1,1,2)$, $(2,2,4), (4,4,1), (5,0,5), (6,0,6), (6,6,5)$ and $\tau_{\theta_2} = 8$.

$*$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$
$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{B}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{B}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{G}$

(c) Cayley colour table for θ_3. Schur triples: $(0,5,5)(5,0,5)$ and $\tau_{\theta_3} = 2$.

$*$	$\mathfrak{Y}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{B}$
$\mathfrak{Y}$	$\mathfrak{Y}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{B}$
$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{B}$	$\mathfrak{Y}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{R}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{R}$	$\mathfrak{G}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{G}$
$\mathfrak{Y}$	$\mathfrak{Y}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$

(d) Cayley colour table for θ_4. Schur triples: $(4,4,1)(5,5,3)$ and $\tau_{\theta_4} = 2$.

$*$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{Y}$	$\mathfrak{G}$
$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{Y}$	$\mathfrak{G}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{R}$
$\mathfrak{B}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{R}$	$\mathfrak{G}$
$\mathfrak{Y}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{B}$
$\mathfrak{G}$	$\mathfrak{G}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$
$\mathfrak{Y}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{R}$	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{G}$
$\mathfrak{G}$	$\mathfrak{G}$	R	$\mathfrak{G}$	$\mathfrak{B}$	$\mathfrak{Y}$	$\mathfrak{G}$	$\mathfrak{Y}$

3.3 The relation between Schur colourings and The Cayley graphs of a finite Semigroup.

Theorem 3.24. *Let θ be a t-colouring of the finite Semigroup $\widehat{\mathcal{G}}$. θ is not a Schur colouring if θ is a proper colouring of the Cayley graph $Cay(\widehat{\mathcal{G}}, M_x^\theta)$ for each $x \in \widehat{\mathcal{G}}$. Where M_x^θ is the colour class containing x under θ.*

Proof. Let θ be proper vertex colouring of the Cayley graph $Cay(\widehat{\mathcal{G}}, M_x^\theta)$ for each $x \in \widehat{\mathcal{G}}$. Suppose on the contrary that θ is a Schur colouring. Then there exists $x, y, z \in \widehat{\mathcal{G}}$ and a colour $s \in \mathbb{Z}_t$ such that $\theta(x) = \theta(y) = \theta(z) = s$ & $xy = z$. But then $x, y, z \in M_x^\theta$ and $\theta(M_x^\theta) = s$. Also there is an edge $(y, z) \in E(Cay(\widehat{\mathcal{G}}, M_x^\theta))$. But this shows that θ is not a proper colouring for the graph $Cay(\widehat{\mathcal{G}}, M_x^\theta)$. This contradicts our assumption and proves the theorem. $\square$

That is the if θ is Schur colouring then there exists an $x \in \mathcal{G}$ such that θ is not a proper colouring of $Cay(\mathcal{G}, M_x^\theta)$. We illustrate this in the following example.

Example 3.25. Take $\mathcal{G} = \mathbb{Z}_5$ and $\theta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ Y & B & Y & G & B. \end{pmatrix}$. Then the colour classes are $M_B = \{1, 3\}$, $M_Y = \{0, 2\}$, $M_G = \{3\}$. Since $2 + 0 = 2$ and $\theta(2) = \theta(0) = Y$, θ is a Schur colouring. Then by the Theorem 3.24 there must exist a colour class such that Cayley graph associated to this colour class is not a proper colouring. We find that θ is not a proper colouring for $Cay(\mathcal{G}, M_B)$. **Figure** 1 illustrates the graph $Cay(\mathcal{G}, M_B)$.

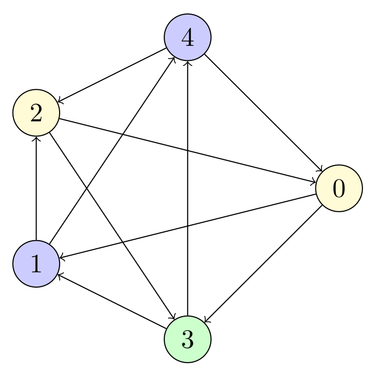

Figure 1. $Cay(\mathcal{G}, M_B)$ coloured by θ.

Theorem 3.26. *Let $\widehat{\mathcal{G}}$ be a finite semigroup such that $\widehat{\mathcal{G}} = ST = \{st \mid s \in S, t \in T\}$ for some $S, T \subset \widehat{\mathcal{G}}$ and $S \subset M_x^\theta$ for some $x \in \widehat{\mathcal{G}}$. If θ is a non-proper colouring of $Cay(\widehat{\mathcal{G}}, S)$ and $\pi : \mathbb{Z}_t \longrightarrow \mathbb{Z}_t$ such that $(\pi \circ \theta)(xy) = \theta(x)$ for $x \in S$, then $\pi \circ \theta$ is a Schur colouring.*

Proof. Let θ be non-proper colouring of the Cayley graph $Cay(\widehat{\mathcal{G}}, S)$ for a subset S holding the initial conditions of the theorem. Then there exists $(x, y) \in E(Cay(\widehat{\mathcal{G}}, S))$ such that $\theta(x) = \theta(y)$. Also there exists an $z \in S$ and $y = zx$. Let $\theta(S) = \theta(M_x^\theta) = r$. Then $\theta(z) = s$.
Now $(\pi \circ \theta)(y) = (\pi \circ \theta)(zx) = \theta(z) = r$. Since $\mathcal{G} = ST$, there exists $s_1, s_2 \in S$ and $t_1, t_2 \in T$ such that $y = s_1 t_1$ and $x = s_2 t_2$. Then

$$(\pi \circ \theta)(y) = (\pi \circ \theta)(s_1 t_1) = \theta(s_1) = r$$

$$(\pi \circ \theta)(x) = (\pi \circ \theta)(s_2 t_2) = \theta(s_2) = r$$

That is

$$(\pi \circ \theta)(y) = (\pi \circ \theta)(x) = (\pi \circ \theta)(z) = r$$

This implies that $(\pi \circ \theta)$ is a Schur colouring. □

Theorem 3.27. *Let $\widehat{\mathcal{G}}$ be finite (right)left-zero semigroup of size at least 3 and $M \subseteq \widehat{\mathcal{G}}$ such that $Cay(\widehat{\mathcal{G}}, M)$ is undirected and $\chi(\widehat{\mathcal{G}}) = 2$. Then for any proper 2-colouring θ, $\tau_\theta(\widehat{\mathcal{G}}) \geq 2$. If both colour classes contain number of elements m_θ and n_θ respectively then,*

$$\tau_\theta(\widehat{\mathcal{G}}) = m_\theta^2 + n_\theta^2 - m_\theta - n_\theta. \tag{3.18}$$

Proof. Let $\{g, h\}$ be an non-edge of $Cay(\widehat{\mathcal{G}}, M)$. This is possible by the fact that $|\widehat{\mathcal{G}}| \geq 3$. Since θ is a proper 2-colouring, $Cay(\widehat{\mathcal{G}}, M)$ is bipartite. Here the two colour classes form a bi-partition. Then $\{g, h\}$ is a subset of one of the colour classes under θ. That is $\theta(g) = \theta(h)$. Now $gh = g$ and $hg = h$ by the definition of left-zero semigroup. That is $\theta(g) = \theta(gh) = \theta(hg) = \theta(h)$. This gives two Schur triples (g, h, g) and (h, g, h). This implies $\tau_\theta(\widehat{\mathcal{G}}) \geq 2$. Now if both colour classes H and K contains elements m_θ and n_θ respectively then there are m_θ^2 products which produce Schur triples of the form (x, y, x) such that $x, y \in H$ and n_θ^2 products which produce Schur triples of the form (x, y, x) such that $x, y \in K$. Since $\widehat{\mathcal{G}}$ is left-zero semigroup, $x^2 = x$ for all $x \in \widehat{\mathcal{G}}$. Then (x, x, x) and (y, y, y) are not Schur triples for $x \in H$ and $y \in K$. Note that products st such that $s \in H$ and $t \in K$ never gives rise to a Schur triple. Since $st = s$ and $\theta(s) \neq \theta(t)$, (s, t, s) cannot be a Schur triple. There are m_θ number of triples of the form $(x, x, x),\ x \in H$ and n_θ number of triples of the form $(y, y, y),\ y \in K$. By avoiding these triples carefully we get the total number of Schur triples,

$$\tau_\theta(\widehat{\mathcal{G}}) = m_\theta^2 + n_\theta^2 - m_\theta - n_\theta.$$

□

4 Application in Physics: Quantum Spin Systems and Frustrated Magnetism

1. **Quantum Spin Systems**
   In quantum mechanics, especially in the study of magnetic systems, the behavior of spins on a lattice plays a crucial role. These spins can be described using algebraic groups and graph-theoretical models.[16, 17]

2. **Frustrated Magnetism**
   Frustrated magnetism occurs in systems where not all spin interactions can be simultaneously minimized, leading to a complex energy landscape and rich physical phenomena. The study of such systems often involves examining the interactions between spins and how they can be represented mathematically.[18]

4.1 Spin Glasses and Frustrated Lattices

1. **Coloring in Spin Glasses:**
   In a spin glass, each site on a lattice (often modeled as $\mathbb{Z}_n \times \mathbb{Z}_n$) contains a spin that can point in different directions (e.g., up or down in the Ising model)[19]. Coloring the elements of this group can represent different spin states or the presence of different types of magnetic interactions (e.g., ferromagnetic or antiferromagnetic).

2. **Schur Triples and Interaction Energies:**
   Consider a lattice where each site is connected to its neighbors, and the interactions are described by a Hamiltonian. A Schur triple (x, y, z) in this context could represent a triplet of spins whose interaction energies are related. For instance, in a system where the Hamiltonian includes terms like $J_{ij}S_iS_j$ (with J_{ij} being the interaction strength and S_i, S_j being the spins), Schur triples could be used to identify specific configurations of spins that minimize or characterize the energy states.

4.2 Triangular Lattice Antiferromagnets

In a triangular lattice antiferromagnet, each spin interacts with its neighbors in such a way that it is impossible to satisfy all interactions simultaneously, leading to frustration [20]. This can be modeled using group theory:

1. **Triangular Lattice and Group Representation**
   - The lattice can be represented using a finite group, where each lattice point corresponds to an element of the group, and the spin interactions are described by relations in the group.

2. **Coloring and Schur Triples**
   - Different spin states (e.g., $\uparrow, \downarrow, \rightarrow$) can be represented by different colors or group elements.
   - Schur triples (x, y, z) can represent triplets of interacting spins whose combined energy (or total spin state) is of interest. For example, in a frustrated triangular lattice, you might look for configurations where the spins satisfy certain energy minimization conditions, akin to solving for $x + y = z$ in the group.

4.3 Importance in Physics

1. **Phase Transitions**
   Understanding the configurations of spins, especially in frustrated systems, can provide insights into phase transitions and critical phenomena. Schur triples can help identify key configurations that drive these transitions.

2. **Disordered Systems**
   In disordered magnetic systems (spin glasses), the complex energy landscape can be better understood by examining the algebraic structures and configurations represented by Schur triples.

3. **Quantum Computing**
   In quantum computing, particularly in the design of quantum algorithms and error-correcting codes, the principles of coloring and Schur triples can be applied to optimize quantum state configurations and minimize error rates.

In a study of a triangular lattice with antiferromagnetic interactions, researchers might use graph coloring to represent different spin states and identify Schur triples that correspond to low-energy configurations. By analyzing these configurations, they can make predictions about the macroscopic properties of the material, such as its susceptibility to external magnetic fields or its heat capacity at various temperatures.

In summary, the concepts of coloring elements in a finite algebraic group and Schur triples have significant applications in physics, particularly in the study of quantum spin systems and frustrated magnetism. These mathematical tools help physicists understand complex interactions and predict the behavior of materials under various conditions.

5 Questions

We have addressed a limited number of problems regarding coloured algebraic groups. Here is a couple of problems for further study.

1. Is there an efficient algorithm to list Schur triples for a finite group $\mathcal{G}$ and a given colouring θ?

2. Given a finite group $\mathcal{G}$ and a positive integer k such that $1 < k < O(\mathcal{G})^2$. Is there a colouring θ such that $\tau_\theta(\mathcal{G}) = k$? If yes, how many of them?

6 Conclusion

In this paper, we presented and investigated the concept of finite colouring of finite groups and related definitions regarding group colouring. Especially the idea of Schur colourings and their properties. We have found weak lower bounds for the number of Schur colourings of a finite group. We have also established results connecting Schur colourings and vertex colourings of Cayley graphs associated to respective finite groups. We have established weaker bounds regarding the number Schur triples. It has applications in physical sciences where group structure arises. In physics, they model symmetries in gauge theories and condensed matter systems, Finite group coloring and the concept of Schur triples provide a powerful framework for studying the symmetry and interaction patterns in complex physical systems such as spin glasses. By linking spin states to group elements and analyzing Schur triples, one can gain insights into the ground state properties, frustration effects, and symmetry-breaking phenomena in these systems. This approach enriches the study of disordered magnetic materials and contributes to the broader understanding of condensed matter physics. In combinatorics and graph theory, they address problems in graph colourings and combinatorial designs. Overall, colored algebraic groups provide a versatile framework for understanding complex symmetries and interactions in systems with multiple elements or states.

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