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Viscosity Approximation Methods for Generalized Modification of the System of Equilibrium Problem and Fixed Point Problems of an Infinite Family of Nonexpansive Mappings

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Abstract Fixed points (FP) of infinite families of nonexpansive mappings find diverse applications across various disciplines. In economics, they help to find stable prices and quantities in markets. In game theory, fixed points help to find Nash equilibria. In computer science, fixed points are used to understand program meanings and help in making better algorithms for tasks like data analysis, checking models, and improving compilers. Solutions to equilibrium problems have practical uses in various areas. For instance, in physics, these solutions assist in analyzing systems at rest or in motion. In engineering, they aid in designing structures that can withstand forces without collapsing, ensuring safety and stability in construction projects. The main aim of the article is to present the concept of generalized modification of the system of equilibrium problems (GMSEP) for an infinite family of nonexpansive mappings. In this paper, we study viscosity approximation methods and present a new algorithm to find a common element of the fixed point of an infinite family of nonexpansive mappings and the set of solutions of generalized modification of the system of equilibrium problem in the setting of Hilbert spaces. Under some conditions, we prove that the sequence generated by the algorithm converges strongly to this common solution.

1 Introduction

In 1966, Hartman and Stampacchia [1] introduced variational inequality (VI) theory as a method for studying partial differential equations with applications, primarily in mechanics. The variational inequality problem (VIP) has a wide range of applications in nonlinear analysis and optimization theory. In 1980, Dafermos [2] recognized that the traffic network equilibrium conditions stated by Smith [3] had the structure of a VI, which was a breakthrough in finite-dimensional case.

The VIP is defined as: for a nonempty closed convex subset $\mathcal{E}$ of Hilbert space Σ and a mapping $\chi : \mathcal{E} \to \mathcal{E}$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ the VIP is to find a point $\eta \in D$ such that

$$\langle \chi(\eta), \zeta - \eta \rangle \geq 0 \ \ \forall \zeta \in D. \tag{1.1}$$

The set of solutions of VIP (1.1) will be denoted by $VI(\chi, \Sigma)$. If the mapping χ is L-Lipschitz continuous and k-strongly monotone with $L, k > 0$, then the above VIP has a unique solution [4, 5]. There are many iterative methods for solving VIP (1.1) [6–10, 17, 18].

Suppose $\mathcal{E} \neq \emptyset$ be a subset of a Hilbert space Σ, $\Phi : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ a bifunction. Then a wider class of optimization problem, including convex minimization, variational inequality, fixed point and Nash equilibrium problems can be defined as the equilibrium problem (EP) associated with bifunction Φ and $\mathcal{E}$ [11, 12]

$$\text{find } \zeta \in \mathcal{E} \text{ such that } \Phi(\zeta, \eta) \geq 0 \ \ \forall \eta \in \mathcal{E}.$$

The point $\zeta \in \mathcal{E}$ which solves this problem is called an equilibrium point. We denote set of equilibrium points of above problem by $EP(\Phi)$. There are a number of algorithms available in the literature for analyzing the existence and approximation of a solution to EPs in linear spaces.

In 1999, Verma [13] introduced the new system of variational inequalities problem (NSVIP) which is to find $(\zeta^*, \eta^*) \in \mathcal{E} \times \mathcal{E}$ such that

$$\begin{aligned} &\langle \lambda G(\eta^*) + \zeta^* - \eta^*, \zeta - \zeta^* \rangle \geq 0, \text{ for all } \zeta \in \mathcal{E}, \\ &\langle \mu G(\zeta^*) + \eta^* - \zeta^*, \zeta - \eta^* \rangle \geq 0, \text{ for all } \zeta \in \mathcal{E}. \end{aligned} \tag{1.2}$$

Here $A : \mathcal{E} \to \Sigma$ and $\mu, \lambda > 0$ are constants. If $\zeta^* = \eta^*$, the $VI(\chi, \Sigma)$ (1.1) is a special case of (NSVIP) (1.2). Recently, Saechou and Kangtunyakarn [14] combined the concept of $EP(\Phi)$ and $NSVIP$ and named it modification of equilibrium problem MEP and in 2023, Saechou and Kangtunyakarn [15] again generalized the concept of MEP and introduced the generalized modification of the system of equilibrium problem $GMSEP$, which is to find $(\zeta^*, \eta^*) \in \mathcal{E} \times \mathcal{E}$, such that

$$\begin{cases} \Phi(\zeta^*, \eta) + \frac{1}{r}\langle \eta - \zeta^*, \zeta^* - \eta^* + \gamma_1 G(\eta^*) \rangle \geq 0, \text{ for all } \eta \in \mathcal{E}, \\ \Psi(\eta^*, \zeta) + \frac{1}{s}\langle \zeta - \eta^*, \eta^* - \zeta^* + \gamma_2 H(\zeta^*) \rangle \geq 0, \text{ for all } \zeta \in \mathcal{E}, \end{cases} \tag{1.3}$$

where $G, H : \mathcal{E} \to \Sigma$ are mappings, $\Phi, \Psi : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ are two bifunctions, and $\gamma_1, \gamma_2, s, r > 0$ are constants.

In this paper, motivated by Saechou and Kangtunyakarn [14] and Saechou and Kangtunyakarn [15] we introduced the algorithm for an infinite family of nonexpansive mappings and proved a strong convergence result. The result shows that the sequence generated by the algorithm converges strongly to a common element of the set of an infinite family of nonexpansive mappings and the set of solution of generalized modification of the system of equilibrium problem (GMSEP).

2 Preliminaries

In this section, we recall some basic definitions and useful facts. Suppose Σ be a real Hilbert space equipped with the norm $\|.\|$ induced by inner product $\langle ., . \rangle$ and $\mathcal{E} \neq \emptyset$ a subset of Σ. Then for all $\zeta, \eta \in \mathcal{E}$ a mapping $G : \mathcal{E} \to \Sigma$ is said to be

(1) monotone if
$$\langle G(\zeta) - G(\eta), \zeta - \eta \rangle \geq 0,$$

(2) nonexpansive if
$$\|G(\zeta) - G(\eta)\| \leq \|\zeta - \eta\|,$$

(3) γ-inverse strongly monotone if there exists a positive real number γ such that
$$\langle G(\zeta) - G(\eta), \zeta - \eta \rangle \geq \gamma \|G(\zeta) - G(\eta)\|^2.$$

We denote the set of fixed points of a mapping G by $F(G)$.

Lemma 2.1. [11] *Suppose $\mathcal{E} \neq \emptyset$ be a closed and convex subset of Σ and suppose $\Phi : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ a bifunction satisfying following conditions*

(A_1) $\Phi(\zeta, \zeta) = 0, \forall \zeta \in \mathcal{E}$,

(A_2) Φ *is monotone i.e.* $\Phi(\zeta, \eta) + \Phi(\eta, \zeta) \leq 0, \forall \zeta, \eta \in \mathcal{E}$,

(A_3) $\forall \zeta, \eta, \vartheta \in \mathcal{E}$, $\lim\limits_{\varepsilon \to 0^+} \Phi(\varepsilon\vartheta + (1 - \varepsilon)\zeta, \eta) \leq \Phi(\zeta, \eta)$,

(A_4) $\forall \zeta \in \mathcal{E}$, $\eta \mapsto \Phi(\zeta, \eta)$ *is convex and lower semicontinuous.*

Suppose $r > 0$, $\zeta \in \Sigma$, *then* $\exists\, \vartheta \in \mathcal{E}$ *such that*

$$\Phi(\vartheta, \eta) + \frac{1}{r}\langle \eta - \vartheta, \vartheta - \zeta\rangle \geq 0, \;\; \forall \eta \in \mathcal{E}.$$

Lemma 2.2. *The mapping* $\phi : \mathcal{E} \to \mathcal{E}$ *defined by* $\phi(\zeta) = \chi_r(I - \gamma_1 G)\chi_s(I - \gamma_2 H)(\zeta)$ *for all* $\gamma_1, \gamma_2 > 0$ *is a nonexpansive mapping.*

Proof. If we follow the first part of [20, Theorem 3.1] we can easily get that the mapping ϕ is a nonexpansive mapping. □

Lemma 2.3. [21]. *Suppose* $\{\lambda_n\}$ *and* $\{\rho_n\}$ *be the sequences of nonnegative real numbers such that*

$$\lambda_{n+1} \leq (1 - \rho_n)\lambda_n + \omega_n + \gamma_n, n \geq 1,$$

where $\{\rho_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\omega_n\}$ *is a real sequence. Suppose* $\sum_{n=0}^{\infty} \gamma_n < \infty$. *Then following hold:*

(1) *If* $\omega_n \leq \rho_n C$ *for some* $C \geq 0$, *then* $\{\lambda_n\}$ *is a bounded sequence.*

(2) *If* $\sum_{n=0}^{\infty} \rho_n = \infty$ *and* $\limsup_{n\to\infty} \frac{\omega_n}{\rho_n} \leq 0$, *then* $\lim_{n\to\infty} \lambda_n = 0$.

Lemma 2.4. [25]. *Suppose* $\zeta \in \Sigma$ *and* $\eta \in \mathcal{E}$. *Then* $P_{\mathcal{E}}\zeta = \eta$ *if and only if following inequality holds*

$$\langle \zeta - \eta, \eta - \mu\rangle, \;\; \forall \mu \in \mathcal{E}.$$

Lemma 2.5. [22]. *Suppose* $\Phi : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *a bifunction satisfying* $(A_1) - (A_4)$ *and for each* $r > 0$, $\zeta \in \Sigma$, *define a mapping* $\chi_r : \Sigma \to \mathcal{E}$ *as follows:*

$$\chi_r(\zeta) = \{\vartheta \in \mathcal{E} : \Phi(\vartheta, \eta) + \frac{1}{r}\langle \eta - \vartheta, \vartheta - \zeta\rangle \geq 0, \forall \eta \in \mathcal{E}\},$$

then the mapping χ_r *is a single valued, firmly nonexpansive mapping, that is*

$$\chi_r(\zeta) - \chi_r(\eta)\|^2 \leq \langle \chi_r(\zeta) - \chi_r(\eta), \zeta - \eta\rangle, \;\; \forall \zeta, \eta \in \Sigma.$$

Moreover, $F(\chi_r) = EP(\Phi)$ *is closed and convex.*

Lemma 2.6. [15]. *Suppose* $\Phi, \Psi : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *be two bifunctions satisfying the conditions* $(A_1) - (A_4)$ *of Lemma 2.1. Suppose* $s, r > 0$, *then following are equivalent:*

(1) (ζ^*, η^*) *is a solution of* (1.3),

(2) ζ^* *is a FP of mapping* $\phi : \mathcal{E} \to \mathcal{E}$ *defined by* $\phi(\zeta) = \chi_r(I - \gamma_1 G)\chi_s(I - \gamma_2 H)(\zeta)$ *for all* $\gamma_1, \gamma_2 > 0$ *and* $\zeta \in \mathcal{E}$, *where* $\eta^* = \chi_s(I - \gamma_2 H)\zeta^*$.

Lemma 2.7. [19]. *Suppose,* $\mathcal{X}$ *be a real inner product space. Then*

(1) $\|\zeta + \eta\|^2 \leq \|\zeta\|^2 + 2\langle \eta, \zeta + \eta\rangle, \;\; \forall \zeta, \eta \in \mathcal{X}$,

(2) $\|t\zeta + s\eta\|^2 = t(t + s)\|\zeta\|^2 + s(t + s)\|\eta\|^2 - st\|\zeta - \eta\|^2, \;\; \forall \zeta, \eta \in \mathcal{X}, \;\; \forall s, t \in \mathbb{R}$.

Definition 2.8. [23]. A Banach space $\mathcal{X}$ satisfies the Opial property if, for each weakly convergent sequence $\{\zeta_n\}$ with the weak limit $\zeta \in \mathcal{X}$, it holds

$$\liminf_{n\to\infty} \|\zeta_n - \zeta\| < \liminf_{n\to\infty} \|\zeta_n - \eta\|$$

$\forall\, \eta \in \mathcal{X}$ with $\zeta \neq \eta$.

All Banach spaces of finite dimension, Hilbert spaces, ℓ^p $(1 \leq p < \infty)$ spaces satisfy the Opial property. A Banach space with a weakly sequentially continuous duality mapping also has the Opial property. But L_p $(0 < p < \infty, \; p \neq 2)$ space does not satisfy the Opial property.

Definition 2.9. [16]. Suppose $\mathcal{E} \neq \emptyset$ a subset of a Banach space $\mathcal{X}$, and $\forall\, \zeta \in \mathcal{X}$, $\exists$ a $\eta \in \mathcal{E}$ such that $\forall\, \vartheta \in \mathcal{E}$,

$$\|\eta - \zeta\| \leq \|\vartheta - \zeta\|.$$

Then η is called a metric projection of ζ onto $\mathcal{E}$ and denoted by $P_{\mathcal{E}}(\zeta)$. If $P_{\mathcal{E}}(\zeta)$ exists and determines uniquely $\forall\, \zeta \in \mathcal{E}$, then the mapping $P_{\mathcal{E}} : \mathcal{X} \to \mathcal{E}$ is called the metric projection onto $\mathcal{E}$.

3 Main Results

Theorem 3.1. *Suppose Σ be a Hilbert space and $\mathcal{E} \neq \emptyset$ convex and closed subset of Σ. Suppose $\{\chi_i\}_{i\in\mathbb{N}}$ be a family of nonexpansive mappings from $\mathcal{E}$ into itself and $G, H : \mathcal{E} \to \Sigma$ be a_1, a_2-inverse strongly monotone with positive real numbers a_1, a_2, respectively. Suppose $\Phi, \Psi : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ be bifunctions satisfying conditions (A_1)-(A_4), $\phi : \mathcal{E} \to \mathcal{E}$ a mapping defined by $\phi(\zeta) = \chi_r(I - \gamma_1 G)\chi_s(I - \gamma_2 H)(\zeta)$ for all $\gamma_1, \gamma_2 > 0$ and $\chi_r, \chi_s : \Sigma \to \mathcal{E}$ are mappings defined as $\chi_r(\zeta) = \{\vartheta \in \mathcal{E} : \Phi(\vartheta, \eta) + \frac{1}{r}\langle \eta - \vartheta, \vartheta - \zeta\rangle \geq 0, \forall \eta \in \mathcal{E}\}$ and $\chi_s(\zeta) = \{\vartheta \in \mathcal{E} : \Psi(\vartheta, \eta) + \frac{1}{s}\langle \eta - \vartheta, \vartheta - \zeta\rangle \geq 0, \forall \eta \in \mathcal{E}\}$. Assume that $\xi = \bigcap_{i\in\mathbb{N}} F(\chi_i) \bigcap F(\phi) \neq \emptyset$. For given $\zeta_1, \varpi \in \mathcal{E}$, and let the sequences $\{\zeta_n\}$ and $\{\eta_n\}$ be generated by*

$$\begin{cases} \eta_n = (1 - \omega_n)\zeta_n + \omega_n \chi_i(\zeta_n) \\ \zeta_{n+1} = \rho_n \varpi + \lambda_n \eta_n + \omega_n \phi(\zeta_n), \end{cases} \tag{3.1}$$

$\forall\, n \in \mathbb{N}$, and for $\delta = \min\{a_1, a_2\}$, $\gamma_1, \gamma_2 \in (0, 2\delta)$. $\{\lambda_n\}$, $\{\beta_n\}$,$\{\delta_n\}$ are the sequences of real numbers such that $\lambda_n + \omega_n + \rho_n \leq 1$. Assume that the following conditions hold

(1) $\lim_{n\to\infty} \rho_n = 0$ *and* $\sum_{n=1}^{\infty} \rho_n = \infty$,

(2) $0 \leq c \leq \lambda_n, \omega_n \leq d < 1$ *for all* $n \in \mathbb{N}$,

(3) $\sum_{n=1}^{\infty} (1 - \lambda_n - \omega_n - \rho_n) < \infty$,

(4) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |\omega_{n+1} - \omega_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$.

Then the sequence $\{\zeta_n\}$ generated by (3.1) *converges strongly to $\zeta_0 = P_\xi \varpi$, which (ζ_0, η_0) is solution of the GMSEP, where $\eta_0 = \chi_s(I - \gamma_2 H)\zeta_0$.*

Proof. First, using Lemma 2.2 we can say that the mapping ϕ is a nonexpansive mapping. Since $\bigcap_{i\in\mathbb{N}} F(\chi_i) \bigcap F(\phi) \neq \emptyset$, suppose $\zeta^\dagger \in \bigcap_{i\in\mathbb{N}} F(\chi_i) \bigcap F(\phi)$. Now we prove that the sequence $\{\zeta_n\}$ is bounded.

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\| &= \|\rho_n \varpi + \lambda_n \eta_n + \omega_n \phi(\zeta_n) - \zeta^\dagger\| \\
&\leq \rho_n \|\varpi - \zeta^\dagger\| + \lambda_n \|\eta_n - \zeta^\dagger\| + \omega_n \|\phi(\zeta_n) - \zeta^\dagger\| + (1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\| \\
&\leq \rho_n \|\varpi - \zeta^\dagger\| + \lambda_n \|\eta_n - \zeta^\dagger\| + \omega_n \|\zeta_n - \zeta^\dagger\| + (1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\| \\
&\leq \rho_n \|\varpi - \zeta^\dagger\| + \lambda_n \|(1 - \omega_n)\zeta_n + \omega_n \chi_i(\zeta_n) - \zeta^\dagger\| + \omega_n \|\zeta_n - \zeta^\dagger\| \\
&+ (1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\| \\
&= \rho_n \|\varpi - \zeta^\dagger\| + \lambda_n \{(1 - \omega_n)\|\zeta_n - \zeta^\dagger\| + \omega_n \|\chi_n(\zeta_n) - \zeta^\dagger\|\} + \omega_n \|\zeta_n - \zeta^\dagger\| \\
&+ (1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\| \\
&\leq \rho_n \|\varpi - \zeta^\dagger\| + \lambda_n \{(1 - \omega_n)\|\zeta_n - \zeta^\dagger\| + \omega_n \|\zeta_n - \zeta^\dagger\|\} + \omega_n \|\zeta_n - \zeta^\dagger\| \\
&+ (1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\| \\
&= \rho_n \|\varpi - \zeta^\dagger\| + (\lambda_n + \omega_n)\|\zeta_n - \zeta^\dagger\| + (1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\| \\
&\leq \rho_n \|\varpi - \zeta^\dagger\| + (1 - \rho_n)\|\zeta_n - \zeta^\dagger\| + (1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger
\end{aligned} \tag{3.2}$$

If we apply Lemma 2.3 and condition (3) to the above equation we get that the sequence $\{\zeta_n\}$ is bounded and hence $\{\phi(\zeta_n)\}$ and $\{\chi_i(\zeta_n)\}$ are also bounded. Next we prove that $\lim_{n\to\infty} \|\zeta_{n+1} - \zeta_n\| = 0$. We have

$$\begin{aligned}
\|\eta_n - \eta_{n-1}\| &= \|(1 - \omega_n)\zeta_n + \omega_n \chi_i(\zeta_n) - ((1 - \omega_{n-1})\zeta_{n-1} + \omega_{n-1}\chi_i(\zeta_{n-1}))\| \\
&\leq (1 - \omega_n)\|\zeta_n - \zeta_{n-1}\| + |\omega_n - \omega_{n-1}|\|\zeta_{n-1}\| + \omega_n \|\chi_i(\zeta_n) - \chi_i(\zeta_{n-1})\| \\
&+ |\omega_n - \omega_{n-1}|\|\chi_i(\zeta_{n-1})\| \\
&\leq (1 - \omega_n)\|\zeta_n - \zeta_{n-1}\| + |\omega_n - \omega_{n-1}|\|\zeta_{n-1}\| + \omega_n \|\zeta_n - \zeta_{n-1}\| \\
&+ |\omega_n - \omega_{n-1}|\|\chi_i(\zeta_{n-1})\| \\
&= \|\zeta_n - \zeta_{n-1}\| + |\omega_n - \omega_{n-1}|\{\|\zeta_{n-1}\| + \|\chi_i(\zeta_{n-1})\|\}.
\end{aligned} \tag{3.3}$$

Next, using the definition of $\{\zeta_n\}$ and above equation we get

$$\begin{aligned}\|\zeta_{n+1}-\zeta_n\| &= \|\rho_n\varpi+\lambda_n\eta_n+\omega_n\phi(\zeta_n)-\{\rho_{n-1}\varpi+\lambda_{n-1}\eta_{n-1}+\omega_{n-1}\phi(\zeta_{n-1})\}\|\\ &\le |\rho_n-\rho_{n-1}|\|\varpi|+\lambda_n\|\eta_n-\eta_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|\eta_{n-1}\|\\ &+\omega_n\|\zeta_n-\zeta_{n-1}\|+|\omega_n-\omega_{n-1}|\|\phi(\zeta_{n-1})\|\\ &= |\rho_n-\rho_{n-1}|\|\varpi|+\lambda_n\{\|\zeta_n-\zeta_{n-1}\|+|\omega_n-\omega_{n-1}|\{\|\zeta_{n-1}\|+\|\chi_i(\zeta_{n-1})\|\}\}\\ &+|\lambda_n-\lambda_{n-1}|\|\eta_{n-1}\|+\omega_n\|\zeta_n-\zeta_{n-1}\|+|\omega_n-\omega_{n-1}|\|\phi(\zeta_{n-1})\|\\ &\le |\rho_n-\rho_{n-1}|\|\varpi\|+|\lambda_n-\lambda_{n-1}|\|\eta_{n-1}\|+(\lambda_n+\omega_n)\|\zeta_n-\zeta_{n-1}\|\\ &+|\omega_n-\omega_{n-1}|\{\lambda_n\|\zeta_{n-1}\|+\lambda_n\|\chi_i(\zeta_{n-1})\|+\|\phi(\zeta_{n-1})\|\}\\ &\le |\rho_n-\rho_{n-1}|\|\varpi\|+|\lambda_n-\lambda_{n-1}|\|\eta_{n-1}\|+(1-\rho_n)\|\zeta_n-\zeta_{n-1}\|\\ &+|\omega_n-\omega_{n-1}|\{\|\zeta_{n-1}\|+\|\chi_i(\zeta_{n-1})\|+\|\phi(\zeta_{n-1})\|\}.\end{aligned}$$

If we apply Lemma 2.3 and use conditions (1) and (4) we get

$$\lim_{n\to\infty}\|\zeta_{n+1}-\zeta_n\|=0. \tag{3.4}$$

Next we prove that $\lim\limits_{n\to\infty}\|\chi_i(\zeta_n)-\zeta_n\|=0$. Now we have

$$\begin{aligned}\|\zeta_{n+1}-\zeta^\dagger\|^2 &= \|\rho_n(\varpi-\zeta^\dagger)+\lambda_n(\eta_n-\zeta^\dagger)+\omega_n(\phi(\zeta_n)-\zeta^\dagger)-(1-\lambda_n-\omega_n-\rho_n)\zeta^\dagger\|^2\\ &\le \|\rho_n(\varpi-\zeta^\dagger)+\lambda_n(\eta_n-\zeta^\dagger)+\omega_n(\phi(\zeta_n)-\zeta^\dagger)\|^2\\ &-2(1-\lambda_n-\omega_n-\rho_n)\langle\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\\ &\le \|\lambda_n(\eta_n-\zeta^\dagger)+\rho_n(\varpi-\zeta^\dagger)+\omega_n(\phi(\zeta_n)-\zeta^\dagger)\|^2\\ &+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta^\dagger\|\|\zeta_{n+1}-\zeta^\dagger\|\\ &= \left\|\lambda_n(\eta_n-\zeta^\dagger)+\omega_n\left(\frac{\rho_n}{\omega_n}(\varpi-\zeta^\dagger)+\phi(\zeta_n)-\zeta^\dagger\right)\right\|^2\\ &+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta^\dagger\|\|\zeta_{n+1}-\zeta^\dagger\|\\ &= \lambda_n(\lambda_n+\omega_n)\|\eta_n-\zeta^\dagger\|^2+\omega_n(\lambda_n+\omega_n)\left\|\frac{\rho_n}{\omega_n}(\varpi-\zeta^\dagger)+\phi(\zeta_n)-\zeta^\dagger\right\|\\ &-\lambda_n\omega_n\left\|\eta_n-\frac{\rho_n}{\omega_n}(\varpi-\zeta^\dagger)-\phi(\zeta_n)\right\|^2+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta^\dagger\|\|\zeta_{n+1}-\zeta^\dagger\|\end{aligned}$$

$$\begin{aligned}\|\zeta_{n+1}-\zeta^\dagger\|^2 &\le \lambda_n(\lambda_n+\omega_n)\|\eta_n-\zeta^\dagger\|^2+\omega_n(\lambda_n+\omega_n)\left(\frac{\rho_n}{\omega_n}\left(\frac{\rho_n}{\omega_n}+1\right)\|\varpi-\zeta^\dagger\|^2\right.\\ &\left.+\left(\frac{\rho_n}{\omega_n}+1\right)\|\phi(\zeta_n)-\zeta^\dagger\|^2-\frac{\rho_n}{\omega_n}\|\varpi-\phi(\zeta_n)\|^2\right)\\ &+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta^\dagger\|\|\zeta_{n+1}-\zeta^\dagger\|\\ &\le \lambda_n(\lambda_n+\omega_n)\|\eta_n-\zeta^\dagger\|^2+\omega_n(\lambda_n+\omega_n)\frac{\rho_n}{\omega_n}\left(\frac{\rho_n}{\omega_n}+1\right)\|\varpi-\zeta^\dagger\|^2\\ &+\omega_n(\lambda_n+\omega_n)\left(\frac{\rho_n}{\omega_n}+1\right)\|\zeta_n-\zeta^\dagger\|^2\\ &+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta^\dagger\|\|\zeta_{n+1}-\zeta^\dagger\|.\end{aligned} \tag{3.5}$$

Now using the definition of $\{\eta_n\}$, we get

$$\begin{aligned}\|\eta_n-\zeta^\dagger\|^2 &= (1-\omega_n)\|\zeta_n-\zeta^\dagger\|^2+\omega_n\|\chi_i(\zeta_n)-\zeta^\dagger\|^2-\omega_n(1-\omega_n)\|\chi_i(\zeta_n)-\zeta_n\|^2\\ &\le \|\zeta_n-\zeta^\dagger\|^2-\omega_n(1-\omega_n)\|\chi_i(\zeta_n)-\zeta_n\|^2.\end{aligned} \tag{3.6}$$

If we substitute (3.6) to (3.5), we get

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\| &\leq \lambda_n(\lambda_n + \omega_n)\|\eta_n - \zeta^\dagger\|^2 + \omega_n(\lambda_n + \omega_n)\frac{\rho_n}{\omega_n}\left(\frac{\rho_n}{\omega_n} + 1\right)\|\varpi - \zeta^\dagger\|^2 \\
&+ \omega_n(\lambda_n + \omega_n)\left(\frac{\rho_n}{\omega_n} + 1\right)\|\zeta_n - \zeta^\dagger\|^2 + 2(1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\|\|\zeta_{n+1} - \zeta^\dagger\| \\
&\leq \lambda_n(\lambda_n + \omega_n)\{\|\zeta_n - \zeta^\dagger\|^2 - \omega_n(1 - \omega_n)\|\chi_i(\zeta_n) - \zeta_n\|^2\} \\
&+ \rho_n(\lambda_n + \omega_n)\left(\frac{\rho_n}{\omega_n} + 1\right)\|\varpi - \zeta^\dagger\|^2 + \omega_n(\lambda_n + \omega_n)\left(\frac{\rho_n}{\omega_n} + 1\right)\|\zeta_n - \zeta^\dagger\|^2 \\
&+ 2(1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\|\|\zeta_{n+1} - \zeta^\dagger\| \\
&= (\lambda_n + \omega_n)(\lambda_n + \omega_n + \rho_n)\|\zeta_n - \zeta^\dagger\|^2 - \lambda_n\omega_n(\lambda_n + \omega_n)(1 - \omega_n)\|\chi_i(\zeta_n) - \zeta_n\|^2 \\
&+ \rho_n(\lambda_n + \omega_n)\left(\frac{\rho_n}{\omega_n} + 1\right)\|\varpi - \zeta^\dagger\|^2 + 2(1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\|\|\zeta_{n+1} - \zeta^\dagger\| \\
&\leq \|\zeta_n - \zeta^\dagger\|^2 - \lambda_n\omega_n(\lambda_n + \omega_n)(1 - \omega_n)\|\chi_i(\zeta_n) - \zeta_n\|^2 \\
&+ \rho_n(\lambda_n + \omega_n)\left(\frac{\rho_n}{\omega_n} + 1\right)\|\varpi - \zeta^\dagger\|^2 + 2(1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\|\|\zeta_{n+1} - \zeta^\dagger\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\lambda_n\omega_n(\lambda_n + \omega_n)(1 - \omega_n)\|\chi_i(\zeta_n) - \zeta_n\|^2 \leq \|\zeta_n - \zeta^\dagger\|^2 - \|\zeta_{n+1} - \zeta^\dagger\|^2 \\
&+ \rho_n(\lambda_n + \omega_n)\left(\frac{\rho_n}{\omega_n} + 1\right)\|\varpi - \zeta^\dagger\|^2 + 2(1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\|\|\zeta_{n+1} - \zeta^\dagger\| \\
&\leq \|\zeta_n - \zeta_{n+1}\|(\|\zeta_n - \zeta^\dagger\| + \|\zeta_{n+1} - \zeta^\dagger\|) + \rho_n(\lambda_n + \omega_n)\left(\frac{\rho_n}{\omega_n} + 1\right)\|\varpi - \zeta^\dagger\|^2 \\
&+ 2(1 - \lambda_n - \omega_n - \rho_n)\|\zeta^\dagger\|\|\zeta_{n+1} - \zeta^\dagger\|.
\end{aligned}$$

Using (3.4) and conditions (1) and (3) we get

$$\lim_{n\to\infty}\|\chi_i(\zeta_n) - \zeta_n\| = 0. \tag{3.7}$$

Now we prove that $\lim_{n\to\infty}\|\phi(\zeta_n) - \zeta_n\| = 0$. Since we have

$$\begin{aligned}
\eta_n - \zeta_n &= (1 - \omega_n)\zeta_n + \omega_n\chi_i(\zeta_n) - \zeta_n \\
&= \omega_n(\chi_i(\zeta_n) - \zeta_n).
\end{aligned}$$

So

$$\|\eta_n - \zeta_n\| = \omega_n\|\chi_i(\zeta_n) - \zeta_n\|.$$

From (3.7) we get

$$\lim_{n\to\infty}\|\eta_n - \zeta_n\| = 0. \tag{3.8}$$

Since

$$\begin{aligned}
\zeta_{n+1} - \zeta_n &= \rho_n\varpi + \lambda_n\eta_n + \omega_n\phi(\zeta_n) - \zeta_n \\
&= \rho_n(\varpi - \zeta_n) + \lambda_n(\eta_n - \zeta_n) + \omega_n(\phi(\zeta_n) - \zeta_n) - (1 - \lambda_n - \omega_n - \rho_n)\zeta_n.
\end{aligned}$$

Applying (3.4) and (3.8) and conditions (1) and (3) to the above equation we get

$$\lim_{n\to\infty}\|\phi(\zeta_n) - \zeta_n\| = 0 \tag{3.9}$$

Now, we prove that

$$\limsup_{n\to\infty}\langle\varpi - \zeta_0, \zeta_n - \zeta_0\rangle \leq 0, \text{ where } \zeta_0 \in P_\xi\varpi.$$

Indeed, we take a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that

$$\limsup_{n\to\infty}\langle\varpi - \zeta_0, \zeta_n - \zeta_0\rangle = \lim_{k\to\infty}\langle\varpi - \zeta_0, \zeta_{n_k} - \zeta_0\rangle.$$

Since $\{\zeta_n\}$ is the bounded sequence, w.l.g., we can assume $\zeta_{n_k} \rightharpoonup \zeta^*$ as $k \to \infty$. Suppose $\zeta^* \neq \phi(\zeta^*)$. Now using (3.9), nonexpansiveness of the mapping ϕ, the Opial property, we get

$$\begin{aligned}\liminf_{k\to\infty} \|\zeta_{n_k} - \zeta^*\| &< \liminf_{k\to\infty} \|\zeta_{n_k} - \phi(\zeta^*)\| \\ &\leq \liminf_{k\to\infty}(\|\zeta_{n_k} - \phi(\zeta_{n_k})\| + \|\phi(\zeta_{n_k}) - \phi(\zeta^*)\|) \\ &\leq \liminf_{k\to\infty}(\|\zeta_{n_k} - \phi(\zeta_{n_k})\| + \|\zeta_{n_k} - \zeta^\dagger\|) \\ &\leq \liminf_{k\to\infty} \|\zeta_{n_k} - \zeta^*\|.\end{aligned}$$

And we get a contradiction so we have

$$\zeta^* \in F(\phi). \tag{3.10}$$

Now let us assume that $\zeta^* \neq \chi_i(\zeta^*)$. Again using (3.7), nonexpansiveness of the family of mappings χ_i and Opial property, we have

$$\begin{aligned}\liminf_{k\to\infty} \|\zeta_{n_k} - \zeta^*\| &< \liminf_{k\to\infty} \|\zeta_{n_k} - \chi_i(\zeta^*)\| \\ &\leq \liminf_{k\to\infty}(\|\zeta_{n_k} - \chi_i(\zeta_{n_k})\| + \|\chi_i(\zeta_{n_k}) - \chi_i(\zeta^*)\|) \\ &\leq \liminf_{k\to\infty}(\|\zeta_{n_k} - \chi_i(\zeta_{n_k})\| + \|\zeta_{n_k} - \zeta^*\|) \\ &\leq \liminf_{k\to\infty} \|\zeta_{n_k} - \zeta^*\|.\end{aligned}$$

And again we get a contradiction. So we have

$$\zeta^* \in \bigcap_{i\in\mathbb{N}} F(\chi_i). \tag{3.11}$$

From the equations (3.10) and (3.11) we can say

$$\zeta^* \in \bigcap_{i\in\mathbb{N}} F(\chi_i) \bigcap F(\phi). \tag{3.12}$$

Since $\zeta_{n_k} \rightharpoonup \zeta^*$ as $k \to \infty$ using Lemma 2.4 and (3.12), we get

$$\begin{aligned}\limsup_{n\to\infty}\langle\varpi - \zeta_0, \zeta_n - \zeta_0\rangle &= \limsup_{k\to\infty}\langle\varpi - \zeta_0, \zeta_{n_k} - \zeta_0\rangle \\ &= \langle\varpi - \zeta_0, \zeta^* - \zeta_0\rangle \leq 0,\end{aligned}$$

where $\zeta_0 = P_\xi\varpi$. Now lastly, we prove that the sequence $\{\zeta_n\}$ converges strongly to $\zeta_0 = P_\xi\varpi$. Now, we have

$$\begin{aligned}\|\zeta_{n+1} - \zeta_0\|^2 &\leq \|\rho_n\varpi + \lambda_n\eta_n + \omega_n\phi(\zeta_n) - \zeta_0\|^2 \\ &= \|\rho_n(\varpi - \zeta_0) + \lambda_n(\eta_n - \zeta_0) + \omega_n(\phi(\zeta_n) - \zeta_0) - (1 - \lambda_n - \omega_n - \rho_n)\zeta_0\|^2 \\ &\leq \|\lambda_n(\eta_n - \zeta_0) + \omega_n(\phi(\zeta_n) - \zeta_0) - (1 - \lambda_n - \omega_n - \rho_n)\zeta_0\|^2 \\ &+ 2\rho_n\langle\varpi - \zeta_0, \zeta_{n+1} - \zeta_0\rangle \\ &\leq \|\lambda_n(\eta_n - \zeta_0) + \omega_n(\phi(\zeta_n) - \zeta_0)\|^2 \\ &- 2(1 - \lambda_n - \omega_n - \rho_n)\langle\zeta_0, \lambda_n(\eta_n - \zeta_0) + \omega_n(\phi(\zeta_n) - \zeta_0) - (1 - \lambda_n - \omega_n - \rho_n)\zeta_0\rangle \\ &+ 2\rho_n\langle\varpi - \zeta_0, \zeta_{n+1} - \zeta_0\rangle \\ &\leq \|\lambda_n(\eta_n - \zeta_0) + \omega_n(\phi(\zeta_n) - \zeta_0)\|^2 \\ &+ 2(1 - \lambda_n - \omega_n - \rho_n)\|\zeta_0\|\|\lambda_n(\eta_n - \zeta_0) + \omega_n(\phi(\zeta_n) - \zeta_0) - (1 - \lambda_n - \omega_n - \rho_n)\zeta_0\| \\ &+ 2\rho_n\langle\varpi - \zeta_0, \zeta_{n+1} - \zeta_0\rangle \\ &= \lambda_n(\lambda_n + \omega_n)\|\eta_n - \zeta_0\|^2 + \omega_n(\lambda_n + \omega_n)\|\phi(\zeta_n) - \zeta_0\|^2 - \lambda_n\omega_n\|\eta_n - \phi(\zeta_n)\|^2 \\ &+ 2(1 - \lambda_n - \omega_n - \rho_n)\|\zeta_0\|\|\lambda_n(\eta_n - \zeta_0) + \omega_n(\phi(\zeta_n) - \zeta_0) - (1 - \lambda_n - \omega_n - \rho_n)\zeta_0\| \\ &+ 2\rho_n\langle\varpi - \zeta_0, \zeta_{n+1} - \zeta_0\rangle\end{aligned}$$

$$
\begin{aligned}
\|\zeta_{n+1}-\zeta_0\|^2 &\le \lambda_n(\lambda_n+\omega_n)\|\eta_n-\zeta_0\|^2+\omega_n(\lambda_n+\omega_n)\|\zeta_n-\zeta_0\|^2\\
&+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta_0\|\|\lambda_n(\eta_n-\zeta_0)+\omega_n(\phi(\zeta_n)-\zeta_0)-(1-\lambda_n-\omega_n-\rho_n)\zeta_0\|\\
&+2\rho_n\langle\varpi-\zeta_0,\zeta_{n+1}-\zeta_0\rangle\\
&\le \lambda_n(\lambda_n+\omega_n)\|\zeta_n-\zeta_0\|^2+\omega_n(\lambda_n+\omega_n)\|\zeta_n-\zeta_0\|^2\\
&+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta_0\|\|\lambda_n(\eta_n-\zeta_0)+\omega_n(\phi(\zeta_n)-\zeta_0)-(1-\lambda_n-\omega_n-\rho_n)\zeta_0\|\\
&+2\rho_n\langle\varpi-\zeta_0,\zeta_{n+1}-\zeta_0\rangle\\
&=(\lambda_n+\omega_n)^2\|\zeta_n-\zeta_0\|^2+2\rho_n\langle\varpi-\zeta_0,\zeta_{n+1}-\zeta_0\rangle\\
&+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta_0\|\|\lambda_n(\eta_n-\zeta_0)+\omega_n(\phi(\zeta_n)-\zeta_0)-(1-\lambda_n-\omega_n-\rho_n)\zeta_0\|\\
&\le(1-\rho_n)^2\|\zeta_n-\zeta_0\|^2+2\rho_n\langle\varpi-\zeta_0,\zeta_{n+1}-\zeta_0\rangle\\
&+2(1-\lambda_n-\omega_n-\rho_n)\|\zeta_0\|\|\lambda_n(\eta_n-\zeta_0)+\omega_n(\phi(\zeta_n)-\zeta_0)-(1-\lambda_n-\omega_n-\rho_n)\zeta_0\| \qquad (3.13)
\end{aligned}
$$

Using (3.12), Lemma 2.3 and condition (3) in (3.13) we get the sequence $\{\zeta_n\}$ converges strongly to $\zeta_0 = P_\xi\varpi$. Using Lemma 2.6, we get (ζ_0, η_0) is a solution of the GMSEP, where $\eta_0 = \chi_s(I-\gamma_2 H)(\zeta_0)$. It completes the proof. ☐

Conclusion

In this paper, we have presented a new algorithm to find a common element of the fixed point of an infinite family of nonexpansive mappings and the set of solutions of GMSEP in the setting of Hilbert spaces. We proved that the sequence generated by this algorithm converges strongly to this common solution.

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