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The Locating and Local Locating Domination of Prism Family Graphs

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Abstract

In the fields o f c ombinatorics a nd g raph t heory, prism graphs are very important. They provide insights into the structural features of many real-world networks and act as a model for them. In graph theory, the study of dominant sets is essential for a variety of applications, including social network research and network design. A dominating set in a graph G is a subset D of vertices V having the property that each vertex w belongs to $V - D$ is neighbouring to at least one vertex D. Determining the minimum cardinal number of dominating sets, locating dominating sets, and local locating dominating sets is of critical importance in such fields as network design and social network analysis. In this paper, we determine these minimum cardinal bounds for families of prism graphs. The study adds to the basic understanding of graph theory by methodically disentangling the intricate relationships between dominating sets in prism graphs. The exploration of lowest cardinal value of locating dominating sets yields solutions to optimisation issues in network design. In this work, we determine the upper bounds of locating domination and local locating domination for prsim, antiprism, crossed prism and circulant ladder prism graph.

Keywords Domination, Locating Domination, Local Locating Domination, Prism Graphs

1 Introduction

A *dominating set* in a graph $G = (V, E)$ is a set of vertices $D \subseteq V$ having the property that each vertex $w \in V - D$ is neighbouring to at least one vertex $v \in D$. The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of a dominating set in G.

Although the roots of domination concept in graphs dates back to the mid-to-late 1800s in the game of chess, the first mathematical and graph theoretical treatment of domination in graphs was due to König in 1936 [1]. The next graph theory books to discuss domination in graphs were due to Berge in 1958 and later in 1962 [2] and Ore in 1962 [3]. The definitions and basics are from [4]. The first book to deal entirely with domination in graphs was published in 1998 by Haynes, Hedetniemi and Slater [5]. Since then the field of domination in graphs has seen explosive growth, to the point where today there are more than 5,000 papers on this subject.

The idea of a dominating set has numerous practical applications, from planning effective wireless communication networks to comprehending how illnesses propagate through a community. However, the problem of computing the domination number of a graph is a difficult and computationally fascinating subject that has drawn the interest of researchers, engineers, mathematicians, and computer scientists.

The study of interconnection networks for parallel and distributed computing systems often uses prism graphs. A basic type of *prism* consists of two copies G_1 and G_2 of a given graph G, which are connected to each other by the edges of a perfect matching, that is, every vertex in G_1 is connected by an edge to its corresponding vertex in G_2. Prism graphs provide a basis for comprehending the connectivity and communication patterns present in these systems, assisting in the optimization of information transfer across processors.

In this paper we study the techniques and approaches used to determine the domination numbers of graphs in several families of prisms. Imagine, for example, installing a wireless sensor network in a farmland to keep an eye on the soil quality. The main function of the networks many sensors, which are dispersed across several places, is to gather and send information regarding soil moisture levels. But in order to maintain effective network operation, it is crucial to select a minimal dominating set of sensors in real-time due to power limitations and the necessity for efficient data collecting. The objective in this real-time scenario is to locate a minimal dominating set of sensors in the wireless sensor network to enable effective data gathering, while reducing energy consumption. This is a dynamic and real-time optimization problem because the dominating set must be chosen in response to shifting monitoring requirements.

Locating dominating sets were introduced by Slater [6] in 1987 and were extended by him in 1988 [7]. Bounds on the locating domination numbers of cycles and paths were examined by Bertrand et al. in 2004 [8]. For bipartite graphs the bounds were discussed in [9] by Argiroffo et al. in 2015. In 2019 locating-dominating sets of functigraphs were examined by Murtaza et al. [10]. In 2020 Pribadi and Saputro investigated the locating domination number for comb product of graphs [11]. In 2021 Gafur and Saputro discussed locating domination in regular graphs [12]. In 2022 locating dominating sets in circulant graphs were studied by Givens et al. [13]. And in 2023 Bosquet et al. studied bounds on locating domination numbers in oriented graphs [14]. In the parts that follow, we will present a thorough analysis of the existing literature, suggest novel methods of proof and offer experimental findings that illuminate the viability and effectiveness of our suggested techniques. By the end of this article, readers will have a thorough grasp of locating and local locating domination and how it plays a vital role in solving problems in a wide range of fields, ultimately advancing network theory and its useful applications.

In the field of urban planning, building strong and efficient urban infrastructure heavily depends on the design and optimisation of transportation networks. Prism graphs are a mathematical representation of interconnected systems that closely resembles the complexity of urban transportation networks. They are produced by taking the Cartesian product of a cycle graph C_n (which represents, for example, road segments) and a path graph P_2 (which may represent, perhaps, the connectivity of neighbourhoods). Therefore, "locating domination" refers to the deliberate identification of groups of crucial nodes inside the prism network. These nodes might stand in for important locations in the transportation network, including service centres, crossroads, or main traffic hubs, in the urban planning scenario. The aim is to position these vital nodes in a way that guarantees thorough network coverage. Urban planners can maximise the positioning of these crucial nodes to accomplish a number of goals by using locating domination. Improving traffic management greatly depends on the strategic identification and placement of dominating sets in transportation networks. Urban planners can implement targeted monitoring and optimisation methods by concentrating on important intersections and hubs. This method reduces bottlenecks and enhances traffic flow throughout. By taking proactive measures to manage these vital nodes, we can improve the quality of urban mobility generally and create a more efficient transport network with shorter travel times and fewer delays. The end product is a smoothly integrated system that supports a more resilient and sustainable urban environment while meeting the changing demands of urban commuters. An organised method for optimising the location of crucial nodes in transport networks is made possible by the application of locating domination in the context of prism graphs for urban design. This strategy tackles the difficulties brought on by the dynamic and linked character of urban settings by promoting the development of more effective, robust, and well organized urban infrastructure.

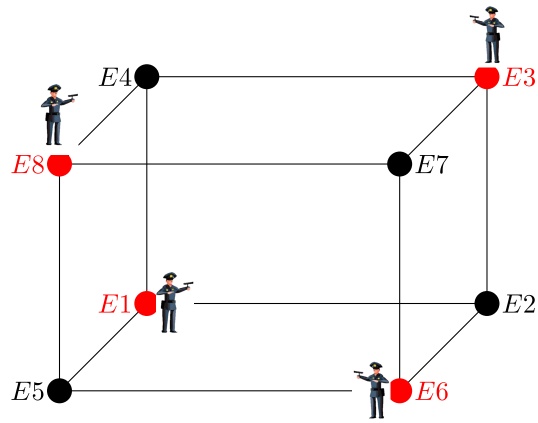

Figure 1. Model of prism graph constructed from the building

In order to demonstrate how to locate domination vertices in prism network CL_4, let's look at a real-world example as shown in the Figure 1, security guards monitoring a building with several entry points $E_1, E_2, \ldots, E_8$. Consider a structure with two floors and four entrances on each floor. The structure is laid out in a rectangle, with entrances on each corner. To further create a prism-like form, stairways connect matching access locations on the two floors. Place a guard at one entry point on the ground floor and its non-corresponding immediate entry point on the upper floor . This guard will effectively cover both floors. Similarly, place another guard at a different pair of entry points to ensure comprehensive coverage of both floors. Each guard covers the entry points on their floor and the opposite entry point on the other floor to ensure that every entry point is either guarded or near one. Because they can monitor multiple entry points simultaneously, the corners provide the finest guard coverage. We minimise redundancy and maximise coverage by carefully positioning guards at designated access points. To minimise needless effort duplication, guards are positioned to protect as many entry points as possible. Every access point into the building is either directly guarded or under the attentive surveillance of a guard because to the presence of guards stationed at domination vertices. This guarantees complete security inside the structure. Locating domination vertices in the prism like building layout involves guards are positioned strategically at designated access points to optimise coverage, guarantee all encompassing protection, and reduce redundancy.

2 Preliminaries

The fundamental results and necessary definitions have been provided in this part to set the stage for theorem proofs. This material acts as a precondition by providing important insights and laying out a conceptual framework that is essential for the later clarification and verification of the theorems that are being discussed.

Definition 1. *Dominating set (DS) [15] for a graph G = (V, E) is a subset D of V such that each vertex not in D is neighbouring to at least one number of D. The domination number $\gamma(G)$ is number of vertices in a smallest dominating set for G.*

Definition 2. *The open neighbourhood [16] $N(v) = \{u \in V : uv \in E\}$ of a vertex $v \in V$ is the set of vertices u adjacent to v.*

Definition 3. *A dominating set $S \subset V$ in a graph $G = (V, E)$ is defined as a locating dominating set (LDS) [17], if the following constraint is satisfied for every pair of vertices $u, v \in V - S$ $N(u) \cap S \neq N(v) \cap S$. The locating domination number $\gamma_L(G)$ equals the minimum cardinality of a locating dominating set in G.*

Definition 4. *A dominating set $S \subset V$ in a graph $G = (V, E)$ is definded to be a local locating dominating set (LLDS) , if the following constraint is satisfied for every pair of vertices $u, v \in V$ $N(u) \cap S \neq N(v) \cap S$. The local locating domination number $\gamma_{LL}(G)$ equals the minimum cardinality of a local locating dominating set in G*

Definition 5. *The Cartesian product $G \Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u_1, v_1) and (u_2, v_2) are connected if either $u_1 = u_2$ and v_1 and v_2 are adjacent in H, or $v_1 = v_2$ and u_1 and u_2 are adjacent in G.*

Definition 6. *Given a graph G, the prism [18] of G is the graph $G \Box K_2$, which consists of two disjoint copies G_1 and G_2 of G, having an edge u_1u_2 for every pair of vertices u_1 and u_2 corresponding to a vertex u in G. A prism graph of the form $G = C_n \Box K_2 = CL_n$, is called a circular ladder graph, and has $2n$ vertices and $3n$ edges. Here C_n is a cycle of length n and K_2 is a complete graph of order 2.*

Definition 7. *An antiprism graph [19] Q_n is a prism of the form $C_n \Box K_2$, having two cycles C_n with vertices labelled $u_0, u_1, \ldots, u_{n-1}$ and $v_0, v_1, \ldots, v_{n-1}$, to which are added $2n$ additional edges of the form u_i, v_i and u_i, v_{i+1}, mod n. An n-sided antiprism graph A_n therefore has $2n$ vertices and $4n$ edges. Here C_n is a cycle of length n and K_2 is a complete graph of order 2.*

Definition 8. *Let $n \geq 4$ be a positive even integer. An n-crossed prism graph [20] is a graph obtained by taking two disjoint cycles on n vertices, $C1_n$ and $C2_n$, where*

$$\begin{aligned} V(C1_n) &= \{x_1, x_2, \ldots, x_n\} \\ V(C2_n) &= \{w_1, w_2, \ldots, w_n\} \\ E(C1_n) &= (x_ix_{i+1}, x_1x_n), i = 1, 2, \ldots, n-1 \\ E(C2_n) &= \{w_iw_{i+1}, w_1w_n\}, i = 1, 2, \ldots, n-1 \end{aligned}$$

and adding edges w_sx_{s+1}, for $s \in \{1, 3, \ldots, n-1\}$ and w_tx_{t-1} for $t \in \{2, 4, \ldots, n\}$.

Definition 9. *The pentagonal circular ladder [21] PCL_n is obtained from the graph of prism by adding a new vertex x_i between y_i and y_{i+1}, for $i = 1, 2, 3, \ldots, n$. The pentagonal circular ladder PCL_n consists of vertex set and edge set in the following form*

$$\begin{aligned} V(R_n) &= \{x_i, y_i, z_i : 1 \leq i \leq n\} \\ E(R_n) &= \{z_iz_{i+1}; 1 \leq i \leq n\} \\ &\quad \cup \{x_iy_i, y_iz_i, x_iy_{i+1}; 1 \leq i \leq n\}. \end{aligned}$$

Definition 10. *The addition property [22] of any two sets A and B is*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Theorem 2.1. *[23] The domination number of prism graph CL_n, for $n \geq 4$, is*

1. $\gamma(CL_n) = \frac{n}{2}$, *if* $n \equiv 0 \bmod 4$
2. $\gamma(CL_n) = \lceil \frac{n}{2} \rceil$, *if* $n \equiv 1 \bmod 4$
3. $\gamma(CL_n) = \frac{n+2}{2}$, *if* $n \equiv 2 \bmod 4$
4. $\gamma(CL_n) = \lceil \frac{n}{2} \rceil$, *if* $n \equiv 3 \bmod 4$

Theorem 2.2. *[23] The domination number of the antiprism graph Q_n, where $n \geq 4$, is $\gamma(Q_n) = \lceil \frac{2n}{5} \rceil$.*

Theorem 2.3. *[23] The domination number of the crossed prism graph R_n, where $n \geq 4$ is*

1. $\gamma(R_n) = \frac{n}{2}$, *if* $n \equiv 0 \mod 4$
2. $\gamma(R_n) = \frac{n+2}{2}$, *if* $n \equiv 2 \mod 4$.

Remark 1. *For any graph $G = (V, E)$,*

$$\gamma(G) \leq \gamma_L(G).$$

3 Methodology

In a graph, the neighbourhood of each vertex $N(v)$ must be defined in order to determine the locating dominant set. This is figuring out which set of vertices, in accordance with the given definition are next to each vertex. This hard procedure makes it easier to construct a locating dominating set with distinct adjacency configurations later on. To determine the local domination set, a systematic technique of modulo division was based on the intrinsic graph properties. Using this approach, the structure of the graph was carefully examined. Applying the addition property selectively allowed for the achievement of the intended outcome of locating domination number.

4 Computation of Locating and Local Locating Domination Number

The boundaries for detecting and proving local dominance in prism family graphs such as prism, antiprism, crossed prism, and pentagonal circulant prism have been established in this section. By exploring the spatial relationships inside these graph structures, this analysis sheds information on their distinct qualities and positional dominance traits.

Theorem 4.1. *The locating domination number of prism graph $CL_n, n \geq 3$ is*

$$\gamma_L(CL_n) \leq \begin{cases} n & \text{for } n \equiv 0 \text{ mod } 2 \\ n-1 & \text{for } n \equiv 1 \text{ mod } 2 \end{cases}$$

Proof. Let CL_n be a prism graph with vertex set $V = \{u_i : i = 1, 2, \ldots, n\} \cup \{v_i : i = 1, 2, \ldots, n\} \implies |V| = 2n$ and edge set E with cardinality $|E| = 3n$. The graph consists of two cycles in which the inner cycle has the vertex set $\{u_i : i = 1, 2, \ldots, n\}$ and the outer cycle has the vertex set $\{v_i : i = 1, 2, \ldots, n\}$. Each u_i is adjacent with corresponding v_i. The open neighbourhoods are

$$\begin{aligned} N(u_1) &= \{u_2, u_n, v_1\}, \\ N(u_i) &= \{u_{i+1}, u_{i-1}, v_i\}, i = 2, 3, \ldots n-1, \\ N(u_n) &= \{u_1, u_{n-1}, v_n\} \\ N(v_1) &= \{v_2, v_n, v_1\}, \\ N(v_i) &= \{v_{i+1}, v_{i-1}, u_i\}, i = 2, 3, \ldots n-1, \\ N(v_n) &= \{v_1, v_{n-1}, u_n\} \end{aligned}$$

The domination number of the prism graph CL_n, for $n \geq 4$, is $\gamma(CL_n) \leq \frac{n+2}{2}$ [Theorem 2.1].

Case 1: When $n \equiv 0 \; mod \; 2$ consider vertex u_1 as first vertex we get the locating dominating set as

$$\begin{aligned} LDS &= \{u_i : i = 1, 3, \ldots, n-1\} \\ &\cup \{v_j : j = 2, 4, \ldots, n\} \\ &= \{u_i : i = 2k-1, k = 1, \ldots, \frac{n}{2}\} \\ &\cup \{v_j : j = 2k, k = 1, 2, \ldots, \frac{n}{2}\} \end{aligned}$$

By the definition 10. of addition property, we obtain

$$\begin{aligned} |LDS| &= |\{u_i : i = 2k-1, k = 1, \ldots, \frac{n}{2}\}| \\ &+ |\{v_j : j = 2k, k = 1, 2, \ldots, \frac{n}{2}\}| \\ |LDS| &= \frac{n}{2} + \frac{n}{2} = n \end{aligned}$$

Thus the upper bound is

$$\gamma_L(CL_n) \leq n.$$

Case 2: When $n \equiv 1 \; mod \; 2$ by considering the vertex u_1 as a first vertex , we get the locating dominating set as

$$\begin{aligned} LDS &= \{u_i : i = 1, 3, \ldots, n-2\} \\ &\cup \{v_j : j = 2, 4, \ldots, n-1\} \\ &= \{u_i : i = 2k-1, k = 1, 2, \ldots, \frac{n-1}{2}\} \\ &\cup \{v_j : j = 2k, k = 1, 2, \ldots, \frac{n-1}{2}\} \end{aligned}$$

Using definition 10., it follows that

$$\begin{aligned} |LDS| &= |\{u_i : i = 2k-1, k = 1, 2, \ldots, \frac{n-1}{2}\}| \\ &+ |\{v_j : j = 2k, k = 1, 2, \ldots, \frac{n-1}{2}\}| \\ |LDS| &= \frac{n-1}{2} + \frac{n-1}{2} \end{aligned}$$

Thus the upper bound of locating dominating set of prism graph is obtained as $\gamma_L(CL_n) \leq n-1$. By Remark 1 and Theorem 2.1 it is verified that $\gamma(CL_n) \leq \gamma_L(CL_n)$. □

Theorem 4.2. *The locating domination number of antiprism graph Q_n where $n \geq 4$ is given as $\gamma_L(Q_n) \leq n-1$.*

Proof. Let Q_n be the antiprism graph having vertex set

$$V = \{u_i : i = 1, 2, \ldots, n\} \cup \{v_j : j = 1, 2, \ldots, n\}$$

which has $2n$ number of elements and edge set E with number of elements $|E| = 4n$. The antiprism graph consists of two cycles. Let the inner cycle vertex set be $\{u_i : i = 1, 2, \ldots, n\}$ and the vertex set of outer cycle be $\{v_j : j = 1, 2, \ldots, n\}$. Each u_i is adjacent with v_i and v_{i-1} except $i = 1$. u_1 is adjacent with v_n and v_1. The open neighbourhood of each vertex of Q_n is

$$\begin{aligned} N(u_1) &= \{u_2, u_8, v_1, v_2\}, \\ N(u_i) &= \{u_{i-1}, u_{i+1}, v_{i-1}, v_i\} \\ &\quad \text{here } i = 2, 3, \ldots, n-1, \\ N(u_n) &= \{u_1, u_{n-1}, v_1, v_n\} \text{ and} \\ N(v_1) &= \{u_1, u_2, v_2, v_n\}, \\ N(v_n) &= \{u_1, u_n, v_1, v_7\}, \\ &\text{for } i = 2, 3, \ldots, n-1 \text{ we have} \\ N(v_i) &= \{u_i, u_{i+1}, v_{i-1}, v_{i+1}\} \end{aligned}$$

The domination number of the antiprism graph Q_n, where $n \geq 4$, is $\gamma(Q_n) = \lceil \frac{2n}{5} \rceil$ [Theorem 2.2].

Case 1: When n is even considering u_1 as first vertex we obtain the locating dominating set as follows

$$\begin{aligned} LDS &= \{u_i : i = 1, 3, \ldots, n-1\} \\ &\cup \{v_i : i = 2, 4, \ldots, n-2\} \\ &= \{u_i : i = 2k-1, k = 1, 2, \ldots, \frac{n}{2}\} \\ &\cup \{v_i : i = 2k, k = 1, 2, \ldots, \frac{n-2}{2}\} \end{aligned}$$

According to the definition 10. of the addition property, it becomes

$$|LDS| = |\{u_i : i = 2k-1, k = 1, 2, \ldots, \frac{n}{2}\}| + |\{v_i : i = 2k, k = 1, 2, \ldots, \frac{n-2}{2}\}|$$
$$|LDS| = \frac{n}{2} + \frac{n-2}{2} = n - 1$$

Hence the upper bound is found to be $\gamma_L(Q_n) \leq n - 1$. By Remark 1 and Theorem 2.2 it is verified that $\gamma(Q_n) \leq \gamma_L(Q_n)$.

Case 2: When n is odd the locating dominating set is obtained as

$$\begin{aligned} LDS &= \{u_i : i = 1, 3, \ldots, n-2\} \\ &\cup \{v_i : i = 2, 4, \ldots, n-3\} \cup v_n \\ &= \{u_i : i = 2k-1, k = 1, 2, \ldots, \frac{n-1}{2}\} \\ &\cup \{v_i : i = 2k, k = 1, 2, \ldots, \frac{n-3}{2}\} \cup v_n \end{aligned}$$

Using definition 10., it follows that

$$|LDS| = |\{u_i : i = 2k-1, k = 1, 2, \ldots, \frac{n-1}{2}\}| + |\{v_i : i = 2k, k = 1, 2, \ldots, \frac{n-3}{2}\}| + |v_n|$$
$$|LDS| = \frac{n-1}{2} + \frac{n-3}{2} + 1 = n - 1$$

Thus the upper bound is $\gamma_L(Q_n) \leq n - 1$. □

Theorem 4.3. *The upper bound for the locating domination number of crossed prism graph R_n, n is even and $n \leq 4$ is given as*

$$\gamma_L(R_n) \leq \begin{cases} \frac{2n+3}{3} & \text{for } n \equiv 0 \text{ mod } 6 \\ \frac{2n+2}{3} & \text{for } n \equiv 2 \text{ mod } 6 \\ \frac{2n+4}{3} & \text{for } n \equiv 4 \text{ mod } 6. \end{cases}$$

Proof. Let R_n be the crossed prism graph with vertex set

$$V = \{u_i : i = 1, 2, \ldots, n\} \cup \{v_i : i = 1, 2, \ldots, n\}$$

such that $|V| = 2n$ and edge set $|E| = 3n$. The graph consists of two cycles C_n vertex set of inner and outer cycles which are given as $\{u_i : i = 1, 2, \ldots, n\}$ and $\{v_i : i = 1, 2, \ldots, n\}$. When i is even u_i is adjacent with $v_{i-1 \text{ mod } n}$ and when i is odd u_i is adjacent with $v_{i+1 \text{ mod } n}$. Here the subscript bound is $0 < i \leq n$. The open neighbourhood of crossed prism graph is given as $N(u_1) = \{u_1, u_n, v_n\}$,

$N(u_i) = \{u_{n-1}, u_{n+1}, v_{n+1}\}$ when i is even
$N(u_j) = \{u_{n-1}, u_{n+1}, v_{n-1}\}$ when j is odd ,
$N(u_n) = \{u_1, u_{n-1}, v_1\}$ and
$N(v_1) = \{u_n, v_2, v_n\}$,
$N(v_i) = \{u_{i+1}, v_{i-1}, v_{1+1}\}$ when i is even
$N(v_j) = \{u_{i-1}, v_{i-1}, v_{i+1}\}$ is odd ,
$N(v_n) = \{u_1, v_1, v_{n-1}\}$

The domination number of the crossed prism graph R_n, is $\gamma(R_n) \leq \frac{n+2}{2}$ where $n \geq 4$ and even [Theorem 2.3].

Case 1: When $n \equiv 0$ *mod* 6 the locating dominating set is

$$\begin{aligned} LDS &= \{u_i : i = 1, 4, \ldots, n-2\} \\ &\cup \{u_n\} \cup \{v_i : i = 1, 4, \ldots, n-2\} \\ &= \{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n}{3}\} \\ &\cup \{u_n\} \\ &\cup \{v_i : i = 3k-2, k = 1, 2, \ldots, \frac{n}{3}\} \end{aligned}$$

By the definition 10. of addition property we have

$$|LDS| = |\{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n}{3}\}| + |\{v_i : i = 3k-2, k = 1, 2, \ldots, \frac{n}{3}\}| + |u_n|$$
$$|LDS| = \frac{n}{3} + 1 + \frac{n}{3} = \frac{2n+3}{3}$$

Thus the upper bound is found to be

$$\gamma_L(R_n) \leq \frac{2n+3}{3}.$$

Case 2: When $n \equiv 2$ *mod* 6 the locating dominating set is

$$\begin{aligned} LDS &= \{u_i : i = 1, 4, \ldots, n-4\} \\ &\cup \{u_{n-2}\} \\ &\cup \{v_i : i = 1, 4, \ldots, n-1\} \\ &= \{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n-2}{3}\} \\ &\cup \{u_{n-2}\} \\ &\cup \{v_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+1}{3}\} \end{aligned}$$

Hence by definition 10

$$|LDS| = |\{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n-2}{3}\}| + |\{v_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+1}{3}\}| + |\{u_{n-2}\}|$$
$$|LDS| = \frac{n-2}{3} + 1 + \frac{n+1}{3} = \frac{2n+2}{3}$$

The locating domination number of crossed prism graph is bounded above by

$$\gamma_L(R_n) \leq \frac{2n+2}{3}.$$

Case 3: When $n \equiv 4$ *mod* 6 the locating dominating set is

$$\begin{aligned} LDS &= \{u_i : i = 1, 4, \ldots, n\} \\ &\cup \{v_i : i = 1, 4, \ldots, n\} \\ &= \{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+2}{3}\} \\ &\cup \{v_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+2}{3}\} \end{aligned}$$

Using definition 10., it follows that

$$|LDS| = |\{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+2}{3}\}| + |\{v_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+2}{3}\}|$$

$$|LDS| = \frac{n+2}{3} + \frac{n+2}{3} = \frac{2n+4}{3}$$

Hence the upper bound is obtained as $\gamma_L(R_n) \leq \frac{2n+4}{3}$. By Remark 1 and Theorem 2.3 it is verified that $\gamma(R_n) \leq \gamma_L(R_n)$. □

Theorem 4.4. *The locating domination number of pentagonal circulant ladder graph $PCL_n, n \geq 3$ is bounded above by $\gamma_L(PCL_n) \leq n$.*

Proof. Let PCL_n is a pentagonal circulant ladder graph with vertex set

$$V = \{v_i : i = 1, 2, \ldots, n\} \cup \{u_i : i = 1, 2, \ldots, n\} \cup \{w_i : i = 1, 2, \ldots, n\}$$

and edge set E having cardinality $|E| = 3n$. The pentagonal circulant ladder graph consists of two cycles of length C_n the inner cycle and C_{2n} the outer cycle each vertex u_i from inner cycle is adjacent with alternative vertex of outer cycle. The open neighbourhoods of pentagonal circulant ladder graph are given by

$$N(u_1) = \{u_2, u_n, v_1\},$$
$$N(u_i) = \{u_{n-1}, u_{n+1}, v_i\}, i = 2, 3, \ldots, n-1\},$$
$$N(u_n) = \{u_1, u_{n-1}, v_n\}$$
$$N(v_1) = \{w_1, w_n, v_1\},$$
$$N(v_i) = \{w_i, w_{i-1}, u_i\},$$
$$N(w_i) = \{w_i, w_{i+1}\}, i = 1, 2, \ldots, n-1,$$
$$N(w_n) = \{v_1, v_n\}$$

The domination number of pentagonal circulant ladder graph PCL_n is $\gamma(PCL_n) = n$.

The locating dominating set for pentagonal circulant ladder graph is the subset of V given as

$$LDS = \{u_1, u_2, \ldots, u_n\}$$
$$\implies |LDS| = n$$

Therefore the locating domination number of pentagonal graph is bounded above by $\gamma_L(PCL_n) \leq n$. By Remark 1 it is verified that $\gamma(PCL_n) \leq \gamma_L(PCL_n)$. □

Theorem 4.5. *The local locating domination number of prism graph CL_n where $n \geq 3$ is bounded above by $\gamma_{LL}(CL_n) \leq n-1$.*

Proof. Let CL_n be a prism graph with vertex set $V = \{u_i : i = 1, 2, \ldots, n\} \cup \{v_i : i = 1, 2, \ldots, n\} \implies |V| = 2n$ and edge set E with cardinality $|E| = 3n$. The antiprism graph consists of two cycles. Let the inner cycle vertex set be $\{u_i : i = 1, 2, \ldots, n\}$ and the vertex set of outer cycle be $\{v_j : j = 1, 2, \ldots, n\}$. Each u_i is adjacent with v_i and v_{i-1} except $i = 1$. u_1 is adjacent with v_n and v_1. The open neighbourhoods are given by

$$N(u_1) = \{u_2, u_n, v_1\},$$
$$N(u_i) = \{u_{i+1}, u_{i-1}, v_i\}, i = 2, 3, \ldots n-1,$$
$$N(u_n) = \{u_1, u_{n-1}, v_n\}$$
$$N(v_1) = \{v_2, v_n, v_1\},$$
$$N(v_i) = \{v_{i+1}, v_{i-1}, u_i\}, i = 2, 3, \ldots n-1,$$
$$N(v_n) = \{v_1, v_{n-1}, u_n\}$$

Case 1: When n is odd consider u_1 as the first vertex we obtain the local locating dominating set as follows

$$LLDS = \{u_i : i = 1, 3, \ldots, n-2\} \cup \{v_i : i = 1, 3, \ldots, n-2\}$$
$$= \{u_i : i = 2j-1, j = 1, 2, \ldots, \frac{n-1}{2}\} \cup \{v_i : i = 2j-1, j = 1, 2, \ldots, \frac{n-1}{2}\}$$

By the definition 10. of addition property, we have

$$|LLDS| = |\{u_i : i = 2j-1, j = 1, 2, \ldots, \frac{n-1}{2}\}| + |\{v_i : i = 2j-1, j = 1, 2, \ldots, \frac{n-1}{2}\}|$$
$$\implies |LLDS| = \frac{n-1}{2} + \frac{n-1}{2}$$

Hence the local locating domination number is bounded above by

$$\gamma_{LL}(CL_n) \leq n-1.$$

Case 2: When n is even we get the local locating dominating set as

$$LLDS = \{u_1, u_3, \ldots, u_{n-3}\} \cup \{v_1, v_3, \ldots, v_{n-1}\}$$
$$= \{u_i : i = 1, 3, \ldots, n-3\} \cup \{v_i : i = 1, 3, \ldots, n-1\}$$
$$= \{u_i : i = 2j-1, j = 1, 2, \ldots, \frac{n-2}{2}\} \cup \{v_i : i = 2j-1, j = 1, 2, \ldots, \frac{n}{2}\}$$

By definition 10. we have

$$|LLDS| = |\{u_i : i = 2j-1, j = 1, 2, \ldots, \frac{n-2}{2}\}| + |\{v_i : i = 2j-1, j = 1, 2, \ldots, \frac{n}{2}\}|$$

$$= \frac{n-2}{2} + \frac{n}{2}$$

Thus the upper bound for is obtained as

$$\gamma_{LL}(CL_n) \leq n-1.$$

□

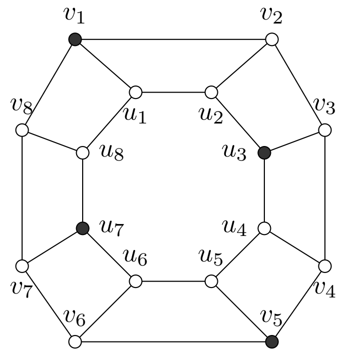

Figure 2: DS of CL_8

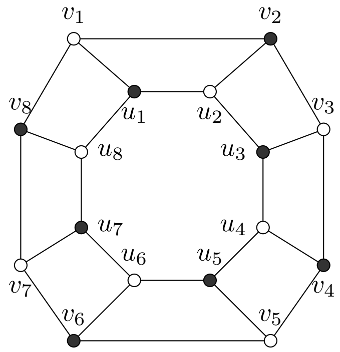

Figure 3: LDS of CL_8

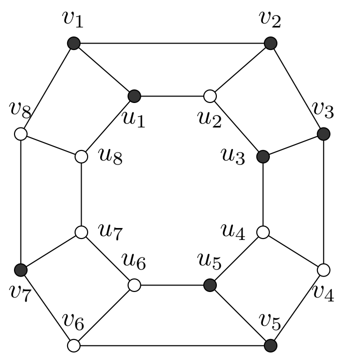

Figure 4: LLDS of CL_8

The sets indicated in Figures 2, 3 and 4 illustrate dominating sets, locating dominating sets and local locating dominating sets in the prism graph CL_8.

Theorem 4.6. *The local locating domination number of antiprism graph $Q_n, n \geq 4$ is bounded above by*

$$\gamma_{LL}(Q_n) \leq \begin{cases} \frac{2n}{3} & \text{for } n \equiv 0 \; mod \; 3 \\ \frac{2n+4}{3} & \text{for } n \equiv 1 \; mod \; 3 \\ \frac{2n+2}{3} & \text{for } n \equiv 2 \; mod \; 3 \end{cases}$$

Proof. Let Q_n be the antiprism graph having vertex set $V = \{u_i : i = 1, 2, \ldots, n\} \cup \{v_j : j = 1, 2, \ldots, n\}$ and edge set E with number of elements $|E| = 4n$. The antiprism graph consists of two cycles. Let the inner cycle vertex set be $\{u_i : i = 1, 2, \ldots, n\}$ and the vertex set of outer cycle be $\{v_j : j = 1, 2, \ldots, n\}$. Each u_i is adjacent with v_i and v_{i-1} except $i = 1$. u_1 is adjacent with v_n and v_1. The open neighbourhood of each vertex of Q_n is

$$\begin{aligned} N(u_1) &= \{u_2, u_8, v_1, v_2\}, \\ N(u_i) &= \{u_{i-1}, u_{i+1}, v_{i-1}, v_i\} \\ &\quad \text{here } i = 2, 3, \ldots, n-1, \\ N(u_n) &= \{u_1, u_{n-1}, v_1, v_n\}, \\ N(v_1) &= \{u_1, u_2, v_2, v_n\}, \\ N(v_n) &= \{u_1, u_n, v_1, v_7\}, \\ N(v_i) &= \{u_i, u_{i+1}, v_{i-1}, v_{i+1}\} \\ &\quad \text{for } i = 2, 3, \ldots, n-1. \end{aligned}$$

Case 1: When $n \equiv 0 \; mod \; 3$ consider u_1 as the first vertex we get the local locating dominating set as follows

$$\begin{aligned} LLDS &= \{u_1, u_4, \ldots, u_{n-2}\} \\ &\quad \cup \{v_3, v_6, \ldots, v_n\} \\ &= \{u_i : i = 1, 4, \ldots, n-2\} \\ &\quad \cup \{v_i : i = 3, 6, \ldots, n\} \\ &= \{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n}{3}\} \\ &\quad \cup \{u_i : i = 3k, k = 1, 2, \ldots, \frac{n}{3}\} \end{aligned}$$

From the definition 10. of addition property

$$\begin{aligned} |LLDS| &= |\{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n}{3}\}| \\ &\quad + |\{u_i : i = 3k, k = 1, 2, \ldots, \frac{n}{3}\}| \\ &= \frac{n}{3} + \frac{n}{3} = \frac{2n}{3} \end{aligned}$$

Thus the local locating domination number is bounded above by $\gamma_{LL}(Q_n) \leq \frac{2n}{3}$.

Case 2: When $n \equiv 1 \; mod \; 3$ the local locating dominating set is obtained as

$$\begin{aligned} LLDS &= \{u_1, u_4, \ldots, u_n\} \\ &\quad \cup \{v_3, v_6, \ldots, v_{n-1}\} \cup v_n \\ &= \{u_i : i = 1, 4, \ldots, n\} \\ &\quad \cup \{v_i : i = 3, 6, \ldots, n-1\} \cup v_n \\ &= \{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+2}{3}\} \\ &\quad \cup \{u_i : i = 3k, k = 1, 2, \ldots, \frac{n-1}{3}\} \cup v_n \end{aligned}$$

Using the definition 10 of addition property

$$\begin{aligned} |LLDS| &= |\{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+2}{3}\}| \\ &\quad + |\{u_i : i = 3k, k = 1, 2, \ldots, \frac{n-1}{3}\}| + 1 \\ &= \frac{n+2}{3} + \frac{n-1}{3} + 1 = \frac{2n+4}{3} \end{aligned}$$

The upper bound is given by

$$\gamma_{LL}(Q_n) \le \frac{2n+4}{3}.$$

Case 3: When $n \equiv 2\ mod\ 3$ consider u_1 as the first vertex we get the local locating dominating set as

$$\begin{aligned} LLDS &= \{u_1, u_4, \ldots, u_{n-1}\} \\ &\quad \cup \{v_3, v_6, \ldots, v_{n-2}\} \cup \{v_n\} \\ &= \{u_i : i = 1, 4, \ldots, n-1\} \\ &\quad \cup \{v_i : i = 3, 6, \ldots, n-2\} \cup \{v_n\} \\ &= \{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+1}{3}\} \\ &\quad \cup \{u_i : i = 3k, k = 1, 2, \ldots, \frac{n-2}{3}\} \cup \{v_n\} \end{aligned}$$

By the definition 10. of addition property,

$$\begin{aligned} |LLDS| &= |\{u_i : i = 3k-2, k = 1, 2, \ldots, \frac{n+1}{3}\}| \\ &\quad + |\{u_i : i = 3k, k = 1, 2, \ldots, \frac{n-2}{3}\}| + 1 \\ &= \frac{n+1}{3} + \frac{n-2}{3} + 1 = \frac{2n+2}{3} \end{aligned}$$

Thus the local locating number is bounded above by

$$\gamma_{LL}(Q_n) \le \frac{2n+2}{3}.$$

□

The sets indicated in Figures 5, 6 and 7 illustrate dominating sets, locating dominating sets and local locating dominating sets in the antiprism graph Q_5.

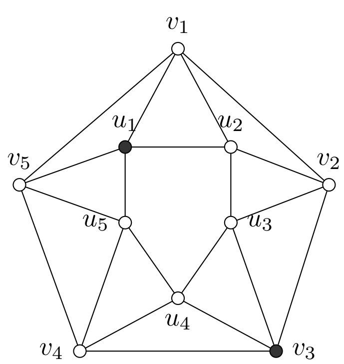

Figure 5: DS of Graph Q_5

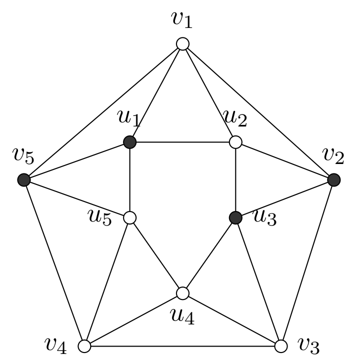

Figure 6: LDS of Q_5

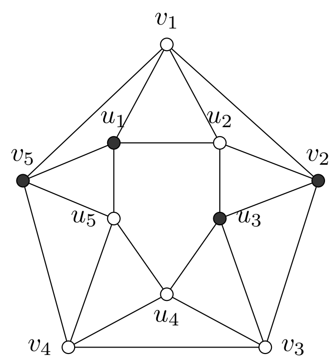

Figure 7: LLDS of Q_5

Theorem 4.7. *The local locating domination number of the crossed prism graph R_n is at most $\gamma_{LL}(R_n) \le n-1$.*

Proof. Let R_n be the crossed prism graph with vertex set $V = \{u_i : i = 1, 2, \ldots, n\} \cup \{v_i : i = 1, 2, \ldots, n\}$ such that $|V| = 2n$ and edge set $|E| = 3n$. The graph consists of two cycles C_n vertex set of inner and outer cycles are given as $\{u_i : i = 1, 2, \ldots, n\}$ and $\{v_i : i = 1, 2, \ldots, n\}$. When i is even u_i is adjacent with $v_{i-1\ mod\ n}$ and when i is odd u_i is adjacent with $v_{i+1\ mod\ n}$. Here the subscript bound is $0 < i \le n$. The open neighbourhood of crossed prism graph is given as

$$\begin{aligned} N(u_1) &= \{u_1, u_n, v_n\}, \\ N(u_i) &= \{u_{n-1}, u_{n+1}, v_{n+1}\} \text{ when } i \text{ is even }, \\ N(u_j) &= \{u_{n-1}, u_{n+1}, v_{n-1}\} \text{ when } j \text{ is odd }, \\ N(u_n) &= \{u_1, u_{n-1}, v_1\}, \\ N(v_1) &= \{u_n, v_2, v_n\}, \\ N(v_i) &= \{u_{i+1}, v_{i-1}, v_{1+1}\} \text{ when } i \text{ is even }, \\ N(v_j) &= \{u_{i-1}, v_{i-1}, v_{i+1}\} \text{ is odd }, \\ N(v_n) &= \{u_1, v_1, v_{n-1} \end{aligned}$$

The local locating dominating set of crossed prism graph is found to be

$$\begin{aligned} LLDS &= \{u_1, u_3, \ldots, u_{n-1}\} \\ &\quad \cup \{v_2, v_4, \ldots, v_{n-2}\} \\ &= \{u_i : i = 1, 3, \ldots, n-1\} \\ &\quad \cup \{v_i : i = 2, 4, \ldots, n-2\} \\ &= \{u_i : i = 2k-1, k = 1, 2, \ldots, \frac{n}{2}\} \\ &\quad \cup \{v_i : i = 2k, k = 1, 2, \ldots, \frac{n-2}{2}\} \end{aligned}$$

By the definition 10. of addition property,

$$\begin{aligned} |LLDS| &= |\{u_i : i = 2k-1, k = 1, 2, \ldots, \frac{n}{2}\}| \\ &\quad + |\{v_i : i = 2k, k = 1, 2, \ldots, \frac{n-2}{2}\}| \\ &= \frac{n}{2} + \frac{n-2}{2} = n - 1 \end{aligned}$$

Thus the upper bound of local locating dominating set is given as

$$\gamma_{LL}(R_n) \leq n - 1$$

□

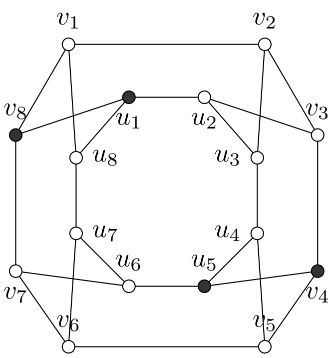

Figure 8: DS of R_8

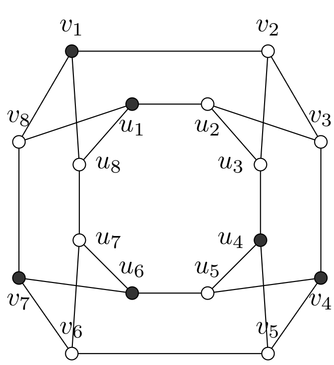

Figure 9: LDS of R_8

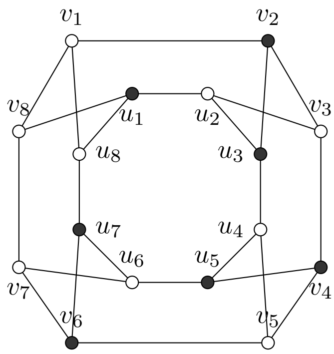

Figure 10: LLDS of R_8

Figures 8, 9 and 10 illustrate dominating, locating dominating and local locating dominating sets of the crossed graph R_8.

Theorem 4.8. *The local locating domination number of the pentagonal circulant ladder PCL_n is bounded above by $\gamma_{LL}(PCL_n) \leq 2(n-1)$.*

Proof. Let PCL_n is a pentagonal circulant ladder graph with vertex set

$$\begin{aligned} V &= \{v_i : i = 1, 2, \ldots, n\} \\ &\quad \cup \{u_i : i = 1, 2, \ldots, n\} \\ &\quad \cup \{w_i : i = 1, 2, \ldots, n\} \end{aligned}$$

and edge set E having cardinality $|E| = 3n$. The pentagonal circulant ladder graph consists of two cycles of length C_n the inner cycle and C_{2n} the outer cycle each vertex u_i from inner cycle is adjacent with alternative vertex of outer cycle. The open neighbourhoods of pentagonal circulant ladder graph are given by

$$\begin{aligned} N(u_1) &= \{u_2, u_n, v_1\}, \\ N(u_i) &= \{u_{n-1}, u_{n+1}, v_i\}, i = 2, 3, \ldots, n-1, \\ N(u_n) &= \{u_1, u_{n-1}, v_n\}, \\ N(v_1) &= \{w_1, w_n, v_1\}, \\ N(v_i) &= \{w_i, w_{i-1}, u_i\}, \\ N(w_i) &= \{w_i, w_{i+1}\}, i = 1, 2, \ldots, n-1, \\ N(w_n) &= \{v_1, v_n\} \end{aligned}$$

The local locating dominating set for pentagonal prism graph is given by

$$\begin{aligned} LLDS &= \{u_1\} \cup \{v_1, v_2, \ldots, v_{n-1}\} \\ &\quad \cup \{w_2, w_3, \ldots, w_{n-1}\} \\ &= \{u_1\} \cup \{v_i : i = 1, 2, \ldots, n-1, \} \\ &\quad \cup \{w_j : j = 2, 3, \ldots, n-1\} \end{aligned}$$

By the definition 10. of addition property,

$$\begin{aligned} |LLDS| &= |\{u_1\}| + |\{v_i : i = 1, 2, \ldots, n-1, \}| \\ &\quad + |\{w_j : j = 2, 3, \ldots, n-1\}| \\ &= 1 + n - 1 + n - 2 = 2n - 2 \end{aligned}$$

Hence the minimum cardinality is given by

$$\gamma_{LL}(PCL_n) \leq 2(n-1)$$

□

Figures 11, 12 and 13 illustrate dominating, locating dominating and local locating dominating sets of pentagonal prism graph PCL_5.

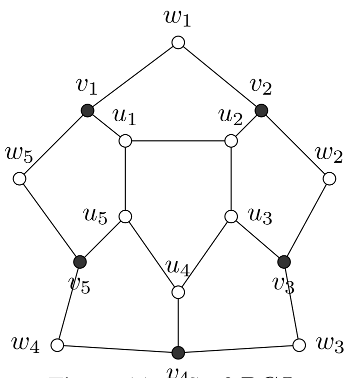

Figure 11: DS of PCL_5

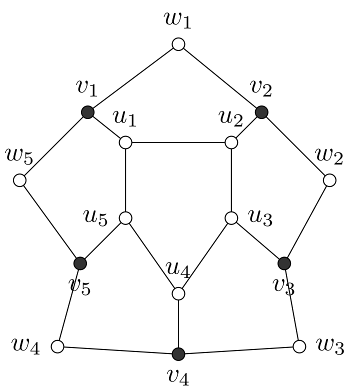

Figure 12: LDS of PCL_5

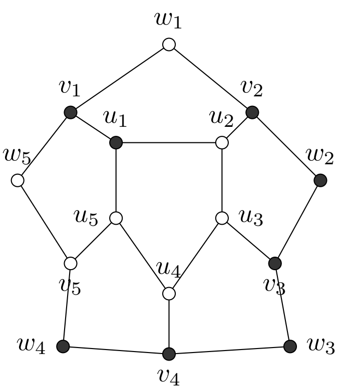

Figure 13: LLDS of PCL_5

5 Conclusions

Our work has advanced the topic of locating domination number identification and contributed significant new information to graph theory in general. The results that are provided here shed light on the particular properties of prism graphs and lay the groundwork for more general applications. We hypothesise that future research into the locating dominating qualities of other graph types will be sparked by the findings and techniques presented in this work. Prism graphs' intrinsic complexity offers an ideal environment for experimenting with new algorithms and analytical techniques, which may open up new directions for combinatorial optimisation. Beyond prism graphs, we see a cascading effect where the techniques and knowledge gained from this research will be applied in ways that go beyond their immediate use. Pushing the bounds of our understanding of locating dominating qualities will be beneficial to the larger graph theory community. The possibility of developing efficient algorithms for related combinatorial optimisation problems is gaining momentum, leading to real-world applications in a variety of domains, including resource allocation, network design, and decision-making procedures. To put it another way, our work advances the field of prism graphs and facilitates the advancement of graph theory in general and its various applications in particular with regard to identifying domination.

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