Mathematics and Statistics 12(4): 303-313, 2024 http://www.hrpub.org
DOI: 10.13189/ms.2024.120401

# Construction of Bivariate Transmuted Frechet Distribution with its Properties

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*Received February 4, 2024; Revised May 14, 2024; Accepted June 16, 2024*

***Cite This Paper in the Following Citation Styles***

***(a):*** *[1] Hayfa Abdul Jawad Saieed, Mhasen Saleh Altalib, Safwan Nathem Rashed, Manaf Hazim Ahmed , "Construction of Bivariate Transmuted Frechet Distribution with its Properties," Mathematics and Statistics, Vol. 12, No. 4, pp. 303 - 313, 2024. DOI: 10.13189/ms.2024.120401.*

***(b):*** *Hayfa Abdul Jawad Saieed, Mhasen Saleh Altalib, Safwan Nathem Rashed, Manaf Hazim Ahmed (2024). Construction of Bivariate Transmuted Frechet Distribution with its Properties. Mathematics and Statistics, 12(4), 303 - 313. DOI: 10.13189/ms.2024.120401.*



**Abstract** In multivariate data modeling, the statistical analyst can desire to construct a multivariate distribution with correlated variables. For this reason, there is a need to generalize univariate distributions, but this generalization is not easy. Many methods have been presented for construction of continuous multivariate families with univariate distributions. Some of these methods are based on a single baseline, while others are based on more than one baseline, so that their variables are dependent. Some authors were interested in expanding a univariate transmuted family into multivariate case. Some suggestions were made about extension of univariate quadratic transmuted (QT) family to bivariate ones, and another modification was made to this family by replacing the (c.d.f.) with exponentiated (c.d.f.). Another construction of bivariate family is based on probability distribution of paired order Statistics for a sample size two drawn from quadratic ranked transmuted (QRT) margin, and this bivariate family allows for positive and negative dependence between variables. Another family proposed an extension of univariate mixture of standard continuous uniform, with decreasing densities to a bivariate case. Our proposed ($\mathrm{CT}_2$) reduces to a bivariate quadratic transmuted ($\mathrm{QT}_2$) family if the cubic transmutation parameters equal to zero. ($\mathrm{CT}_2$) family can be used for modeling positive and negative correlated variables. Some statistical properties of ($\mathrm{CT}_2$) family have been studied which comprise joint, marginal and conditional (c.d.f., p.d.f), joint, marginal and conditional moments, data generation and dependence coefficients. It is seen that (joint, marginal and conditional) moments depend on raw moments of (baseline variables and largest order statistics of samples sizes 2 and 3). The Egyptian bivariate economic data are fitted by ($\mathrm{CT}_2\mathrm{Fr}$), (FGMFr), ($\mathrm{T}_2\mathrm{Fr}$) and (DSASFr). The ($\mathrm{CT}_2\mathrm{Fr}$) is the fit to which has smallest (AIC) and (BIC) criteria.

**Keywords** Frechet Distribution, K-transmuted Family, Bivariate Quadratic Family, Bivariate Cubic Transmuted Family, Bivariate T-X Family, Two-stage Maximum Likelihood Method

## 1. Introduction

Many recent literatures have presented different methodologies for constructing bivariate and multivariate distributions. These distributions were formed in many ways. Some authors expanded univariate families to multivariate distributions. Bourguignon et al. [1] were the first who gave a generalization of univariate quadratic rank transmuted family to a bivariate case named a bivariate quadratic transmuted (BQT) family. Alizadeh et al. [2] introduced a univariate generalization of transmuted family whose baseline (c.d.f.) has an exponentiated form and extended it to a bivariate family. Alizadeh et al. [3] gave an examination of (BQT) family proposed by Bourguignon et al. [1] and introduced an alternative method based on mixing distributions of pairs order statistics for a sample of size two drawn from marginal density of (QRT) family, which allows for positive and negative dependence between variables. Another transmuted family proposed by

Bakouch et al. [4] which is a mixture of a standard u(0,1) density and a decreasing density gave a generalization to a bivariate case. This article reviews methods of generating multivariate distributions which contain many recent developments. Some of these developments based on joint distributions of order statistics, another methods based on mixtures of conditional specified distributions, multivariate skew families and many other methods such as construction of copulas, transformations, marginal replacements and other methods see [5]. In most of these ways, the bivariate and multivariate distributions are not unique. For this reason, continuous studies focus on proposing new ways to configure multivariate probability distributions which possess greater flexibility in applications of known classical distributions. The simplest way of obtaining bivariate distributions from univariate baseline distributions is a family introduced by Gumbel [6] which is called a bivariate Gumbel family. The joint c.d.f. of bivariate Gumbel family is given as:

$$F(x,y) = G(x)G(y)\left(1+\lambda\left(1-G(x)\big(1-G(y)\big)\right)\right) \quad ;\mathrm{X},\mathrm{Y}\in\mathrm{R}^2 \tag{1}$$

Where $G(X)$ and $G(y)$ are the c.d.f.'s of x, y in baseline distribution and λ is the association parameter such that $-1 \le \lambda \le 1$. The family in (1) is used to generate many bivariate distributions. Sarabia et al. [7] proposed three bivariate classes of Beta G families, and these families were constructed by definitions of three bivariate distributions, namely the first one a stochastic representation of inverse functions of teo genuine c.d.f., the second class induced by bivariate Beta distribution having univariate Beta G marginals, the third family more general than the marginal distributions that allow for any linear correlation coefficients. Ganji et al. [8] have generalized the transformed transformer (T-X) family introduced by Ayman et al. [9] to a bivariate (T-X) family of distributions. The bivariate c.d.f. of this family is:

$$F(x,y) = \int_0^{G_2(Y)}\int_0^{G_1(X)} r(u_1,u_2)du_1du_2 \qquad ;0<u_1,u_1<1;\mathrm{x},\mathrm{y}\in\mathrm{R}^2 \tag{2}$$

Where $r(\underline{u})$, $\underline{\mathrm{u}} = (\mathrm{u}_1,\mathrm{u}_2)'$ defined on support $[0,1]\times[0,1]$ to range $[0,1]$ and $G_1(X)$, $G_2(Y)$ are cumulative c.d.f.'s of x and y in the baseline distributions respectively. The Gumbel family defined in (1) can be found in (2) when the joint p.d.f. $r(u_1,u_2)$ is:

$$r(u_1,u_2) = 1+\lambda(1-2u_1)(1-2u_2) \;\; ;a<u_1,u_2<1$$

A new bivariate quadratic transmuted family has been proposed by Darwish et al. [10] who defined $r(\underline{u})$ as:

$$r(\underline{u}) = 1+\lambda_1(1-2u_1)+\lambda_2(1-2u_2)+2\lambda_3(1-u_1-u_2) \quad ;0<u_1,u_1<1 \tag{3}$$

Where $\lambda_1,\lambda_2,\lambda_3$ are transmutation parameters, such that $\lambda_i\in[-1,1]$ for i = 1,2,3 and $-1\le\lambda_1+\lambda_3\le 1$ and $-1\le\lambda_2+\lambda_3\le 1$. Darwish et al. [11] gave a generalization of a bivariate transmuted family to a p-variate transmuted family, the p.d.f. $r(\underline{u}),\underline{u}=(u_1,u_2,u_3,\cdots,u_p)'$ are defined as:

$$r(\underline{u}) = 1+\sum_{i=1}^{p}\lambda_i(1-2u_i)+\lambda_{p+1}\left(p-2\sum_{i=1}^{p}u_i\right) \tag{4}$$

Where $\underline{\lambda}=(\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_{p+1})'$ is a vector of transmutation parameters, such that $-1\le\lambda_i\le 1$ for $i=1,2,\dots,p+1$, $-1\le\lambda_i+\lambda_{p+1}\le 1$ for $i=1,2,\dots,p$ and $-1\le\sum_{i=1}^{p}\lambda_i+p\lambda_{p+1}\le 1$.

Depending on multivariate T-X family, the multivariate c.d.f. of $\underline{x}=(x_1,x_2,\cdots,x_p)'$ is:

$$F(\underline{x}) = \int_0^{G_p(X_p)}\cdots\int_0^{G_2(X_2)}\int_0^{G_1(X_1)} r(\underline{u})d\underline{u} \tag{5}$$

Substituting (4) into (5), we get the joint c.d.f. of $\underline{\mathrm{X}}$ as:

$$F(\underline{x}) = \prod_{i=1}^{p}G_i(x_i)\left(1+\sum_{i=1}^{p}(\lambda_i+\lambda_{p+1})\big(1-G_i(x_i)\big)\right) \;\; ;\underline{\mathrm{x}}\in\mathrm{R}^{\mathrm{p}} \tag{6}$$

Where $G_i(x_i)$ is a c.d.f. of $\mathrm{X}_\mathrm{i}$ in the baseline distribution. The p-variate joint c.d.f. defined in [6] reduces to a bivariate transmuted family if *p=2* [10], but if *p=1* and $\lambda_{p+1}=0$, the multivariate transmuted family reduces to univariate quadratic transmuted family which is a special case of the K-transmuted family [12].

In this paper, we introduce a new bivariate transmuted family of distribution, this family is a generalization of [10] which includes a cubic order of the baseline c.d.f. This new family called a bivariate cubic transmuted (*BCT*) family of distributions, and it can be used for any baseline distribution that has a closed form of the c.d.f.

This paper is organized as: Section 2 introduces a new bivariate cubic transmuted family of distributions. Section 3 contains the construction of a bivariate cubic transmuted Frechet distribution with its properties. Section 4 deals with application of real data set. The last Section contains important conclusions.

## 2. New Bivariate Cubic Transmuted Family of Distributions

It is well known that a bivariate T-X family defined in (4) generates different forms of bivariate distributions. This generation depends on the choice of the mathematical form of the bivariate joint p.d.f. $r(u_1, u_2)$ . We introduced a new form of this function:

$$r(u_1,u_2) = 1 + \lambda_{11}(1-2u_1) + \lambda_{21}(1-2u_2) + \lambda_3(2-2(u_1+u_2)) + 2\lambda_{12}u_1u_2(2-3u_1) + 2\lambda_{22}u_1u_2(2-3u_2) \quad (7)$$

Where $\underline{u} \in [0,1]^2$ and $-1 \le \lambda_{ij}, \lambda_3 \le 1$ for $i,j = 1,2$ , $-1 \le \lambda_{11} + \lambda_3 \le 1$ , $-1 \le \lambda_{21} + \lambda_3 \le 1$ , $-2 \le \lambda_{11} + \lambda_{12} + 2\lambda_3 \le 1$ , $-2 \le \lambda_{21} + \lambda_{22} + 2\lambda_3 \le 1$ This function defined in (7) generates a new bivariate cubic transmuted family. After putting (7) into (4), the bivariate c.d.f. of a random vector $\underline{X} = (X_1, X_2)'$ is:

$$F(\underline{x}) = G_1(x_1)G_2(x_2)\Big(1 + (\lambda_{11}+\lambda_3)\big(1-G_1(x_1)\big) + (\lambda_{21}+\lambda_3)\big(1-G_2(x_2)\big) + \lambda_{12}G_1(x_1)G_2(x_2)\big(1-G_1(x_1)\big) + \lambda_{22}G_1(x_1)G_2(x_2)\big(1-G_2(x_2)\big)\Big) \quad (8)$$

Where $\underline{x} \in R^2$, $G_i(X_i)$ is a baseline c.d.f. of a random variable $X_i$ for $i = 1,2$ and $\underline{\lambda} = (\lambda_{11}\ \ \lambda_{21}\ \ \lambda_3\ \ \lambda_{12}\ \ \lambda_{22})'$ is a vector of transmutation parameters.

Taking the second partial derivative with respect to $X_1$ and $X_2$ to both sides of (8), the joint p.d.f. of $\underline{x}$ is:

$$f(X_1,X_2) = g_1(X_1)g_2(X_2)\Big(1 + (\lambda_{11}+\lambda_3)\big(1-2G_1(X_1)\big) + (\lambda_{21}+\lambda_3)\big(1-2G_2(X_2)\big) + 2\lambda_{12}G_1(X_1)G_2(X_2)\big(2-3G_1(X_1)\big) + 2\lambda_{22}G_1(X_1)G_2(X_2)\big(2-3G_2(X_2)\big)\Big) \quad (9)$$

Where $g_1(X_1), g_2(X_2)$ are a baseline p.d.f. for $x_1$ and $x_2$ respectively. A bivariate random vector $\underline{x}$ whose joint c.d.f. and p.d.f. are defined in (8,9) respectively is a bivariate cubic transmuted $CT_2$ family and denoted by $\underline{x} \sim CT_2(\underline{\lambda}, \underline{\theta})$ where $\underline{\theta}$ contains the parameters of two baseline distributions, i.e. $\underline{\theta} = \big(\underline{\theta_1}', \underline{\theta_2}'\big)'$.[10] is a special case of $CT_2$ family when $\lambda_{12} = \lambda_{22} = 0$.

### 2.1. Marginal Distribution

$$F(x_1) = \lim_{x_2\to\infty} F(\underline{x}) = G_1(x_1)\Big(1 + (\lambda_{11}+\lambda_3)\big(1-G_1(x_1)\big) + \lambda_{12}G_1(x_1)\big(1-G_1(x_1)\big)\Big);\ \ x_1 \in R \quad (10)$$

In the same way, the marginal c.d.f. of $X_2$ is:

$$F(x_2) = \lim_{x_2\to\infty} F(\underline{x}) = G_2(x_2)\Big(1 + (\lambda_{21}+\lambda_3)\big(1-G_2(x_2)\big) + \lambda_{22}G_2(X_2)\big(1-G_2(x_2)\big)\Big);\ \ X_2 \in R \quad (11)$$

Taking the first derivative with respect to $X_1$ to both sides of (10), the marginal p.d.f. of $X_1$ is:

$$f(X_1) = g_1(X_1)\Big(1 + (\lambda_{11}+\lambda_3)\big(1-2G_1(X_1)\big) + \lambda_{12}G_1(X_1)\big(2-3G_1(X_1)\big)\Big);\ \ X_1 \in R \quad (12)$$

And zero otherwise.

Similarly, the marginal p.d.f. of $X_2$ is:

$$f(X_2) = g_2(X_2)\Big(1 + (\lambda_{21}+\lambda_3)\big(1-2G_2(X_2)\big) + \lambda_{22}G_2(X_2)\big(2-3G_2(X_2)\big)\Big);\ \ X_2 \in R \quad (13)$$

And zero otherwise.

The marginal c.d.f. and p.d.f. for each variable $x_i$ for $i = 1,2$ are a univariate cubic transmuted family which is a special case of the general transmuted family of distribution introduced by [12].

### 2.2. Conditional Distributions

The conditional p.d.f. of $X_1|X_2 = x_2$ is:

$$f(X_1|X_2 = x_2) = C_1(X_2)g_1(X_1)\Big(1 + (\lambda_{11}+\lambda_3)\big(1-2G_1(X_1)\big) + (\lambda_{21}+\lambda_3)\big(1-2G_2(X_2)\big) + 2\lambda_{12}G_1(X_1)G_2(X_2)\big(2-3G_1(X_1)\big) + 2\lambda_{22}G_1(X_1)G_2(X_2)\big(2-3G_2(X_2)\big)\Big) \quad (14)$$

where

$$C_1(X_2) = \left(1 + (\lambda_{21} + \lambda_3)\big(1 - 2G_2(X_2)\big) + \lambda_{22}G_2(X_2)\big(2 - 3G_2^2(X_2)\big)\right)^{-1} \quad (15)$$

The conditional p.d.f. of $(X_2|X_1 = x_1)$ is:

$$f(X_2|X_1 = x_1) = d_1(X_1)g_2(X_2)\Big(1 + (\lambda_{11} + \lambda_3)\big(1 - 2G_1(X_1)\big) + (\lambda_{21} + \lambda_3)\big(1 - 2G_2(X_2)\big) + 2\lambda_{12}G_1(X_1)G_2(X_2)\big(2 - 3G_1(X_1)\big) + 2\lambda_{22}G_1(X_1)G_2(X_2)\big(2 - 3G_2(X_2)\big)\Big) \quad (16)$$

where

$$d_1(X_1) = \left(1 + (\lambda_{11} + \lambda_3)\big(1 - 2G_1(X_1)\big) + \lambda_{12}G_1(X_1)\big(2 - 3G_1^2(X_1)\big)\right)^{-1} \quad (17)$$

The conditional c.d.f. of $(X_1|X_2 = x_2)$ is:

$$\begin{aligned} F(x_1|X_2 = x_2) &= \int_0^{x_1} f(X_2|X_1 = x_1)dX_1 \\ &= C_1(X_2)\Bigg(\int_0^{x_1} g_1(X_1)dX_1 \ \Bigg(1 + (\lambda_{11} + \lambda_3)\int_0^{x_1}\big(1 - 2G_1(X_1)\big)g_1(X_1)dX_1 \\ &+ (\lambda_{21} + \lambda_3)\big(1 - 2G_2(X_2)\big)\int_0^{x_1} g_1(X_1)dX_1 \\ &+ 2\lambda_{12}G_2(X_2)\left(2\int_0^{x_1} g_1(X_1)\ G_1(X_1)dX_1 - 3\int_0^{x_1} g_1(X_1)G_1^2(X_1)dX_1\right) \\ &+ 2\lambda_{22}G_2(X_2)\big(2 - 3G_2(X_2)\big)\int_0^{x_1} g_1(X_1)G_1(X_1)dX_1\Bigg)\Bigg) \end{aligned} \quad (18)$$

Define a one-to-one transformation $V_1 = G_1(X_1),\ dV_1 = g_1(X_1)dX_1$. After some mathematical simplifications of (18), the conditional c.d.f. of $X_1|X_2 = x_2$ is:

$$\begin{aligned} F(x_1|X_2 = x_2) &= C_1(X_2)G_1(x_1)\Big(1 + (\lambda_{11} + \lambda_3)\big(1 - G_1(x_1)\big) + 2\lambda_{12}G_1(x_1)G_2(x_2)\big(1 - G_1(x_1)\big) \\ &+ (\lambda_{21} + \lambda_3)\big(1 - 2G_2(x_2)\big) + \lambda_{22}G_1(x_1)G_2(x_2)\big(2 - 3G_2(x_2)\big)\Big) \end{aligned} \quad (19)$$

In the same way, the conditional c.d.f. of $X_2|X_1 = x_1$ is:

$$\begin{aligned} F(x_2|X_1 = x_1) &= d_1(X_1)G_2(x_2)\Big(1 + (\lambda_{21} + \lambda_3)\big(1 - G_2(x_2)\big) + (\lambda_{11} + \lambda_3)\big(1 - 2G_1(x_1)\big) \\ &+ \lambda_{12}G_1(x_1)G_2(x_2)\big(2 - 3G_1(x_1)\big) + 2\lambda_{22}G_1(x_1)G_2(x_2)\big(1 - G_2(x_2)\big)\Big) \end{aligned} \quad (20)$$

Marginal and conditional distributions are very important in simulating $CT_2$ independent bivariate vectors of observations. We proposed a new algorithm to generate random samples from $CT_2$ family. The steps of this algorithm are:

1- Generate a random sample of size (n) from standard continuous uniform distribution i.e. $V_{1i} \sim U(0,1)$ for $i = 1,2,\cdots,n$.
2- Equate random observations $V_{1i}$ generated in step (1) with the marginal c.d.f. $R(u_1)$ and solve the result of the nonlinear equation with respect to $u_i$ where:

$$R(u_1) = \int_0^{u_1}\int_0^1 r(u_1, u_2)\,du_2du_1 = u_1\big(1 + (\lambda_{11} + \lambda_3)(1 - u_1) + \lambda_{12}u_1(1 - u_1)\big) \quad (21)$$

3- Equate $u_{1i}$ found in step (2) with the marginal c.d.f. of $X_1$ defined in (10) and solve the nonlinear equation $u_{1i} = F(x_{1i})$ with respect to $x_{1i}$.
4- Put $u_{1i}$ in the conditional c.d.f. of $u_2|u_1$ where:

$$\begin{aligned} R(u_2|u_1 = v_1) &= b_1(u_1)u_2\big(1 + (\lambda_{11} + \lambda_3)(1 - 2u_1) + (\lambda_{21} + \lambda_3)u_1(1 - u_2) + \lambda_{12}u_1u_2(2 - 3u_1) \\ &+ 2\lambda_{22}u_1u_2(1 - u_2)\big) \end{aligned} \quad (22)$$

Where

$$b_1(u_1) = 1 + (\lambda_{11} + \lambda_3)(1 - 2u_1) + \lambda_{12}u_1(2 - 3u_1) \tag{23}$$

5- Generate random observations of size (n) from $U(0,1)$ and equate them with (22).

$$V_{2i} = \ R(u_{2i}|u_2 = v_{2i}) \tag{24}$$

Solve (24) with respect to $u_{2i}$.

6- Equate the results of step (5) with $F(x_2|X_1 = x_1)$ after substituting the result of step (3) in it and solving the non-linear equation $V_{2i} = F(x_{2i}|X_1 = x_{1i})$ with respect to $X_2$.

7- The results of steps (3and 6) represent the bivariate vectors of random observations $(X_{1i}\ \ X_{2i})$ for $i = 1,2,\cdots,n$.

### 2.3. Moments

In this section, we find the r,sth joint moments between $X_1$ and $X_2$, the marginal and conditional moments of some orders. The joint r,sth moment between $X_1$ and $X_2$ is:

$$E(X_1^r X_2^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1^r\ \ X_2^s\ \ f(X_1, X_2)dX_1 dX_2 \tag{25}$$

Substituting (5) into (25) and making some mathematical simplifications, the r,sth joint moment is:

$$\begin{aligned} &E(X_1^r X_2^s) \\ &= E^*X_1^r\ \ E^*X_2^s + (\lambda_{11} + \lambda_3)E^*X_1^r\ \ E^*X_2^s - (\lambda_{11} + \lambda_3)E^*X_{1(2,2)}^r\ \ E^*X_2^s + (\lambda_{21} + \lambda_3)E^*X_1^r\ \ E^*X_2^s \\ &- (\lambda_{21} + \lambda_3)E^*X_1^r\ \ E^*X_{2(2,2)}^s + \lambda_{12}E^*X_{1(2,2)}^r E^*X_{2(2,2)}^s - \lambda_{12}E^*X_{1(3,3)}^r E^*X_{2(2,2)}^s + \lambda_{22}E^*X_{1(2,2)}^r E^*X_{2(2,2)}^s \\ &- \lambda_{22}E^*X_{1(2,2)}^r E^*X_{2(3,3)}^s \end{aligned} \tag{26}$$

Where $E^*X_1^r, E^*X_2^s$ are the rth and sth moments around zero of $X_1$ and $X_2$ depending on baseline distributions, and $E^*X_{1(2,2)}^r E^*X_{2(2,2)}^s$ are the rth and sth moments of $X_1$ , $X_2$ based on the largest order statistic of random samples of size 2 taken from the baseline distributions. Also $E^*X_{1(3,3)}^r E^*X_{2(3,3)}^s$ are the rth and sth moments around zero for $X_1$ and $X_2$ based on largest order statistic of random samples of size 3 taken from the baseline distributions.

Let $s = 0$ in (26), the rth moment around zero of $X_1$ is:

$$E(X_1^r) = (1 + \lambda_{11} + \lambda_3)E^*X_1^r + \big(\lambda_{12} - (\lambda_{11} + \lambda_3)\big)E^*X_{1(2,2)}^r - \lambda_{12}E^*X_{1(3,3)}^r \tag{27}$$

In the same way, put $r = 0$, the sth moment around zero of $X_2$ is:

$$E(X_2^s) = (1 + \lambda_{21} + \lambda_3)\ \ E^*X_2^s + \big(\lambda_{22} - (\lambda_{21} + \lambda_3)\big)\ \ E^*X_{2(2,2)}^s - \lambda_{22}E^*X_{2(3,3)}^s \tag{28}$$

The rth conditional moment around zero of $X_1|X_2 = x_2$ is:

$$E(X_1^r|X_2 = x_2) = \int_{-\infty}^{\infty} X_1^r\ \ f(X_1|X_2 = x_2)dX_1 \tag{29}$$

Substituting (14) into (29), we get:

$$\begin{aligned} E(X_1^r|X_2 = x_2) = C_1(X_2)\Big(&(1 + \lambda_{11} + \lambda_{21} + 2\lambda_3)E^*X_1^r - 2G_2(X_2)(\lambda_{21} + \lambda_3)E^*X_1^r + \big(2G_2(X_2)(\lambda_{12} + \lambda_{22}) - \\ &3\lambda_{22}G_2^2(X_2) - (\lambda_{11} + \lambda_3)\big)E^*X_{1(2,2)}^r - 2\lambda_{12}G_2(X_2)E^*X_{1(3,3)}^r\Big) \end{aligned} \tag{30}$$

Similarly, the sth conditional moment around zero of $X_2|X_1 = x_1$ is:

$$\begin{aligned} E(X_2^s|X_1 = x_1) = d_1(X_1)\Big(&\big((1 + \lambda_{11} + \lambda_{21} + 2\lambda_3) - 2(\lambda_{11} + \lambda_3)G_1(X_1)\big)E^*X_2^s + \big(2G_1(X_1)(\lambda_{12} + \lambda_{22}) - \\ &(\lambda_{21} + \lambda_3) - 3\lambda_{12}G_1^2(X_1)\big)E^*X_{2(2,2)}^s - 2\lambda_{22}G_1(X_1)E^*X_{2(3,3)}^s\Big) \end{aligned} \tag{31}$$

### 2.4. Independence Measures

There are two measures to study the relationships between the components of a random vector $\underline{X}$. The first measure is Kendall's Tau coefficient in continuous bivariate case [10], which is defined as:

$$\tau_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\underline{X})\ \ f(\underline{X})\partial X_1\,\partial X_2 \tag{32}$$

Substitute (8,9) into (32) and define two one-to-one transformations $w_1 = G_1(X_1)$ and $w_2 = G_2(X_2)$, and integrate, we get:

$$\tau_2 = \frac{1}{45}\left(\lambda_{12}(\lambda_{11}+\lambda_3)+\lambda_{22}(\lambda_{21}+\lambda_3)\right) - \frac{1}{9}\left((\lambda_{12}+\lambda_{22})+2(\lambda_{11}+\lambda_3)(\lambda_{21}+\lambda_3)+\lambda_{22}(\lambda_{11}+\lambda_3)+\lambda_{12}(\lambda_{21}+\lambda_3)+\frac{\lambda_{12}^2}{3}-\frac{2\lambda_{12}\lambda_{22}}{25}\right) \quad (33)$$

The Spearman Rho coefficient between two variables of continuous distribution [10] is defined as:

$$\rho = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\underline{x})\prod_{i=1}^{2} f(x_i)\,\partial X_1\,\partial X_2 \quad (34)$$

Put equations (8,12,13) into (34) above and define two one-to-one transformations $w_1 = G_1(X_1)$ and $w_2 = G_2(X_2)$ and make some mathematical simplifications:

$$\rho = \frac{\begin{array}{c}9-0.5\lambda_{22}-(\lambda_{11}+\lambda_3)(\lambda_{21}+\lambda_3)-0.5(\lambda_{11}+\lambda_3)\lambda_{22}-0.5\lambda_{12}-0.5(\lambda_{21}+\lambda_3)\lambda_{12}\\ -0.35\lambda_{12}\lambda_{22}+0.1(\lambda_{11}+\lambda_3)\lambda_{12}+0.01(\lambda_{11}+\lambda_3)\lambda_{12}\lambda_{22}+0.1(\lambda_{21}+\lambda_3)\lambda_{22}\end{array}}{36} \quad (35)$$

# 3. Bivariate Cubic Transmuted Frechet Distribution

In this section, we use a $CT_2$ family to introduce a new Bivariate Frechet distribution. This new distribution is named as the bivariate cubic transmuted Frechet $(CT_2Fr)$ distribution. The baseline distributions for two random variables $X_1 \sim Fr(\alpha_1,\beta_1)$ and $X_2 \sim Fr(\alpha_2,\beta_2)$ where the p.d.f. and c.d.f. are:

$$\left.\begin{array}{ll} g_1(X_1) = \alpha_1\ \beta_1^{\alpha_1}X_1^{-(\alpha_1+1)}e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}} & ;\alpha_1,\beta_1,X_1>0 \\ g_2(X_2) = \alpha_2\ \beta_2^{\alpha_2}X_2^{-(\alpha_2+1)}e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}} & ;\alpha_2,\beta_2,X_2>0 \\ G_1(X_1) = e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}} \quad ; & G_2(X_2) = e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}} \end{array}\right\} \quad (36)$$

Referring to (8) and (9), the bivariate cubic transmuted Frechet c.d.f. and p.d.f. are:

$$F(\underline{X}) = e^{-\left(\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}\right)}\left(1+(\lambda_{11}+\lambda_3)\left(1-e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right)+(\lambda_{21}+\lambda_3)\left(1-e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right) + \lambda_{12}e^{-\left(\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}\right)}\left(1-e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right)+\lambda_{22}e^{-\left(\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}\right)}\left(1-e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right)\right) \quad (37)$$

$$f(\underline{X}) = \alpha_1\ \beta_1^{\alpha_1}X_1^{-(\alpha_1+1)}e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\ \alpha_2\ \beta_2^{\alpha_2}X_2^{-(\alpha_2+1)}e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}} \left(1+(\lambda_{11}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right)+(\lambda_{21}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right)+2\lambda_{12}e^{-\left(\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}\right)}\left(2-3e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right) + 2\lambda_{22}e^{-\left(\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}\right)}\left(2-3e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right)\right) \quad (38)$$

Where $\alpha_1,\beta_1,X_1>0$ for $i=1,2$ and $-1\le\lambda_{i1}+\lambda_3\le 1$ for $=1,2$ , $-2\le\lambda_{12}\le 1$ and $-2\le\lambda_{22}\le 1$. This distribution whose joint p.d.f. is defined in (38) is denoted by $CT_2Fr\left(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_{ij},\lambda_3\right)$ for $i=1,2$.

### 3.1. Marginal and Conditional Distributions

The marginal c.d.f. and p.d.f. for each variable are:

$$F(X_1) = e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\left(1+(\lambda_{11}+\lambda_3)\left(1-e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right)+\lambda_{12}e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\left(1-e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right)\right) \quad ; X_1 \in R \tag{39}$$

$$F(X_2) = e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\left(1+(\lambda_{21}+\lambda_3)\left(1-e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right)+\lambda_{22}e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\left(1-e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right)\right) \quad ; \mathrm{X}_2 \in \mathrm{R} \tag{40}$$

$$f(X_1) = \alpha_1\ \beta_1^{\alpha_1}X_1^{-(\alpha_1+1)}e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\left(1+(\lambda_{11}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right)+\lambda_{12}\left(2e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}-3e^{-2\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right)\right); X_1 \in R \tag{41}$$

And zero otherwise.

$$f(X_2) = \alpha_2\ \beta_2^{\alpha_2}X_2^{-(\alpha_2+1)}e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\left(1+(\lambda_{21}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right)+\lambda_{22}\left(2e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}-3e^{-2\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right)\right); X_2 \in R \tag{42}$$

And zero otherwise.

Putting (36), (37) into (14-17) the conditional density function for each variable is:

$$\begin{aligned} f(X_1|X_2=x_2) = \alpha_1\ \beta_1^{\alpha_1}X_1^{-(\alpha_1+1)}e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\Bigg(&1+(\lambda_{11}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right) \\ &+2\lambda_{12}e^{-\left(\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}\right)}\left(2-3e^{-\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}}\right)+(\lambda_{21}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right) \\ &+2\lambda_{22}e^{-\left(\left(\frac{\beta_1}{X_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}\right)}\left(2-3e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right)\Bigg)C_1(x_2) \end{aligned} \tag{43}$$

where

$$C_1(x_2) = \left(1+(\lambda_{21}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right)+\lambda_{22}\left(2-3e^{-2\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right)\right)^{-1} \tag{44}$$

and

$$\begin{aligned} f(X_2|X_1=x_1) = \alpha_2\ \beta_2^{\alpha_2}X_2^{-(\alpha_2+1)}e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\Bigg(&1+(\lambda_{11}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right)+(\lambda_{21}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right) \\ &+2\lambda_{12}e^{-\left(\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}\right)}\left(2-3e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right)+2\lambda_{22}e^{-\left(\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}\right)}\left(2-3e^{-\left(\frac{\beta_2}{X_2}\right)^{\alpha_2}}\right)\Bigg)d_1(x_1) \end{aligned} \tag{45}$$

where

$$d_1(x_1) = \left(1+(\lambda_{11}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right)+\lambda_{12}e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\left(2-3e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right)\right)^{-1} \tag{46}$$

Using (37) into (19,20), the conditional c.d.f. of each variable is:

$$\begin{aligned} F(x_1|X_2=x_2) = C_1(x_2)e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\Bigg(&1+(\lambda_{11}+\lambda_3)\left(1-e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right)+2\lambda_{12}e^{-\left(\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}\right)}\left(1-e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right) \\ &+(\lambda_{21}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right)+\lambda_{22}e^{-\left(\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}\right)}\left(2-3e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right)\Bigg) \end{aligned} \tag{47}$$

$$F(x_2|X_1=x_1)=d_1(x_1)e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\left(1+(\lambda_{21}+\lambda_3)\left(1-e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right)+\lambda_{12}e^{-\left(\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}\right)}\left(2-3e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right)\right.$$
$$\left.+(\lambda_{11}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right)+2\lambda_{22}e^{-\left(\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}+\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}\right)}\left(1-e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right)\right) \quad (48)$$

### 3.2. Moments

Depending on rth and sth moments around zero for $X_1$ and $X_2$ respectively based on the p.d.f. $g_1(X_1), g_2(X_2)$ defined in (40) and their rth and sth moments of largest order statistics depending on random samples of size two and three taken from $g_1(X_1)$ and $g_2(X_2)$ respectively, the joint rth, sth moment of $CT_2Fr$ is:

$$E(X_1^rX_2^s)=\beta_1^r\beta_2^s\ \Gamma\left(1-\frac{r}{\alpha_1}\right)\Gamma\left(1-\frac{s}{\alpha_2}\right)\left(1+(\lambda_{11}+\lambda_{21}+\lambda_3)-(\lambda_{11}+\lambda_3)2^{\frac{r}{\alpha_1}}-(\lambda_{21}+\lambda_3)2^{\frac{s}{\alpha_2}}\right.$$
$$\left.+(\lambda_{12}+\lambda_{22})2^{\frac{r}{\alpha_1}+\frac{s}{\alpha_2}}-\lambda_{12}3^{\frac{r}{\alpha_1}}2^{\frac{s}{\alpha_2}}-\lambda_{22}2^{\frac{r}{\alpha_1}}3^{\frac{s}{\alpha_2}}\right) \quad (49)$$

provided that $\frac{r}{\alpha_1}<1$ and $\frac{s}{\alpha_2}<1$.

Put $s=0$ into (49) above, and the rth marginal moment around zero for $X_1$ is:

$$E(X_1^r)=\beta_1^r\ \Gamma\left(1-\frac{r}{\alpha_1}\right)\left(1+(\lambda_{11}+\lambda_3)+(\lambda_{12}-\lambda_{11}-\lambda_3)2^{\frac{r}{\alpha_1}}-\lambda_{12}3^{\frac{r}{\alpha_1}}\right) \quad (50)$$

provided that $\frac{r}{\alpha_1}<1$.

Also put $r=0$ and provided that $\frac{s}{\alpha_2}<1$, the sth marginal moment around zero for $X_2$ is:

$$E(X_2^s)=\beta_2^s\ \Gamma\left(1-\frac{s}{\alpha_2}\right)\left(1+(\lambda_{21}+\lambda_3)+(\lambda_{22}-\lambda_{12}-\lambda_3)2^{\frac{s}{\alpha_2}}+(\lambda_{12}+\lambda_{22})2^{\frac{r}{\alpha_1}+\frac{s}{\alpha_2}}-\lambda_{22}3^{\frac{s}{\alpha_2}}\right) \quad (51)$$

The rth conditional moment around zero of $X_1^r|X_2=x_2$ is:

$$E(X_1^r|X_2=x_2)=C_1(x_2)\beta_1^r\ \Gamma\left(1-\frac{r}{\alpha_1}\right)\left(\left(1+(\lambda_{11}+\lambda_3)+(\lambda_{21}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}\right)\right)\right.$$
$$\left.+2^{\frac{r}{\alpha_1}}\left(2\lambda_{12}e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}+2\lambda_{22}e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}-3\lambda_{22}e^{-2\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}-(\lambda_{11}+\lambda_3)\right)-2\lambda_{12}e^{-\left(\frac{\beta_2}{x_2}\right)^{\alpha_2}}3^{\frac{r}{\alpha_1}}\right) \quad (52)$$

and the sth conditional moment around zero of $X_2^s|X_1=x_1$ is:

$$E(X_2^s|X_1=x_1)=d_1(x_1)\beta_2^s\ \Gamma\left(1-\frac{s}{\alpha_2}\right)\left(\left(1+(\lambda_{21}+\lambda_3)+(\lambda_{11}+\lambda_3)\left(1-2e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right)\right)\right.$$
$$\left.+2^{\frac{s}{\alpha_2}}\left(2\lambda_{12}e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}+2\lambda_{22}e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}-3\lambda_{12}e^{-2\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}-(\lambda_{21}+\lambda_3)\right)-2\lambda_{22}3^{\frac{s}{\alpha_2}}e^{-\left(\frac{\beta_1}{x_1}\right)^{\alpha_1}}\right) \quad (53)$$

The conditional moments defined in (52) and (53) exist when $\frac{r}{\alpha_1}<1$ and $\frac{s}{\alpha_2}<1$.

### 3.3. Estimation

In cases when the numbers of parameters or variables of multivariate distribution are large, the derivation of the maximum likelihood estimators requires an intensive computational burden and is not feasible. To avoid this problem, the two-stage maximum likelihood method [13] can be used to estimate the parameters of the $(CT_2Fr)$ distribution.

Depending on complete random vectors of observations taken from (6), the log-likelihood function is:

$$ln\ L(\underline{\theta}) = \sum_{i=1}^{n} lng_1(X_{1i}, \underline{\theta_1}) + \sum_{i=1}^{n} lng_2(X_{2i}, \underline{\theta_2}) + \sum_{i=1}^{n} ln\Big[\Big(1 + (\lambda_{11} + \lambda_3)\big(1 - 2G_1(X_{1i})\big)\Big) + (\lambda_{21} + \lambda_3)\big(1 - 2G_2(X_{2i})\big) + 2\lambda_{12}G_1(X_{1i})G_2(X_{2i})\big(2 - 3G_1(X_{1i})\big) + 2\lambda_{22}G_1(X_{1i})G_2(X_{2i})\big(2 - 3G_2(X_{2i})\big)\Big] \quad (54)$$

Where $g_1(.), g_2(.), G_1(.), G_2(.)$ are defined in (36) and $\underline{\theta_1} = (\alpha_1, \beta_1)', \underline{\theta_2} = (\alpha_2, \beta_2)', \underline{\theta} = (\theta_1', \theta_2', \underline{\lambda}')$.

In the first stage, the maximum likelihood estimators for $\underline{\theta}_1$ and $\underline{\theta}_2$ for the baseline distributions $g_1(.), g_2(.)$ are obtained separately. These estimators are $\underline{\hat{\theta}}_1, \underline{\hat{\theta}}_2$ computed by solving, $\frac{\partial \ell(\ \theta_1)}{\partial \underline{\theta_1}} = 0$ and $\frac{\partial \ell(\ \theta_1)}{\partial \underline{\theta_1}} = 0$.

where:

$$\ell_j(\underline{\theta_j}) = n\ \ ln(\alpha_j) + n\alpha_j\ \ ln(\beta_j) - (\alpha_j + 1)\sum_{i=1}^{n} ln(X_{ji}) - \left(\frac{\beta_j}{X_j}\right)^{\alpha_j} \quad \text{for} \quad j = 1,2 \quad (55)$$

These estimators satisfy that $\left.\frac{\partial^2 \ell_j(\ \theta_j)}{\partial \underline{\theta}_j, \partial \underline{\theta}_{j'}}\right|_{\underline{\theta}_j = \underline{\hat{\theta}}_j}$ for j=1,2 is a negative definite matrix.

In the second stage, the maximization of $\ell_3(\underline{\theta_1}, \underline{\theta_2}, \underline{\lambda})$ is made via the vector of parameters $\underline{\lambda}$ with constraints on $\underline{\lambda}$.

Where:

$$\ell_3(\underline{\theta_1}, \underline{\theta_2}, \underline{\lambda}) = \sum_{i=1}^{n} ln\left[1 + (\lambda_{11} + \lambda_3)\left(1 - e^{-\left(\frac{\hat{\beta}_1}{X_{1i}}\right)^{\hat{\alpha}_1}}\right) + (\lambda_{21} + \lambda_3)\left(1 - e^{-\left(\frac{\hat{\beta}_2}{X_{2i}}\right)^{\hat{\alpha}_2}}\right) + \lambda_{12} e^{-\left(\left(\frac{\hat{\beta}_1}{X_{1i}}\right)^{\hat{\alpha}_1} + \left(\frac{\hat{\beta}_2}{X_{2i}}\right)^{\hat{\alpha}_2}\right)}\left(1 - e^{-\left(\frac{\hat{\beta}_1}{X_{1i}}\right)^{\hat{\alpha}_1}}\right) + \lambda_{22} e^{-\left(\left(\frac{\hat{\beta}_1}{X_{1i}}\right)^{\hat{\alpha}_1} + \left(\frac{\hat{\beta}_2}{X_{2i}}\right)^{\hat{\alpha}_2}\right)}\left(1 - e^{-\left(\frac{\hat{\beta}_2}{X_{2i}}\right)^{\hat{\alpha}_2}}\right)\right] \quad (56)$$

An estimator $\underline{\tilde{\lambda}}$ that maximizes (56) is found by using constrained nonlinear optimization, where the objective function and linear inequality constraints maximize $\ell_3(\underline{\theta_1}, \underline{\theta_2}, \underline{\lambda})$.

s.t:

$$\left.\begin{matrix} -1 \le \lambda_{i1} + \lambda_3 \le 1 \\ -2 \le \lambda_{i2} \le 2 \\ -1 \le \lambda_{i1}, \lambda_3 \le 1 \end{matrix}\right\} \quad \text{for} \quad i = 1,2$$

The two-stage ML estimator of $\underline{\theta}$ is $\underline{\tilde{\theta}} = (\underline{\hat{\theta}_1'}, \underline{\hat{\theta}_2'}, \underline{\hat{\lambda}}')'$, which satisfies $\left(\frac{\partial \ell_1(\ \theta_1)}{\partial \underline{\theta_1}} \quad \frac{\partial \ell_2(\ \theta_2)}{\partial \underline{\theta_2}} \quad \frac{\partial \ell_3(\hat{\theta}_1, \hat{\theta}_2, \lambda)}{\partial \underline{\lambda}}\right)' = \underline{0}$ .

# 4. Real Data Application

In this section, an economic Egyptian data set for World Bank national accounts. The data was taken from [14]. The data consists of two economic variables. The first variable $X_1$ represents the export of goods and services, the second variable represents the $GDP$ growth. The sample size is (31). The data set is modeled by our proposed $CT_2Fr$ distribution and three other bivariate families, which are as follows:

1- Farlie–Gamble-Morgenstern $(FGM)$ family see [15] whose joint p.d.f. of a bivariate random vector $\underline{X}$ has the following form:

$$f(\underline{X}) = g_1(X_1)g_2(X_2)\Big(1 + \lambda\big(1 - 2G_1(X_1) - 2G_2(X_2)\big) + 4G_1(X_1)G_2(X_2)\Big) \quad where \ \ -1 \le \lambda \le 1 \quad (57)$$

2- Bivariate transmuted family is proposed by [10]. This family is a quadratic transmuted family denoted by $QT_2$ family whose p.d.f. is:

$$f(\underline{X}) = g_1(X_1)g_2(X_2)\Big(1 + (\lambda_{11} + \lambda_3)\big(1 - G_1(X_1)\big) + (\lambda_{21} + \lambda_3)\big(1 - 2G_2(X_2)\big)\Big) \ \ ; \underline{X} \in R^2 \quad (58)$$

where $(\lambda_{11}\ , \lambda_{21}\ \ , \lambda_3) \in [-1,1]$ and $-1 \le \lambda_{i1} + \lambda_3 \le 1$ for i=1,2.

3- A bivariate family introduced by Darwish et al. [16] symbolize this family as DSAS the p.d.f. of this family defined as:

$$f(\underline{X}) = g_1(X_1)g_2(X_2)\left(1 + \lambda_{11}\big(1 - 2G_1(X_1)\big) + \lambda_{21}\big(1 - 2G_2(X_2)\big) + \lambda_3\big(1 - 4G_1(X_1)G_2(X_2)\big)\right); \underline{X} \in R^2 \quad (59)$$

where $-1 \leq \lambda_{i1}$ , $\lambda_3 \leq 1$ for i=1,2; $\lambda_{11} + \lambda_{21} + \lambda_3 \geq -1$ , $-1 \leq \lambda_{ij} + \lambda_3 \leq 1$ for i=1,2.

After putting $g_i(X_i)$ , $G_i(X_i)$ for i=1,2 defined in (36) on the above three families, the parameters were estimated by the two-stage $(ML)$ method, in the first stage the parameters of $g_1(X_1)g_2(X_2)$ were estimated separately by ordinary ML method. In the second stage the transmutation parameters were estimated by using the objective function defined on eq (56) with two sides, there are inequality constraints on transmutation parameters after substitution the values of estimators in the first stage. The results of two stage ML estimators are taken in Table 1. A comparison has been done between the estimators of four bivariate families of Frechet distribution by using two information criteria $(AIC)$ and $(BIC)$ in Table 2.

**Table 1.** The two-stage $(ML)$ estimators of the four bivariate Frechet distributions

| Parameter | $CT_2Fr$ | $FGMFr$ | $QT_2Fr$ | $DSASFr$ |
|---|---|---|---|---|
| $\alpha_1$ | 3.9925 | 3.9925 | 3.9925 | 3.9925 |
| $\alpha_2$ | 1.7968 | 1.7968 | 1.7968 | 1.7968 |
| $\beta_1$ | 19.4831 | 19.4831 | 19.4831 | 19.4831 |
| $\beta_2$ | 3.6548 | 3.6548 | 3.6548 | 3.6548 |
| $\lambda_{11}$ | 0.7 | | 1 | 0.507 |
| $\lambda_{21}$ | -0.816 | | -0.356 | -0.493 |
| $\lambda_{12}$ | 0.214 | | | |
| $\lambda_{22}$ | -1 | | | |
| $\lambda_3$ | -0.184 | | -0.644 | -0.507 |
| $\lambda$ | | 0.212 | | |

**Table 2.** $(AIC)$ and $(BIC)$ of the four bivariate Frechet distributions

| | $AIC$ | $BIC$ |
|---|---|---|
| $CT_2Fr$ | 236.8101 | 243.9801 |
| $FGMFr$ | 300.1424 | 307.3123 |
| $QT_2Fr$ | 259.7115 | 266.8815 |
| $DSASFr$ | 260.2220 | 267.3919 |

It is seen from Table 2 that the $(GT_2Fr)$ is the best fit to the data because it has the smallest $(AIC)$ and $(BIC)$. The second-best fit is the $(QT_2Fr)$ because it has the second smallest criteria. The $(FGMFr)$ does not seem to well fit the data. The joint p.d.f. and joint c.d.f. of $(CT_2Fr)$ and the histogram of the sample data are plotted in Figure 1. The plots are based on two-stage maximum likelihood estimators.

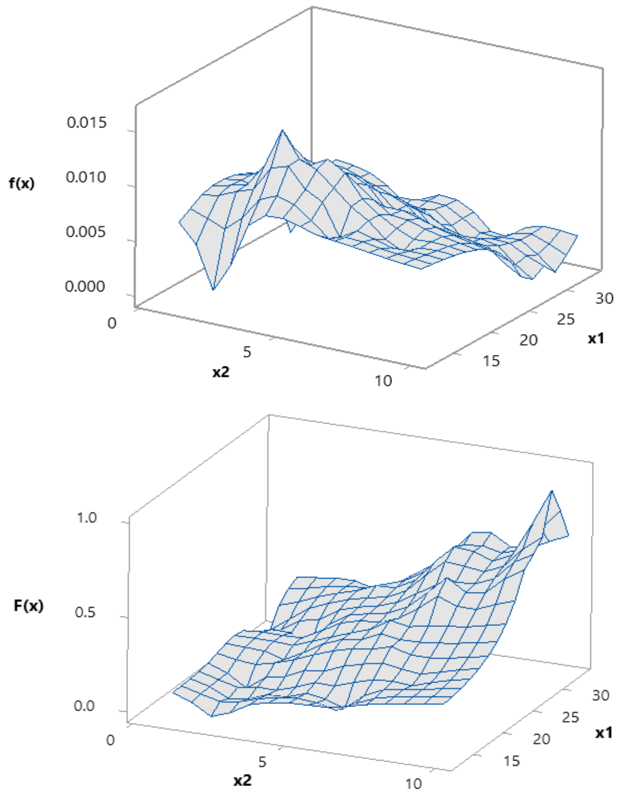


**Figure 1.** Plot the density function and the cumulative function fort $(CT_2Fr)$

## 5. Conclusions

In this article, we have introduced a new bivariate cubic transmuted $(CT_2)$ family of distributions. Various statistical properties of $(CT_2)$ family have been studied. It is seen that the marginal distributions belong to the univariate quadratic family. The $(CT_2)$ family is used on two independent Frechet variables to construct a new $(CT_2Fr)$ distribution. The Egyptian economic data was fitted by $(CT_2Fr, FGMFr, QT_2Fr)$ and $(DSASFr)$ distributions. The $(CT_2Fr)$ is the best fit to the data set. $(CT_2)$ family can be generalized to higher-order transmutation and to a multivariate case.

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