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Hyers-Ulam Stability of the Hexic-Quadratic-Additive Mixed-Type Functional Equation in Non-Archimedean Normed Spaces

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Abstract Functional equations are important and exciting concepts in mathematics. They make it possible to investigate fundamental algebraic operations and create fascinating solutions. The concept of functional equations develops further creative methods and techniques for resolving issues in information theory, finance, geometry, wireless sensor networks, and other domains. These include geometry, algebra, analysis, and so on. In recent decades, several writers in many domains have covered the study of various types of stability. Many authors have studied the stability of various functional equations in great detail, with the traditional case (Archimedean) revealing more fascinating results. Recently, researchers have used NANS to study the equivalent conclusions of stability problems from various functional equations. In this research, we examine the Hyers-Ulam stability of the hexic-quadratic-additive mixed-type functional equation

$$
\begin{aligned}
&g(mx+ny)+g(mx-ny)+g(nx+my)+g(nx-my) \\
&\quad = m^2n^2(m^2+n^2)[g(x+y)+g(x-y)-2g(x)- \\
&g(y)-g(-y)+2[g(mx)+g(nx)+g(my)+g(ny)]- \\
&(m+n)[g(y)-g(-y)]
\end{aligned}
$$

where $m, n \in \mathbb{Z}$, m is fixed such that $m, n \notin \{-1, 0, 1\}$ and $m+n \neq 0$ in NANS and also provided some suitable counterexamples.

Keywords Hyers-Ulam Stability, Additive Mapping, Quadratic Mapping, Hexic Mapping, Non-Archimedean Normed Spaces(NANS)

1 Introduction

The concept of stability in a functional equation emerges when the equation is converted into an inequality, acting as a perturbation that differs from the original equation. Therefore, the stability issue of a functional equation concerns how the solution to the resulting inequality deviates from the solution of the original functional equation. The first query about the stability problem interrogated by Ulam [1], which is stated below

"Let G be a group and H be a metric group with d. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta \quad for \quad all \quad x, y \in G,$$

then there exists a homomorphism $h : G \to H$ with $d(f(x), h(x)) < \epsilon$ for all $x \in G$?"

The answers were developed by many authors, they are Hyers [2] Aoki [3] , Gavruta [4], and Rassias [5]. Moreover, Lee [6] discussed on the solution and stability of the quadratic type of functional equations. Moslehian and Rassias [7] proved the stability of functional equations in NANS in 2007. Arriorla and Beyer [8] studied the stability of the Cauchy functional equation over p-adic fields. Subsequently, Gordji and Savadkouhi [9] provided stability of mixed type of cubic-quartic functional equation in NANS and Gordji, et al.[11] proved AQCQ functional equation in NANS in 2010. Afterward, Ebadian and Zolghari [10] investigated the stability of a mixed additive and cubic functional equation of several variables in NANS.

Later, Kang and Koh [12] proved the stability of hexic Lie*-derivations. Sanam Falini et al.[13] obtained the approximation of the mixed type of additive-quadratic-hexic functional equation in 2019. Thereafter, the solution and stability of a cubic-type functional equation using direct and fixed point methods were proved by Govindan et al. [14].

Wang, Park, and Shin[15] established the additive ρ-functional inequalities in NANS in 2021. In 2022, Abolfathi[16] studied the stability of 2-dimensional Pexider quadratic functional equation in NANS . In 2023, Bodaghi [17] discussed stability results of multi-hexic mappings. Recently, Ramachandran and Sangeetha [18] investigated the stability of the quadratic-quartic (Q2Q4) functional equation over NANS in 2024.

In our work, we generalized the Hyers-Ulam stability of the hexic-quadratic-additive mixed-type functional equation

$$\begin{aligned}&g(mx+ny)+g(mx-ny)+g(nx+my)+g(nx-my)\\&=m^2n^2(m^2+n^2)[g(x+y)+g(x-y)-2g(x)-g(y)-\\&g(-y)]+2[g(mx)+g(nx)+g(my)+g(ny)]-(m+n)\\&[g(y)-g(-y)]\end{aligned} \tag{1}$$

where $m, n \in \mathbb{Z}$, m is fixed such that $m, n \notin \{-1, 0, 1\}$ and $m + n \neq 0$ in NANS.
Let us consider,

$$\begin{aligned}&M(x,y)\\&=g(mx+ny)+g(mx-ny)+g(nx+my)+g(nx-my)\\&=m^2n^2(m^2+n^2)[g(x+y)+g(x-y)-2g(x)-g(y)\\&-g(-y)]+2[g(mx)+g(nx)+g(my)+g(ny)]-(m+n)\\&[g(y)-g(-y)].\end{aligned} \tag{2}$$

In the discussion, let X be a NANS and Y be a complete NANS.

2 Preliminaries

Definition 2.1. [19] Let $|.| : \mathbb{F} \to \mathbb{R}$ which is said to be a non-Archimedean(NA), satisfy the following axioms: (i)$|u| \geq 0$ and equals 0 if and only if $u = 0$
(ii)$|uv| = |u||v|$
(iii)$|u + v| \leq max\{|u|, |v|\}$, where $\mathbb{F}$ is any field.

Definition 2.2. Consider p be a prime and u be a rational number, which can be written as $u = p.\frac{\mathfrak{a}}{\mathfrak{b}}$, where $\mathfrak{a}, \mathfrak{b}, \beta$, are integers in such a way that $\mathfrak{a}$ and $\mathfrak{b}$ are not divisible by p. Then, p-adic valuation can be defined as

$$|u|_p = \frac{1}{p^\beta} \quad if \quad u \neq 0$$
$$|u|_p = 0 \quad if \quad u = 0.$$

Example 2.3. *Let,* $u = \frac{1450}{7}$

$$u = 5^2\frac{58}{7}$$

Therefore, $|u|_p = \frac{1}{5^2}$.

Definition 2.4. [20] Let $\|.\| : X \to \mathbb{F}$ be a function which is said to be NA norm, if the following conditions hold:
(i) $\|u\| \geq 0$ and $\|u\| = 0$ iff $u = 0$ forall $u \in X$.
(ii) $\|\alpha u\| = |\alpha|\|u\|$ forall $u \in X$ and $\alpha \in \mathbb{F}$
(iii) $\|u + v\| \leq max\{\|u\|, \|v\|\}$ forall $u, v \in X$.
where X is a vector space, $\mathbb{F}$ is a field and the space $(X, \|.\|)$ is called the NANS.

Definition 2.5. [19] A Cauchy sequence is a sequence $\{u_t\} \in X$ which satisfies $|u_{t+1} - u_t| \to 0$ as $t \to \infty$ with respect to NA valuation.

Definition 2.6. [20] X is said to be complete, if every Cauchy sequence is convergent in X.

3 Stability of (1): For an odd case

Theorem 3.1. *Let us consider the function ζ: $X \times X \to [0, \infty)$ so that*

$$\lim_{r\to\infty} \frac{1}{|m^r|}\zeta(m^r x, m^r y) = 0 \quad and \tag{3}$$

$$\lim_{r\to\infty} \frac{1}{|4m^r|}\zeta(0, m^{r-1}x) = 0 \tag{4}$$

let for each $x \in X$

$$\lim_{r\to\infty} max\{\frac{1}{|m^j|}\zeta(0, m^j x) : 0 \leq j < r\} \tag{5}$$

say $\zeta_a(x)$ exists. If $g_o : X \to Y$ is an odd function satisfying,

$$\|M_{g_o}(x, y)\| \leq \zeta(x, y) \tag{6}$$

then there is an additive function $A_{dd} : X \to Y$ so that

$$\|g_o(x) - A_{dd}(x)\| \leq \frac{1}{|4m|}\zeta_a(x). \tag{7}$$

Moreover, if

$$\lim_{s\to\infty}\lim_{r\to\infty} max\{\frac{1}{|m^j|}\zeta(0, m^j x) : s \leq j < r + s\} = 0$$

then A_{dd} is unique.

Proof. Replacing (x, y, n) by $(0, x, m)$ in (6)

$$\|g_o(mx) - mg_o(x)\| \leq \frac{1}{|4|}\zeta(0, x). \tag{8}$$

Replacing x by $m^{r-1}x$ and dividing by m^r

$$\|\frac{g_o(m^r x)}{m^r} - \frac{g_o(m^{r-1}x)}{m^{r-1}}\| \leq \frac{1}{|4m^r|}\zeta(0, m^{r-1}x) \tag{9}$$

which implies $\{\frac{g_o(m^r x)}{m^r}\}$ is Cauchy. Since Y is complete.

$$A_{dd}(x) = \lim_{r\to\infty} \frac{1}{|m^r|} g_o(m^r x). \tag{10}$$

By using induction

$$\|\frac{g_o(m^r x)}{m^r} - g_o(x)\| \le \frac{1}{|4m|} max\{\frac{1}{|m^j|}\zeta(0, m^j x) : 0 \le j < r\}. \tag{11}$$

By taking limit $r \to \infty$ in (11), we obtain (7).
Now we show that A_{dd} is additive,

$$\|A_{dd}(mx) - mA_{dd}(x)\| \le \lim_{r\to\infty} \frac{1}{|4m^r|}\zeta_a(0, m^r x).$$

Hence A_{dd} is additive.
By (3), (4) and (10), we get

$$\begin{aligned}\|M_{A_{dd}}(x, y)\| &= \lim_{r\to\infty} \frac{1}{|m^r|}\|M_{g_o}(m^r x, m^r y)\| \\ &= \lim_{r\to\infty} \frac{1}{|m^r|}\zeta(m^r x, m^r y).\end{aligned}$$

Therefore, the function A_{dd} satisfies (1).
Let A'_{dd} be another additive function satisfying (7)

$$\begin{aligned}&\|A_{dd}(x) - A'_{dd}(x)\| \\ &\le \frac{1}{|4m|} \lim_{s\to\infty} \lim_{r\to\infty} max\{\frac{1}{|m^j|}\zeta(0, m^j x) : s \le j < r + s\}.\end{aligned}$$

Therefore, A_{dd}=A'_{dd}. □

Corollary 3.2. *$l, k, \theta \in \mathbb{R}_{>0}$ and let $l + k < 1$. If $g_o : X \to Y$ satisfying*

$$\|M_{g_o}(x, y)\| \le \theta(\|x\|^{l+k} + \|y\|^{l+k} + \|x\|^l\|y\|^k)$$

then there is $A_{dd} : X \to Y$ an additive mapping which is unique, so that

$$\|g_o(x) - A_{dd}(x)\| \le \frac{1}{|4m|}\zeta_a(x)$$

where,

$$\zeta_a(x) = \lim_{r\to\infty} max\{\frac{1}{|m|^j}\zeta_a(0, m^j x) : 0 \le j < n\}$$

and

$$\zeta_a(0, x) = \theta\|x\|^{l+k}.$$

Counter-Example 3.3. *Let $g_o : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $g_o(x) = 1$, $p > 2$, where p is prime, let $|2|_p^r$=1, $\theta > 1$, $m = 2$, and $n = 2$, $r \in \mathbb{Z}$, we have,*

$$\|M_{g_o}(x, y)\| = |4|_p|31|_p \le \theta(\|x\|^{l+k} + \|y\|^{l+k} + \|x\|^l\|y\|^k).$$

and

$$\|\frac{g_o(m^r x)}{m^r} - \frac{g_o(m^{r-1}x)}{m^{r-1}}\| = \frac{1}{|2|_p^r} \ne 0.$$

Hence $\{\frac{g_o(m^r x)}{m^r}\}$ is not Cauchy.

4 Stability of (1): For an even case

Theorem 4.1. *Let us consider the function $\zeta : X \times X \to [0, \infty)$ so that*

$$\lim_{r\to\infty} \frac{1}{|2|^{2r}}\zeta(2^r x, 2^r y) = 0 \quad and \tag{12}$$

$$\lim_{r\to\infty} \frac{1}{|2^{2r+1}m^6|}\hat{\zeta}(2^{r-1}x) = 0 \tag{13}$$

let for each $x \in X$

$$\lim_{r\to\infty} max\{\frac{1}{|2^{2j}|}\hat{\zeta}(2^j x) : 0 \le j < r\} \tag{14}$$

say $\zeta_q(x)$ exists. If $g_e : X \to Y$ is an even function satisfying,

$$\|M_{g_e}(x, y)\| \le \zeta(x, y) \tag{15}$$

then there is a quadratic function $Q_{ua} : X \to Y$ so that

$$\|g_e(2x) - 64g_e(x) - Q_{ua}(x)\| \le \frac{1}{|8m^6|}\zeta_q(x). \tag{16}$$

Moreover, if

$$\lim_{s\to\infty} \lim_{r\to\infty} max\{\frac{1}{|2^{2j}m^6|}\hat{\zeta}(2^j x) : s \le j < s + r\} = 0$$

then Q_{ua} is unique.

Proof. Substitute (x, y, n) by $(0, 0, m)$ in (15)

$$\|130m^6 g_e(0)\| \le \frac{|65m^6|}{|2(m^6 - 1)|}\zeta(0, 0). \tag{17}$$

Replacing (x, y, n) by $(x, x, 2m)$ in (15)

$$\begin{aligned}&\|4g_e(3mx) + 4g_e(mx) - 40m^6[g_e(2x) - 4g_e(x) + g_e(0)] \\ &\quad -8[g_e(mx) + g_e(2mx)]\| \le 2\zeta(x, x).\end{aligned} \tag{18}$$

Replacing $x = y$, $n = 3m$ in (15)

$$\begin{aligned}&\|2g_e(4mx) + 2g_e(2mx) - 90m^6[g_e(2x) - 4g_e(x) + g_e(0)] \\ &\quad -4[g_e(mx) + g_e(3mx)]\| \le \zeta(x, x).\end{aligned} \tag{19}$$

Using (18) and (19)

$$\begin{aligned}&\|2g_e(4mx) + 2g_e(2mx) - 130m^6[g_e(2x) - 4g_e(x)] \\ &-8[g_e(2mx) + g_e(mx)]\| \le max\{2\zeta(x, x), \frac{|65m^6|}{|2(m^6 - 1)|} \\ &\qquad \zeta(0, 0)\}.\end{aligned} \tag{20}$$

Interchanging (x, y, n) into $(x, 2x, m)$ in (15)

$$\begin{aligned}&\|4g_e(3mx) + 4g_e(mx) - 4m^6[g_e(3x) - g_e(x) - 2g_e(2x)] \\ &\quad -8[g_e(mx) + g_e(2mx)]\| \le 2\zeta(x, 2x).\end{aligned} \tag{21}$$

Putting $y = 3x$ and $n = m$ in (15)

$$\|2g_e(4mx) + 2g_e(2mx) - 2m^6[g_e(4x) + g_e(2x) - 2g_e(x) - 2g_e(3x)] - 4[g_e(3mx) + g_e(mx)]\| \leq \zeta(x, 3x). \tag{22}$$

Plugging (21) and (22)

$$\|2g_e(4mx) + 2g_e(2mx) - 2m^6[g_e(4x) - 3g_e(2x) - 4g_e(x)] - 8[g_e(2mx) + g_e(mx)]\| \leq max\{2\zeta(x, 2x), \zeta(x, 3x)\}. \tag{23}$$

Using (20) and (23)

$$\|g_e(4x) - 68g_e(2x) + 256g_e(x)\| \leq \frac{1}{|2m^6|} max\{max\{2\zeta(x, x), \frac{|65m^6|}{|2(m^6-1)|}\zeta(0, 0)\}, 2\zeta(x, 2x), \zeta(x, 3x)\}. \tag{24}$$

Consider the mapping $f : X \to Y$ with $f(x) = g_e(2x) - 64g_e(x)$

$$\|f(2x) - 4f(x)\| \leq \frac{1}{|2m^6|}\hat{\zeta}(x). \tag{25}$$

where $\hat{\zeta}(x) = max\{max\{2\zeta(x, x), \frac{|65m^6|}{|2(m^6-1)|}\zeta(0, 0)\}, 2\zeta(x, 2x), \zeta(x, 3x)\}$.
Replacing x by $2^{r-1}x$ and dividing by 2^{2r}

$$\|\frac{f(2^r x)}{2^{2r}} - \frac{f(2^{r-1}x)}{2^{2(r-1)}}\| \leq \frac{1}{|2^{2r+1}m^6|}\hat{\zeta}(2^{r-1}x) \tag{26}$$

which implies $\{\frac{1}{|2|^{2r}} f(2^r x)\}$ is Cauchy.
Define,

$$Q_{ua}(x) = \lim_{r\to\infty} \frac{1}{|2|^{2r}} f(2^r x). \tag{27}$$

By using induction

$$\|f(x) - \frac{1}{|2|^{2r}} f(2^r x)\| \leq \frac{1}{|2|^3|m^6|} max\{\frac{1}{|2^{2j}|}\hat{\zeta}(2^j x) : 0 \leq j < r\}. \tag{28}$$

By taking limit $r \to \infty$ in (28), we obtain (16).
Now we show that Q_{ua} is quadratic,

$$\|Q_{ua}(2x) - 4Q_{ua}(x)\| \leq \lim_{r\to\infty} \frac{1}{|2^{2r+1}m^6|}\hat{\zeta}(2^r x). \tag{29}$$

Hence Q_{ua} is quadratic.
By (12) and (27), we get,

$$\|M_{Q_{ua}}(x, y)\| = \lim_{r\to\infty} \frac{1}{|2|^{2r}} \|H_f(2^r x, 2^r y)\| \leq \lim_{r\to\infty} \frac{1}{|2|^{2r}} max\{\zeta(2^{r+1}x, 2^{r+1}y), |64|\zeta(2^r x, 2^r y)\}.$$

Therefore, the function Q_{ua} satisfies (1).
Q'_{ua} is another quadratic function satisfying (16),

$$\|Q_{ua}(x) - Q'_{ua}(x)\| \leq \frac{1}{|2|^3} \lim_{s\to\infty} \lim_{r\to\infty} max\{\frac{1}{|2^{2j}m^6|}\hat{\zeta}(2^j x) : s \leq j < s + r\}.$$

Therefore, Q_{ua} $=Q'_{ua}$. □

Corollary 4.2. *$l, k, \theta \in \mathbb{R}_{>0}$ and $l + k < 2$. If $g_e : X \to Y$ is a function satisfying*

$$\|M_{g_e}(x, y)\| \leq \theta(\|x\|^{l+k} + \|y\|^{l+k} + \|x\|^l\|y\|^k)$$

then there is $Q_{ua} : X \to Y$a quadratic mapping which is unique, so that

$$\|g_e(x) - 64g_e(x) - Q_{ua}(x)\| \leq \frac{1}{|8m^6|}\zeta_q(x)$$

where,

$$\zeta_q(x) = \lim_{r\to\infty} max\{\frac{1}{|2|^{2j}}\hat{\zeta}(2^j x) : 0 \leq j < r\}$$

and

$$\hat{\zeta}(x) = max\{3|2|\theta\|x\|^{l+k}, |2|\theta\|x\|^{l+k}[1 + |2|^{l+k} + |2|^k], \theta\|x\|^{l+k}[1 + |3|^{l+k} + |3|^k]\}.$$

Counter-Example 4.3. *Let $g_e : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $g_e(x) = 1$, $p > 2$, where p is prime, let $|2|_p^r$=1, $\theta > 1$, $m = 2$, and $n = 2$, $r \in \mathbb{Z}$, we have,*

$$\|M_{g_e}(x, y)\| = |4|_p|31|_p \leq \theta(\|x\|^{l+k} + \|y\|^{l+k} + \|x\|^l\|y\|^k).$$

and

$$\|\frac{f(2^r x)}{2^{2r}} - \frac{f(2^{r-1}x)}{2^{2(r-1)}}\| = \frac{1}{|2|_p^{2r}}|189|_p \neq 0.$$

Hence $\{\frac{f(2^r x)}{2^{2r}}\}$ is not Cauchy, where $f(x) = g_e(2x) - 64g_e(x)$.

Theorem 4.4. *Let us consider the function $\zeta : X \times X \to [0, \infty)$ so that*

$$\frac{1}{|2|^{6r}}\zeta(2^r x, 2^r y) = 0 \quad and \tag{30}$$

$$\lim_{r\to\infty} \frac{1}{|2^{6r+1}m^6|}\hat{\zeta}(2^{r-1}x) = 0 \tag{31}$$

let for each $x \in X$

$$\lim_{r\to\infty} max\{\frac{1}{|2^{6j}|}\hat{\zeta}(2^j x) : 0 \leq j < r\} \tag{32}$$

say $\zeta_h(x)$ exists. If $g_e : X \to Y$ is an even function satisfying,

$$\|M_{g_e}(x, y)\| \leq \zeta(x, y). \tag{33}$$

then there is a hexic function H_{ex} so that

$$\|g_e(2x) - 4g_e(x) - H_{ex}(x)\| \leq \frac{1}{|128m^6|}\zeta_h(x). \tag{34}$$

Moreover if

$$\lim_{s\to\infty}\lim_{r\to\infty} max\{\frac{1}{|2^{6j}m^6|}\hat{\zeta}(2^j x) : s \leq j < s+r\} = 0$$

then H_{ex} is unique.

Proof. By using (24) from theorem 4.1
Consider the mapping $k : X \to Y$ with $k(x) = g_e(2x) - 4g_e(x)$

$$\|k(2x) - 64k(x)\| \leq \frac{1}{|2m^6|}\hat{\zeta}(x). \tag{35}$$

where $\hat{\zeta}(x) = max\{max\{2\zeta(x,x), \frac{|65m^6|}{|2(m^6-1)|}\zeta(0,0)\}, 2\zeta(x,2x), \zeta(x,3x)\}$.
Replacing x by $2^{r-1}x$ and dividing by 2^{6r}

$$\|\frac{k(2^r x)}{|2|^{6r}} - \frac{k(2^{r-1}x)}{|2|^{6(r-1)}}\| \leq \frac{1}{|2^{6r+1}m^6|}\hat{\zeta}(2^{r-1}x) \tag{36}$$

which implies$\{\frac{1}{|2|^{6r}}k(2^r x)\}$ is Cauchy.
Define,

$$H_{ex}(x) = \lim_{r\to\infty}\frac{1}{|2|^{6r}}k(2^r x). \tag{37}$$

By using induction

$$\begin{aligned}&\|k(x)-\frac{1}{|2|^{6r}}k(2^r x)\| \\ &\leq \frac{1}{|2^7|}max\{\frac{1}{|2^{6j}m^6|}\hat{\zeta}(2^j x) : 0 \leq j < r\}.\end{aligned} \tag{38}$$

By taking limit $r \to \infty$ in (38), we obtain (34).
Now we show that H_{ex} is hexic.

$$\|H_{ex}(2x) - 64H_{ex}(x)\| \leq \lim_{r\to\infty}\frac{1}{|2^{6r+1}m^6|}\hat{\zeta}(2^r x).$$

Hence, H_{ex} is hexic.
By (30) and (37), we get

$$\begin{aligned}\|M_{H_{ex}}(x,y)\| &= \lim_{r\to\infty}\frac{1}{|2|^{6r}}\|M_k(2^r x, 2^r y)\| \\ &\leq \lim_{r\to\infty}\frac{1}{|2|^{6r}}max\{\zeta(2^{r+1}x, 2^{r+1}y), |4|\zeta(2^r x, 2^r y)\}.\end{aligned}$$

Therefore, the function H_{ex} satisfies (1).
H'_{ex} is another hexic function satisfying (34)

$$\begin{aligned}&\|H_{ex}(x) - H'_{ex}(x)\| \\ &\leq \frac{1}{|2|^7}\lim_{s\to\infty}\lim_{r\to\infty} max\{\frac{1}{|2^{6j}m^6|}\hat{\zeta}(2^j x) : s \leq j < s+r\}.\end{aligned}$$

Therefore, H_{ex}=H'_{ex}. □

Corollary 4.5. *Let $l, k, \theta \in \mathbb{R}_{>0}$ and let $l+k < 6$. If $g_e : X \to Y$ satisfying*

$$\|M_{g_e}(x,y)\| \leq \theta(\|x\|^{l+k} + \|y\|^{l+k} + \|x\|^l\|y\|^k)$$

then is $H_{ex} : X \to Y$ a hexic mapping which is unique, so that

$$\|g_e(x) - 4g_e(x) - H_{ex}(x)\| \leq \frac{1}{|128m^6|}\zeta_h(x).$$

where,

$$\zeta_h(x) = \lim_{r\to\infty} max\{\frac{1}{|2|^{6j}}\hat{\zeta}(2^j x) : 0 \leq j < r\}$$

and

$$\begin{aligned}\hat{\zeta}_h(x) = max\{&3|2|\theta\|x\|^{l+k}, |2|\theta\|x\|^{l+k}[1 + |2|^{l+k} + |2|^k], \\ &\theta\|x\|^{l+k}[1 + |3|^{l+k} + |3|^k]\}.\end{aligned}$$

Counter-Example 4.6. *Let $g_e : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $g_e(x) = 1$, let $|2|_p^r = 1$, $\theta > 1$, $p > 2$, where p is prime, $m = 2$, and $n = 2$, $r \in \mathbb{Z}$, we have,*

$$\|M_{g_e}(x,y)\| = |4|_p|31|_p \leq \theta(\|x\|^{l+k} + \|y\|^{l+k} + \|x\|^l\|y\|^k).$$

and

$$\|\frac{k(2^r x)}{2^{6r}} - \frac{k(2^{r-1}x)}{2^{6(r-1)}}\| = \frac{1}{|2|_p^{6r}}|189|_p \neq 0.$$

Hence $\{\frac{k(2^r x)}{2^{6r}}\}$ is not Cauchy, where $k(x) = g_e(2x) - 4g_e(x)$.

Theorem 4.7. *Let us consider the function $\zeta : X \times X \to [0,\infty)$ so that*

$$\frac{1}{|2|^{6r}}\zeta(2^r x, 2^r y) = 0 \tag{39}$$

and let for each $x \in X$

$$\lim_{r\to\infty} max\{\frac{1}{|2^{2j}m^6|}\hat{\zeta}(2^j x) : 0 \leq j < r\} \tag{40}$$

say $\zeta_q(x)$ and

$$\lim_{r\to\infty} max\{\frac{1}{|2^{6j}m^6|}\hat{\zeta}(2^j x) : 0 \leq j < r\} \tag{41}$$

say $\zeta_h(x)$ exist.
If $g_e : X \to Y$ is an even function satisfying,

$$\|M_{g_e}(x,y)\| \leq \zeta(x,y) \tag{42}$$

then there is a quadratic function $Q_{ua} : X \to Y$ and hexic function $H_{ex} : X \to Y$ so that

$$\|g_e(x) - Q_{ua}(x) - H_{ex}(x)\|\frac{1}{|480m^6|}max\{\zeta_q(x), \frac{1}{|16|}\zeta_h(x)\}. \tag{43}$$

Moreover, if

$$\lim_{s\to\infty}\lim_{r\to\infty} max\{\frac{1}{|2^{2j}m^6|}\hat{\zeta}(2^j x) : s \leq j < s+r\} = 0$$

$$\text{and} \lim_{s\to\infty}\lim_{r\to\infty} max\{\frac{1}{|2^{6j}m^6|}\hat{\zeta}(2^j x) : s \leq j < s+r\} = 0$$

Q_{ua} and H_{ex} which are unique.

Proof. By theorem 4.1 and theorem 4.4, there is a quadratic function $Q''_{ua} : X \to Y$ and a hexic function $H''_{ex} : X \to$ Y so that respectively

$$\|g_e(2x) - 64g_e(x) - Q''_{ua}(x)\| \leq \frac{1}{|8m^6|}\zeta_q(x). \quad (44)$$

$$\|g_e(2x) - 4g_e(x) - H''_{ex}(x)\| \leq \frac{1}{|128m^6|}\zeta_h(x). \quad (45)$$

Subtract (45) from (44),

$$\|g_e(x) - Q_{ua}(x) - H_{ex}(x)\| \leq \frac{1}{|480m^6|}max\{\zeta_q(x), \frac{1}{|16|}\zeta_h(x)\}.$$

where $Q_{ua}(x) = \frac{-1}{60}\ Q''_{ua}(x)$ and $H_{ex} = \frac{1}{60}S''_{et}(x)$.
Let Q'''_{ua} be another quadratic function and H'''_{ex} another hexic function,
To prove uniqueness,
$\overline{Q_{ua}}$=$Q_{ua} - Q'''_{ua}$ and $\overline{H_{ex}}$=$H_{ex} - H'''_{ex}$

$$\|\overline{Q_{ua}}(x) + \overline{H_{ex}}(x)\| \leq \lim_{s\to\infty}\lim_{r\to\infty} max\{\frac{1}{|2^{2t+1}m^6|}\hat{\zeta}(2^{t-1}x) : s \leq t < s+r\} = 0.$$

Since,

$$\lim_{s\to\infty}\lim_{r\to\infty} max\{\frac{1}{|2^{2j}m^6|}\hat{\zeta}(2^jx) : s \leq j < s+r\} = 0 = \lim_{s\to\infty}\lim_{r\to\infty} max\{\frac{1}{|2^{6j}m^6|}\hat{\zeta}(2^jx) : s \leq j < s+r\}.$$

we have,

$$\lim_{r\to\infty}\frac{1}{|2^{6r}|}\|\overline{Q_{ua}}(2^rx) + \overline{H_{ex}}(2^rx)\| = 0.$$

Hence, $\overline{Q_{ua}}(x) = 0$ and $\overline{H_{ex}}(x) = 0$. □

5 Stability of mixed type of Hexic-Quadratic-Additive functional equation

Theorem 5.1. *Let us consider the function $\zeta : X \times X \to [0,\infty)$ so that*

$$\lim_{r\to\infty} max\{\frac{1}{|m^j|}\zeta(0, m^jx) : 0 \leq j < r\} \quad (46)$$

say $\zeta_a(x)$,

$$\lim_{r\to\infty} max\{\frac{1}{|2^{2j}m^6|}\hat{\zeta}(2^jx) : 0 \leq j < r\} \quad (47)$$

say $\zeta_q(x)$ and

$$\lim_{r\to\infty} max\{\frac{1}{|2^{6j}m^6|}\hat{\zeta}(2^jx) : 0 \leq j < r\} \quad (48)$$

say $\zeta_h(x)$ exist. If $g : X \to Y$ is a function satisfying,

$$\|M_g(x,y)\| \leq \zeta(x,y) \quad (49)$$

then there is $A_{dd} : X \to Y$ additive function which is a unique, $Q_{ua} : X \to Y$ quadratic function which is a unique and $H_{ex} : X \to Y$ hexic function which is a unique so that

$$\|g(x) - A_{dd}(x) - Q_{ua}(x) - H_{ex}(x)\| \leq \frac{1}{|4m|}max\{\zeta_a(x), \frac{1}{|120m^5|}\zeta_q(x), \frac{1}{|1920m^5|}\zeta_h(x)\}. \quad (50)$$

Proof. From the theorem 3.1,

$$\|g_o(x) - A_{dd}(x)\| \leq \frac{1}{|4m|}\zeta_a(x). \quad (51)$$

From the theorem 4.7, we get,

$$\|g_e(x) - Q_{ua}(x) - H_{ex}(x)\| \leq \frac{1}{|480m^6|}max\{\zeta_q(x), \frac{1}{|16|}\zeta_h(x)\}. \quad (52)$$

From (51) and (52), we have

$$\|g(x) - A_{dd}(x) - Q_{ua}(x) - H_{ex}(x)\| \leq \frac{1}{|4m|}max\{\zeta_a(x), \frac{1}{|120m^5|}\zeta_q(x), \frac{1}{|1920m^5|}\zeta_h(x)\}.$$

Following by the above theorem, similarly we can prove the uniqueness part. □

6 Conclusions

In this article, we discussed the Hyers-Ulam stability of mixed type of hexic-quadratic-additive functional equation in NANS with some suitable counter-examples.

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