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On a Variant Weibull-Weibull Distribution: Theory and Properties

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Abstract In general, distribution theory plays a crucial role in modeling various real-life phenomena, making it a fundamental tool in statistical analysis and decision-making. Over the years, extensive research has been conducted on different statistical distributions and estimation techniques. While the literature abounds with information regarding well-known distributions, there is always room for exploring new variants that can better capture the characteristics of complex phenomena. In this paper, we contribute to the field of distribution theory by introducing novel probability distribution called the Weibull-Weibull distribution. The Weibull-Weibull distribution is derived by compounding two Weibull distributions, and it offers a flexible framework for modeling phenomena that exhibits a complex interplay of factors. By combining the strength of the Weibull distribution with itself, we are able to capture a wider range of shapes and behaviour, providing more accurate representations of real-world occurrences. To facilitate the practical application of the Weibull-Weibull distribution, we employ the maximum likelihood estimation (MLE) approach to estimate its shape and scale parameters. The MLE method is a widely used statistical technique that allows us to determine the most likely values of the parameters based on observed data. By applying this estimation method to the Weibull-Weibull distribution, we enable researchers and practitioners to effectively utilize this new distribution in their analyses and modeling efforts. Furthermore, we delved into a comprehensive study of statistical theory and properties of the Weibull-Weibull distribution. We investigate its moments, cumulative distribution function, probability density function, and other key measures. Through rigorous analysis, we establish the theoretical foundations of the Weibull-Weibull distribution and provide insights into its behaviour and characteristics. This comprehensive examination equips researchers with a solid understanding of the distribution, enabling them make informed decisions and interpretations when working with real-life data. In conclusion, our research introduces the Weibull-Weibull distribution as a valuable addition to the existing repertoire of statistical distributions. By leveraging the power of compounding two Weibull distributions, we provide a flexible and robust framework of modeling complex phenomena. With the use of maximum likelihood estimation and the given Chernoff bound, practitioners are able to estimate the distribution's parameters as well as analyze its tails accurately. Our extensive statistical analysis further enhances the understanding of the Weibull-Weibull distribution, facilitating its practical application in a wide range of fields, including reliability analysis, survival analysis, and risk assessment.

Keywords Weibull Distribution, Survival Function, Hazard Function, Maximum Likelihood estimate, Moments, Weibull-Weibull Distribution, Chernoff Bound

1 Introduction

For centuries, the statistical distribution theory has been developed to explain how random variables behave. It is an extremely potent instrument that enables one to predict the results of events like flipping a coin, rolling a die, or performing an action in a video game. It is a very helpful technique to consider circumstances when there is a lot of uncertainty, such as trying to choose which employment offer is better or determining the acceptable amount of risk while investing money. Additionally, understanding how to calculate statistical probabilities can be useful when making decisions in other spheres of life, such as selecting a spouse or casting a ballot. Finally, an important component of statistical analysis that enables one to make more informed judgments and draw more reliable inferences from data is understanding statistical distributions. Outliers, which are extreme values that can skew the results of a statistical analysis, data sets that violate the assumption of normality, small or moderate sample sizes, data with heteroscedasticity, heavily skewed data, and other issues are problems statisticians face when working with real-world data. Robust statistical distributions have been developed and suggested to handle these types of data circumstances in the literature.

A well-known trend in the field of distribution theory is to modify and generalize existing distributions to obtain some more robust statistical distributions[1]. For instance, the authors of [1],[2], [3], [4], [5], [6], [7], [8], and [9] constructed some variant distributions which have a wider range of applicability in modeling statistical data than the existing relevant ones in the literature. On the other hand, various applications of some existing distributions to various real-life occurrences are given in [10], [11], [12], [13], [14], [15] [16], [17], and [18].

As a result of following the aforementioned trend in this study, we are having a proven capacity to generalize the Exponential-Exponential [19] distribution to a variant two-parameter Weibull-Weibull distribution.

2 The Derivation of the Variant Model, Weibull-Weibull Distribution(WWD)

The probability density of the WWD is derived in this section.

Theorem 2.1. *Suppose X has a Weibull distribution with shape α and scale β. Let the pdf and cdf of the Weibull distribution, denoted by $f(x)$ and $F(x)$, be given as;*

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}};\ x>0,\ \alpha>0,\ \beta>0, \tag{2.1}$$

and

$$F(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}};\ x>0,\ \alpha>0,\ \beta>0. \tag{2.2}$$

Corroborating [1] plus [19], let,

$$g(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}(1-F(x))^{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}\left[\frac{f(x)}{1-F(x)} - \left(\frac{x}{\alpha}\right)^{-1}\left(\frac{\beta-1}{\alpha}\right)\ln(1-F(x))\right] \quad \forall\, x>0 \tag{2.3}$$

be the pdf of the WD-X, where f(x) and 1 - F(x) is the baseline pdf and the survival function of the distribution then inserting (2.1) and (2.2) in (2.3), the Weibull-Weibull pdf is obtained as

$$g(x) = \frac{\beta}{\alpha}\left(\frac{2\beta-1}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{2(\beta-1)} e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}};\quad x>0,\ \alpha>0,\ \beta>0. \tag{2.4}$$

3 The Distributional Characteristics of the Variant Weibull-Weibull Statistical Model

The first four statistical moments, variance, relative standard deviation, moment generating function, characteristic function, skewness, and kurtosis of the variant Weibull-Weibull statistical model are obtained in this section as follows

3.1 Statistical Moments

Theorem 3.1. *If X, is a continuous random variable having density given in (2.4), then the rth non-central moment of X, denoted by μ'_r, is obtained by*

$$\mu'_r = \alpha^{\frac{2\beta r}{2\beta-1}}\beta^{\frac{-r}{2\beta-1}}\Gamma\left(\frac{r}{2\beta-1}+1\right)$$

Proof.

$$\begin{aligned}\mu'_r &:= E[x^r] \\ &:= \int_0^\infty x^r f(x;\alpha,\ \beta)dx \\ &= \int_0^\infty x^r \frac{\beta}{\alpha}\left(\frac{2\beta-1}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{2(\beta-1)} e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}}dx \\ &= \frac{\beta}{\alpha}\left(\frac{2\beta-1}{\alpha}\right)\int_0^\infty x^r\left(\frac{x}{\alpha}\right)^{2(\beta-1)} e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}}dx\end{aligned} \tag{3.1}$$

$$\text{Let, } u = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1},\ x = \alpha\left(\frac{\alpha u}{\beta}\right)^{\frac{1}{2\beta-1}},$$

$$\implies dx = \frac{\alpha^2}{\beta(2\beta-1)}\left(\frac{\alpha u}{\beta}\right)^{\frac{2-2\beta}{2\beta-1}}du,$$

so that (3.1) reduces to

$$\begin{aligned}\mu'_r &= \alpha^r\int_0^\infty \left(\frac{\alpha u}{\beta}\right)^{\frac{r}{2\beta-1}}\left(\frac{\alpha u}{\beta}\right)^{\frac{2(\beta-1)}{2\beta-1}}\left(\frac{\alpha u}{\beta}\right)^{\frac{2-2\beta}{2\beta-1}}e^{-u}du \\ &= \alpha^r\int_0^\infty \left(\frac{\alpha u}{\beta}\right)^{\frac{r+2\beta-2+2-2\beta}{2\beta-1}}e^{-u}du \\ &= \alpha^r\left(\frac{\alpha^{\frac{r}{2\beta-1}}}{\beta^{\frac{r}{2\beta-1}}}\right)\int_0^\infty u^{\frac{r}{2\beta-1}}e^{-u}du \\ &= \alpha^{2\beta r}\beta^{\frac{-r}{2\beta-1}}\int_0^\infty u^{\frac{r}{2\beta-1}}e^{-u}du\end{aligned}$$

But $\Gamma(\alpha+1) = \int_0^\infty u^\alpha e^{-u}du$ then,

$$\mu'_r = \alpha^{\frac{2\beta r}{2\beta-1}}\beta^{\frac{-r}{2\beta-1}}\Gamma\left(\frac{r}{2\beta-1}+1\right). \tag{3.2}$$

□

The first (mean), second, third, and fourth moment for the WWD is obtained by substituting $r = 1, 2, 3,$ and 4, in (3.2).

Also, the variance for the WWD is obtained from the association of the first (mean) and second moments as follows

$$V(x) = \mu_2' - (\mu_1')^2 \tag{3.3}$$

The mean of the WWD (denoted by μ), is equals to the first moment and given as

$$\mu = \mu_1' = \alpha^{\frac{2\beta}{2\beta-1}} \beta^{\frac{-1}{2\beta-1}} \Gamma\left(\frac{2\beta}{2\beta-1}\right) \equiv \alpha^{\frac{2\beta}{2\beta-1}} \beta^{\frac{-1}{2\beta-1}} \Gamma_1 \tag{3.4}$$

$$\mu_2' = \alpha^{\frac{4\beta}{2\beta-1}} \beta^{\frac{-2}{2\beta-1}} \Gamma\left(\frac{2\beta+1}{2\beta-1}\right) \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) we have

$$\text{Variance (denoted by } \sigma^2) = V(x) = \alpha^{\frac{4\beta}{2\beta-1}} \beta^{\frac{-2}{2\beta-1}} \left(\Gamma_2 - \Gamma_1^2\right), \tag{3.6}$$

and the standard deviation, denoted by σ, is given as

$$\sigma = \alpha^{\frac{2\beta}{2\beta-1}} \beta^{\frac{-1}{2\beta-1}} \sqrt{\Gamma_2 - \Gamma_1^2}, \tag{3.7}$$

where $\Gamma_1 = \Gamma\left(\frac{2\beta}{2\beta-1}\right)$ and $\Gamma_2 = \Gamma\left(\frac{2\beta+1}{2\beta-1}\right)$ in (3.4), (3.6) and (3.7).

The third and the fourth moment for the WWD is given as

$$\mu_3' = \alpha^{\frac{6\beta}{2\beta-1}} \beta^{\frac{-3}{2\beta-1}} \Gamma\left(\frac{2\beta+2}{2\beta-1}\right) = \alpha^{\frac{6\beta}{2\beta-1}} \beta^{\frac{-3}{2\beta-1}} \Gamma_3, \tag{3.8}$$

$$\mu_4' = \alpha^{\frac{8\beta}{2\beta-1}} \beta^{\frac{-4}{2\beta-1}} \Gamma\left(\frac{2\beta+3}{2\beta-1}\right) = \alpha^{\frac{8\beta}{2\beta-1}} \beta^{\frac{-4}{2\beta-1}} \Gamma_4, \tag{3.9}$$

where $\Gamma_3 = \Gamma\left(\frac{2\beta+2}{2\beta-1}\right)$ and $\Gamma_4 = \Gamma\left(\frac{2\beta+3}{2\beta-1}\right)$.

3.2 The Relative Standard Deviation or Coefficient of Variation (C.V)

Definition 3.2. This is a standard measure of the dispersion of probability distribution and is given as

$$C.V = \frac{\sigma}{\mu} \tag{3.10}$$

Substituting (3.4) and (3.7) into (3.10) we have;

$$C.V = \frac{\sqrt{\Gamma_2 - \Gamma_1^2}}{\Gamma_1}, \tag{3.11}$$

where $\Gamma_1 = \Gamma\left(\frac{2\beta}{2\beta-1}\right)$ and $\Gamma_2 = \Gamma\left(\frac{2\beta+1}{2\beta-1}\right)$.

3.3 Moment Generating Function or the laplace transform of the WWD Function

Theorem 3.3. *If X is a continuous random variable having density given in (2.4), then the moment generating function of X, denoted by $M_x(t)$, is obtained as*

$$M_x(t) = \sum_{r=0}^{\infty} \frac{\alpha^{\frac{2r\beta}{2\beta-1}} \beta^{\frac{-r}{2\beta-1}}}{r!} t^r \Gamma\left(\frac{r}{2\beta-1}+1\right)$$

Proof.

$$M_x(t) := E(e^{tx}) := \int_0^\infty e^{tx} f(x;\alpha,\beta)dx$$
$$= \int_0^\infty e^{tx} \frac{\beta}{\alpha}\left(\frac{2\beta-1}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{2(\beta-1)} e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}} dx \tag{3.12}$$

Let $u = \frac{x}{\alpha}$, $\implies$ $x = u\alpha$ and $dx = \alpha du$, so that (3.12) reduces to

$$M_x(t) = \frac{\beta}{\alpha}\left(\frac{2\beta-1}{\alpha}\right) \int_0^\infty e^{\alpha t u} u^{2(\beta-1)} e^{-\frac{\beta}{\alpha}u^{2\beta-1}} \alpha du \tag{3.13}$$

Furthermore in (3.13), let $p = \frac{\beta}{\alpha}u^{2\beta-1}$,

$\implies u = \left(\frac{\alpha p}{\beta}\right)^{\frac{1}{2\beta-1}}$ and $du = \frac{\alpha}{\beta(2\beta-1)}\left(\frac{\alpha p}{\beta}\right)^{-\frac{2(\beta-1)}{2\beta-1}} dp$, so that Equation (3.13) reduces to

$$M_x(t) = \int_0^\infty e^{\alpha\left(\frac{\alpha p}{\beta}\right)^{\frac{1}{2\beta-1}} t} e^{-p} dp \tag{3.14}$$

Recall that

$$e^p = 1 + p + \frac{p^2}{2!} + \dots + \frac{p^r}{r!} := \sum_{r=0}^{\infty} \frac{p^r}{r!},$$

which implies that

$$e^{\alpha\left(\frac{\alpha p}{\beta}\right)^{\frac{1}{2\beta-1}} t} = e^{\alpha^{\frac{2\beta}{2\beta-1}} \beta^{\frac{-1}{2\beta-1}} p^{\frac{1}{2\beta-1}} t}$$
$$= \sum_{r=0}^{\infty} \frac{(\alpha^{\frac{2\beta}{2\beta-1}} \beta^{\frac{-1}{2\beta-1}} t)^r p^{\frac{r}{2\beta-1}}}{r!}.$$

Substituting this series expression in (3.14), we have

$$M_x(t) = \int_0^\infty \sum_{r=0}^{\infty} \left[\frac{(\alpha^{\frac{2\beta}{2\beta-1}} \beta^{\frac{-1}{2\beta-1}} t)^r p^{\frac{r}{2\beta-1}}}{r!}\right] e^{-p} dp$$
$$= \sum_{r=0}^{\infty} \left[\frac{(\alpha^{\frac{2\beta}{2\beta-1}} \beta^{\frac{-1}{2\beta-1}} t)^r}{r!}\right] \int_0^\infty p^{\frac{r}{2\beta-1}} e^{-p} dp$$
$$= \sum_{r=0}^{\infty} \frac{\alpha^{\frac{2r\beta}{2\beta-1}} \beta^{\frac{-r}{2\beta-1}}}{r!} t^r \Gamma\left(\frac{r}{2\beta-1}+1\right), \tag{3.15}$$

since,

$$\int_0^\infty p^{\frac{r}{2\beta-1}} e^{-p} dp = \Gamma\left(\frac{r}{2\beta-1}+1\right)$$

□

3.4 Characteristic Function (C.F) or the Fourier Transform of the WWD

Theorem 3.4. *Suppose X has a Weibull distribution with shape α and scale β, the characteristic function of WWD, denoted by $\phi_x(it)$, is defined as*

$$\phi_x(it) = \sum_{r=0}^{\infty} \frac{\alpha^{\frac{2r\beta}{2\beta-1}} \beta^{\frac{-r}{2\beta-1}}}{r!} (it)^r \Gamma\left(\frac{r}{2\beta-1}+1\right)$$

Proof.

$$\phi_x(it) := E(e^{itx}) := \int_0^\infty e^{itx} f(x;\alpha,\beta)dx$$
$$= \int_0^\infty e^{itx}\frac{\beta}{\alpha}\left(\frac{2\beta-1}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{2(\beta-1)} e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}}dx \tag{3.16}$$

Let $u=\frac{x}{\alpha}, \implies x=\alpha u$ and $dx=\alpha du,$ so that (3.16) will reduce to

$$\phi_x(it) = \frac{\beta}{\alpha}\left(\frac{2\beta-1}{\alpha}\right)\int_0^\infty e^{i\alpha t u} u^{2(\beta-1)} e^{-\frac{\beta}{\alpha}u^{2\beta-1}}\alpha du \tag{3.17}$$

Furthermore in Equation (3.17), let $p=\frac{\beta}{\alpha}u^{2\beta-1}$,
$\implies u=\left(\frac{\alpha p}{\beta}\right)^{\frac{1}{2\beta-1}}$ and $du=\frac{\alpha}{\beta(2\beta-1)}\left(\frac{\alpha p}{\beta}\right)^{-\frac{2(\beta-1)}{2\beta-1}}dp,$ so that (3.17) reduce to

$$\phi_x(it) = \int_0^\infty e^{i\alpha\left(\frac{\alpha p}{\beta}\right)^{\frac{1}{2\beta-1}}t} e^{-p}dp \tag{3.18}$$

Recalling that

$$e^p = 1+p+\frac{p^2}{2!}+...+\frac{p^r}{r!} = \sum_{r=0}^\infty \frac{p^r}{r!},$$

we similarly have the series expression

$$e^{i\alpha\left(\frac{\alpha p}{\beta}\right)^{\frac{1}{2\beta-1}}t} = e^{i\alpha^{\frac{2\beta}{2\beta-1}}\beta^{\frac{-1}{2\beta-1}}p^{\frac{1}{2\beta-1}}t}$$
$$= \sum_{r=0}^\infty \frac{(i\alpha^{\frac{2\beta}{2\beta-1}}\beta^{\frac{-1}{2\beta-1}}t)^r p^{\frac{r}{2\beta-1}}}{r!}.$$

Substituting this series into Equation (3.18), we have

$$\phi_x(it) = \int_0^\infty \sum_{r=0}^\infty \left[\frac{(i\alpha^{\frac{2\beta}{2\beta-1}}\beta^{\frac{-1}{2\beta-1}}t)^r p^{\frac{r}{2\beta-1}}}{r!}\right] e^{-p}dp$$
$$= \sum_{r=0}^\infty \left[\frac{(i\alpha^{\frac{2\beta}{2\beta-1}}\beta^{\frac{-1}{2\beta-1}}t)^r}{r!}\right]\int_0^\infty p^{\frac{r}{2\beta-1}}e^{-p}dp$$
$$= \sum_{r=0}^\infty \frac{\alpha^{\frac{2r\beta}{2\beta-1}}\beta^{\frac{-r}{2\beta-1}}}{r!}(it)^r\Gamma\left(\frac{r}{2\beta-1}+1\right) \tag{3.19}$$

since,

$$\int_0^\infty p^{\frac{r}{2\beta-1}}e^{-p}dp = \Gamma\left(\frac{r}{2\beta-1}+1\right)$$

□

3.5 Skewness

Definition 3.5. The amount that a distribution deviates from being symmetrical around its mean is measured by its **skewness**. A distribution is said to have a longer tail on the right side (or positive side) of its mean if the skew value is positive, whereas a distribution is said to have a longer tail on the left side (or negative side) of its mean if the skew value is negative. If the skewness is 0, the distribution is symmetric and it defined as

$$S_k = \frac{E(x-\mu)^3}{\sigma^3} = \frac{\mu_3}{\sigma^3} = \frac{\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}{\sigma^3}. \tag{3.20}$$

Substitute (3.4), (3.5), (3.7) and (3.8) into (3.20) we obtain the skewnss for the WWD as follows

$$S_k = \frac{\alpha^{\frac{6\beta}{2\beta-1}}\beta^{\frac{-3}{2\beta-1}}\Gamma_3 - 3\alpha^{\frac{6\beta}{2\beta-1}}\beta^{\frac{-3}{2\beta-1}}\Gamma_1\Gamma_2 + 2\alpha^{\frac{6\beta}{2\beta-1}}\beta^{\frac{-3}{2\beta-1}}\Gamma_1^3}{\alpha^{\frac{6\beta}{2\beta-1}}\beta^{\frac{-3}{2\beta-1}}(\Gamma_2-\Gamma_1^2)^{\frac{3}{2}}}$$
$$= \frac{\Gamma_3 + 2\Gamma_1^3 - 3\Gamma_1\Gamma_2}{(\Gamma_2-\Gamma_1^2)^{\frac{3}{2}}}, \tag{3.21}$$

where $\Gamma_1=\Gamma\left(\frac{2\beta}{2\beta-1}\right)$, $\Gamma_2=\Gamma\left(\frac{2\beta+1}{2\beta-1}\right)$ and $\Gamma_3=\Gamma\left(\frac{2\beta+2}{2\beta-1}\right)$.

3.6 Excess Kurtosis

Definition 3.6. A distribution's **kurtosis**, denoted by the symbol K, indicates how peaked or flat it is relative to the normal distribution. A distribution with positive kurtosis has a more noticeable peak than a distribution without it, whereas a distribution with negative kurtosis has a flatter shape. A distribution with no kurtosis and the normal distribution have the same shape. The difference between a distribution's kurtosis and the kurtosis of a normal distribution, which has a kurtosis of three, is known as **excess kurtosis**. This is frequently used in place of kurtosis itself since it provides a more observable assessment of the deviation of a distribution from normalcy. It is calculated by subtracting 3 from the kurtosis coefficient, or K, of the distribution. According to the excess kurtosis for the WWD (denoted by EK),

$$\text{EK} \overset{def}{=} \text{K}-3 = \frac{E(x-\mu)^4}{\sigma^4} - 3 = \frac{\mu_4}{\sigma^4} - 3$$
$$= \frac{\mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_4')^4}{\sigma^4} - 3. \tag{3.22}$$

Substitute (3.4), (3.5), (3.7) (3.8) and (3.9) into (3.22) we obtain the excess kurtosis for the WWD as follows

$$\text{EK} = \frac{\Gamma_4 - 4\Gamma_1\Gamma_3 + 6\Gamma_1^2\Gamma_2 - 3\Gamma_1^4}{(\Gamma_2-\Gamma_1^2)^2} - 3$$
$$= \frac{\Gamma_4 - 4\Gamma_1\Gamma_3 + 6\Gamma_1^2\Gamma_2 - 3\Gamma_1^4 - 3(\Gamma_2^2 - 2\Gamma_2\Gamma_1^2 + \Gamma_1^4)}{(\Gamma_2-\Gamma_1^2)^2}$$
$$= \frac{\Gamma_4 - 6\Gamma_1^4 - 3\Gamma_2^2 - 4\Gamma_1\Gamma_3 + 12\Gamma_2\Gamma_1^2}{(\Gamma_2-\Gamma_1^2)^2} \tag{3.23}$$

where $\Gamma_1=\Gamma\left(\frac{2\beta}{2\beta-1}\right)$, $\Gamma_2=\Gamma\left(\frac{2\beta+1}{2\beta-1}\right)$, $\Gamma_3=\Gamma\left(\frac{2\beta+2}{2\beta-1}\right)$ and $\Gamma_4=\Gamma\left(\frac{2\beta+3}{2\beta-1}\right)$.

3.7 Cumulative Distribution Function (CDF)

Definition 3.7. The chance that a random variable X will assume a value less than or equal to a given value x is known as the cumulative distribution function of the random variable and is defined as

$$G(x) \overset{def}{=} P(X \leq x) \overset{def}{=} \int_0^\infty g(x)dx$$

Theorem 3.8. *The Cumulative Distribution Function (CDF), indicated by $G(x)$, is defined as*

$$G(x) = 1 - e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}},$$

if X is a continuous random variable from the Weibull-Weibull Distribution.

Proof: Recall that (2.4) is the probability distribution of the variate Weibull-Weibull Distribution, so,

$$G(x) = \int_0^x \frac{\beta}{\alpha}\frac{(2\beta-1)}{\alpha}\left(\frac{x}{\alpha}\right)^{2(\beta-1)} e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}}dx. \quad (3.24)$$

Let $u = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}$, $\implies$ when $x = 0, u = 0$
and $x = x, u = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}$.
Furthermore, $x = \alpha\left(\frac{\alpha u}{\beta}\right)^{\frac{1}{2\beta-1}}$ and
$dx = \left(\frac{\beta}{\alpha}\right)^{-1}\left(\frac{2\beta-1}{\alpha}\right)^{-1}\left(\frac{\alpha u}{\beta}\right)^{-\frac{2(\beta-1)}{2\beta-1}} du$, so that (3.24) will reduce to

$$G(x) = \int_0^{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}} e^{-u}du = 1 - e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}},$$
$$x > 0,\ \alpha > 0,\ \beta > 0. \quad (3.25)$$

3.8 Survival Function

Definition 3.9. The survival function, denoted by the symbol $\overline{F}$, is also referred to as the reliability function. It is a function that estimates the likelihood that a subject of interest−be it a patient, an object, or anything else−will live past a given time limit and is defined as $\overline{F}(x) = 1 - G(x)$, where $G(x)$ is the cumulative distribution function of the random variable X.

Thus, by deducting (3.25) from 1, we obtain the survival function for the WWD as follows:

$$\overline{F}(x) = 1 - \left(1 - e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}}\right) = e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}} \quad (3.26)$$

3.9 Hazard Function

Definition 3.10. The hazard function, which is also known as the force of mortality, instantaneous failure rate, instantaneous death rate, or age-specific failure rate, is the immediate risk that the event of interest occurs within a very constrained time frame and is described as

$$h(x) = \frac{g(x)}{\overline{F}(x)} = \frac{g(x)}{1 - G(x)}. \quad (3.27)$$

As a result, by replacing (2.4) and (3.26) into (3.27), we obtain the hazard function for the WWD as follows

$$h(x) = \frac{\beta}{\alpha}\frac{(2\beta-1)}{\alpha}\left(\frac{x}{\alpha}\right)^{2(\beta-1)} \quad (3.28)$$

3.10 Cumulative Hazard Function

Definition 3.11. The integral of the hazard function is called the cumulative hazard function. It can be understood as the likelihood of failure at time x given survival up until that point, and it can be expressed as

$$H(x) = W(G(x)) = \int_0^x h(x)dx$$
$$= \int_0^x \frac{g(x)}{1 - G(x)}dx = -log(1 - G(x)).$$

Hence, integrating (3.28) from 0 to x with respect to x, we obtain the cumulative hazard function for the WWD as follows

$$H(x) = \int_0^x \frac{\beta}{\alpha}\frac{(2\beta-1)}{\alpha}\left(\frac{x}{\alpha}\right)^{2(\beta-1)} dx$$
$$= \frac{\beta(2\beta-1)}{\alpha^2\beta}\int_0^x x^{2(\beta-1)}dx = \frac{\beta x^{2\beta-1}}{\alpha^{2\beta}} \quad (3.29)$$

4 The Maximum Likelihood Estimator for Weibull-Weibull Distribution (WWD)

Remark 4.1. Statistical inferences obtained from the variant Weibull-Weibull are as follows

With scale parameter α, and shape parameter β, let $X_1, X_2, \ldots,$ and X_n be a random sample from a Weibull-Weibull distribution (WWD). The likelihood function, indicated by the $L(\alpha, \beta; x_1, x_2, \ldots, x_n)$, is therefore given by

$$L(\alpha, \beta; x_1, x_2, \ldots, x_n) := \prod_{i=1}^n f(x_i, \alpha, \beta)$$
$$= \prod_{i=1}^n \frac{\beta}{\alpha}\frac{(2\beta-1)}{\alpha}\left(\frac{x}{\alpha}\right)^{2(\beta-1)} e^{-\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{2\beta-1}}$$
$$= \frac{\beta^n(2\beta-1)^n}{\alpha^{2\beta n}}\left(\prod_{i=1}^n x_i\right)^{2(\beta-1)} e^{-\frac{\beta}{\alpha^{2\beta}}\sum_{i=1}^n x_i^{2\beta-1}}. \quad (4.1)$$

Taking the natural logarithm of (4.1), we obtain

$$\ln L(\alpha, \beta; x_i) = n\ln\beta + n\ln(2\beta-1) - 2\beta n\ln\alpha$$
$$+ 2(\beta-1)\sum_{i=1}^n \ln(x_i) - \frac{\beta}{\alpha^{2\beta}}\sum_{i=1}^n x_i^{2\beta-1}. \quad (4.2)$$

Additionally, we have that

$$\frac{\partial \ln L(\alpha, \beta; x_i, i = 1, \ldots, n)}{\partial\alpha} = -\frac{2\beta n}{\hat{\alpha}} + \frac{2\beta^2}{\hat{\alpha}^{2\beta+1}}\sum_{i=1}^n x_i^{2\beta-1}$$
$$= 0$$

by differentiating (4.2) with regard to α, and setting to 0. Making $\hat{\alpha}$ the subject of formula, we have the estimator of α

as follows

$$\hat{\alpha} = \left[\frac{\beta \sum_{i=1}^{n} x_i^{2\beta-1}}{n}\right]^{\frac{1}{2\beta}}. \tag{4.3}$$

Now, by replacing α and $\alpha^{2\beta}$ in (4.2) with $\left[\frac{\beta \sum_{i=1}^{n} x_i^{2\beta-1}}{n}\right]^{\frac{1}{2\beta}}$ and $\frac{\beta \sum_{i=1}^{n} x_i^{2\beta-1}}{n}$, respectively, we obtain the partial maximized log-likelihood function (also known as the profile log-likelihood) for the WWD as

$$\ln L(\beta; x_i, i = 1, \ldots, n) = n \ln \beta + n \ln(2\beta - 1) - n \ln\left[\frac{\beta \sum_{i=1}^{n} x_i^{2\beta-1}}{n}\right] + 2(\beta - 1)\sum_{i=1}^{n} \ln(x_i) - n \tag{4.4}$$

Differentiating (4.4) with regard to β and setting to zero, we have

$$\frac{1}{2\hat{\beta} - 1} - \frac{\sum_{i=1}^{n} x_i^{2\beta-1} \ln(x_i)}{\sum_{i=1}^{n} x_i^{2\beta-1}} + \frac{\sum_{i=1}^{n} \ln(x_i)}{n} = 0. \tag{4.5}$$

Equation (4.5) has not generated a numerical solution forβ as a result; as a result, the solution is derived using optimization techniques described in the literature. According to [15], the Newton Rapson approach is the most used one for inferential statistics. Now, using our variate Weibull-Weibull, we apply Newton Raphson's approach to calculate the inverse of the Hessian, indicated by H_f, at each successive value until a convergent estimate emerges for β, and then we write the procedure

$$\beta_{i+1} = \beta_i - \frac{f(\beta_i)}{f'(\beta_i)}, \quad (i = 0, 1, \ldots), \tag{4.6}$$

where i is the succession/iteration.

Equation (4.5) is used as the initial point, denoted by β_0, now, finding the first and second derivatives of (4.4) with regard to β we form $f(\beta_i)$ and $f'(\beta_i)$, respectively as

$$f(\beta_i) = \frac{2n}{2\beta - 1} - 2n\frac{\sum_{i=1}^{n} x_i^{2\beta-1} \ln(x_i)}{\sum_{i=1}^{n} x_i^{2\beta-1}} + 2\sum_{i=1}^{n} \ln(x_i) \tag{4.7}$$

and

$$f'(\beta_i) = -\frac{4n}{(2\beta - 1)^2} - 4n\left[\frac{\sum_{i=1}^{n} x_i^{2\beta-1} \sum_{i=1}^{n} x_i^{2\beta-1} \ln(x_i)^2}{\left(\sum_{i=1}^{n} x_i^{2\beta-1}\right)^2}\right] + 4n\left[\frac{\left(\sum_{i=1}^{n} x_i^{2\beta-1} \ln(x_i)\right)^2}{\left(\sum_{i=1}^{n} x_i^{2\beta-1}\right)^2}\right]. \tag{4.8}$$

The Newton Raphson scheme (4.6) can be accelerated for quadratic convergence with Aitken's scheme by replacing $f'(\beta_i)$ in (4.6) by the finite difference in point β_i, with step $h = f(\beta_i)$, i.e.

$$f'(\beta_i) \approx \frac{f(\beta_i + f(\beta_i)) - f(\beta_i)}{f(\beta_i)}.$$

Hence, we obtain the Aitken's Δ^2 Method

$$\beta_{i+1} = \beta_i - \frac{f(\beta_i)^2}{f(\beta_i + f(\beta_i)) - f(\beta_i)}, \quad (i = 0, 1, \ldots), \tag{4.9}$$

see [20] for more details.

5 The Chernoff Bound for Weibull-Weibull Distribution (WWD)

Theorem 5.1. *Suppose X be a continuous random variable having the moment generating function, denoted by $M_x(t)$, as*

$$M_x(t) = \sum_{r=0}^{\infty} \frac{\alpha^{\frac{2r\beta}{2\beta-1}} \beta^{\frac{-r}{2\beta-1}}}{r!} t^r \Gamma\left(\frac{r}{2\beta - 1} + 1\right) = \sum_{r=0}^{\infty} \mathfrak{O}(r, \alpha, \beta) t^r,$$

where, $\mathfrak{O} = \frac{\alpha^{\frac{2r\beta}{2\beta-1}} \beta^{\frac{-r}{2\beta-1}}}{r!} t^r \Gamma\left(\frac{r}{2\beta - 1} + 1\right).$

Let $A_x(t) = M_x(t)e^{-tc}$. Then for positive t, and $c > 0$,the Chernoff bound on the right tail of X is given by

$$P(X \geq c) \leq \inf_{t>0} A_x(t) \text{ and } \inf_{t>0} A_x(t) = 0.$$

Proof. Letting $A_x(t) = M_x(t)e^{-tc}$, implies that

$$A_x(t) = M_{x-c}(t) = \sum_{r=0}^{\infty} \frac{\alpha^{\frac{2r\beta}{2\beta-1}} \beta^{\frac{-r}{2\beta-1}}}{r!} t^r \Gamma\left(\frac{r}{2\beta - 1} + 1\right) e^{-tc} = \sum_{r=0}^{\infty} \mathfrak{O}(r, \alpha, \beta) t^r e^{-tc}. \tag{5.1}$$

Differentiating (5.1) once and twice with respect to t, we obtain

$$A'_x(t) = \sum_{r=0}^{\infty} \mathfrak{O}(r, \alpha, \beta)\left(rt^{r-1} - ct^r\right) e^{-tc} = \sum_{r=0}^{\infty} \mathfrak{O}(r, \alpha, \beta)\left(\frac{r - tc}{t}\right) t^r e^{-tc} = \sum_{r=0}^{\infty} (r - tc)\mathfrak{O}(r, \alpha, \beta) t^{r-1} e^{-tc}, \tag{5.2}$$

and

$$A''_x(t) = \sum_{r=0}^{\infty} \mathfrak{O}(r, \alpha, \beta)\left(\frac{r^2 - 2rtc + t^2c^2 - r}{t}\right) t^{r-1} e^{-tc} = \sum_{r=0}^{\infty} \left[(tc - r)^2 - r\right] \mathfrak{O}(r, \alpha, \beta) t^{r-2} e^{-tc}, \tag{5.3}$$

respectively.
Now, for some critical points t^* of $A_X(t)$, we have $\sum_{r=0}^{\infty}(r - t^*c)\mathfrak{O}(r,\alpha,\beta)(t^*)^{r-1}e^{-t^*c}$, so that

$$\sum_{r=0}^{\infty} r\mathfrak{O}(r,\alpha,\beta)(t^*)^{r-1} = cM_x(t^*)$$

and

$$A_x'' = e^{-t^*c}\left[-\left(c^2 + \frac{c}{t^*}\right)M_x(t^*) + \sum_{r=0}^{\infty} r^2\mathfrak{O}(r,\alpha,\beta)(t^*)^{r-2}\right].$$

For other critical point t^{**} of $A_X(t)$, arising from e^{-tc} tending to 0 at $t = \infty$, we have

$$\inf_{t>0} A_x(t) = A_x(t^{**}) = 0.$$

Hence, we claim that the Chernoff bound is for the Weibull-Weibull distribution is given as $P(X \geq c) = 0$. $\square$

6 Conclusion

The ability to generalize the Exponential-Exponential [19] distribution to a variant Weibull-Weibull distribution is demonstrated in this study. Additionally, the Exponential, Exponential-Exponential, Weibull, and Weibull-Weibull distributions coincide when when $\alpha = 1$ and $\beta = 1$. The widespread application of the Weibull distribution in several practical fields, as well as the fact that the generalization allows for greater flexibility in the analysis of actual data are the driving forces behind the Weibull-Weibull distribution. Utilizing the approach of maximum likelihood estimation, we develop explicit formulations for the first four moments, moment generating function, characteristics function, skewness, kurtosis, cumulative distribution, survival, hazard functions, and subsequently provided a bound on the right tail of X. In reliability, engineering, and other fields of study, we anticipate that the variant distribution will draw more applications.

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