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Partial Product-Exponential Method of Estimation

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Abstract This research introduces the Partial Product-Exponential Method of Estimation, focusing on utilizing partial auxiliary information for estimating population mean in simple random sampling without replacement. The method proposes novel estimators tailored for situations where only partial auxiliary information is available, particularly when it demonstrates a negative correlation with the study variable within sub-populations. The paper evaluates the performance of the suggested method under two cases: when sub-population weights are known and when they are unknown. Approximate expressions for bias and variance, up to the first order, are derived for the suggested estimators. A comprehensive comparative analysis concludes that the proposed estimators are more efficient than existing estimators, such as mean per unit estimator, partial product estimator, and weighted post-stratified estimator, under specific conditions. Particularly, the proposed estimators outperform the corresponding existing methods when certain conditions are true, demonstrating superiority in both known and unknown weight cases. Furthermore, a simulation study using R software validates the theoretical findings for normal and non-normal populations. The study showcases the practical utility of the proposed estimators, emphasizing their superiority over existing counterparts in real-world applications. Particularly, the proposed estimators are increased accuracy and efficiency in estimating the population mean, enhancing the reliability of sample survey results. In summary, the Partial Product-Exponential Method of Estimation presents a valuable addition to the domain of sample survey methodology, addressing the challenge of partial auxiliary information. The suggested methods demonstrated advantages in efficiency and accuracy, and highlights its potential for practical applications, promising enhanced estimation accuracy in various cases of sample survey.

Keywords Efficiency, Estimation, Partial Auxiliary Information, Simple Random Sampling

1 Introduction

The use of auxiliary information in sample surveys plays a crucial role in improving the accuracy and efficiency of estimators of the population parameter(s), both in the design and estimation stages. In this context, Cochran[1] pioneered the incorporation of auxiliary variables in sample surveys. Wherein he introduced a ratio estimator to estimate the population mean, particularly in situations when there is a strong positive correlation between study and auxiliary variables. Subsequently, Murthy[2] anticipated the product estimator, specifically designed for cases where the correlation is negative. The field has seen significant contributions from researchers such as [3]-[6], among others. They have enriched the domain by developing various estimators for the population mean using auxiliary information. Thereafter, [7]-[9] extended this work by developing product-type exponential estimators, considering auxiliary information for each unit of the population. Recently, [10]-[14] contributed in the theory of sample surveys.

However, practical cases often present challenges when only partial auxiliary information is available or relevant. Addressing this gap, Brar et al.[15] introduced the partial ratio method for estimation when partial auxiliary information is available and is positively correlated with the study variable within a sub-population. Similarly, Brar et al.[16] proposed the partial product method for estimating the population mean when partial auxiliary information is available and exhibits a negative correlation with the study variable within a sub-population.

Thus motivated, this paper introduces a novel estimation technique known as the partial product-type exponential method of estimation of mean. This method is tailored for situations in which the auxiliary variable demonstrates a negative correlation with the study variable within a sub-population. We evaluate the performance of this method under two cases: when the weights of the sub-populations are known and when unknown. Approximate expressions for bias and variance, up to the first order, are derived for suggested estimators. Additionally, a comprehensive comparative analysis is conducted which concludes that the proposed estimators are more efficient than the corresponding comparable estimators. In the last, a simulation study has been conducted to verify the theoretical results using R software.

2 Notations

Consider a population of size N, with variables y and x representing the study and auxiliary variables, respectively. Notably, auxiliary information is relevant or available only for a specific sub-population or domain referred to as class A, encompassing N_1 units. Conversely, class $A^{'}$, with a size of $N-N_1=N_2$, does not have auxiliary information. Let Y_{1i} and Y_{2j} denote the values of y associated with the i^{th} unit in class A and the j^{th} unit in class A', respectively.

- $W_1=\frac{N_1}{N}$ and $W_2=\frac{N-N_1}{N}=1-W_1$ are proportions of units belonging to A and $A^{'}$ respectively.
- $\bar{Y}_1=\frac{1}{N_1}\sum_{i=1}^{N_1}Y_{1i}$, $\bar{Y}_2=\frac{1}{N_2}$ and $\sum_{j=1}^{N_2}Y_{2j}$ are means of study variable in A, $A^{'}$ respectively.
- $\bar{X}_1=\frac{1}{N_1}\sum_{i=1}^{N_1}X_{1i}$ is the mean of auxiliary variable in A respectively.
- S_{1y}^2, S_{1x}^2 and S_{2y}^2 are mean squares of study variable and auxiliary variables for A and $A^{'}$ respectively.
- S_{1yx} be the covariance between study variable and auxiliary variable in A.
- $\rho_1=\frac{S_{1yx}}{S_{1y}S_{1x}}$ is the correlation coefficient between study variable and auxiliary variable in A.
- Further,

$$C_{1x}=\frac{S_{1x}}{\bar{X}_1}, C_{1y}=\frac{S_{1y}}{\bar{Y}_1}, C_{2y}=\frac{S_{2y}}{\bar{Y}_2}, R_1=\frac{\bar{Y}_1}{\bar{X}_1}$$

- Population mean of study variable is defined as

$$\bar{Y}=\frac{1}{N}\sum_{h=1}^{2}\sum_{i=1}^{N_h}Y_{hi}=W_1\bar{Y}_1+W_2\bar{Y}_2$$

- S_y^2 denotes population mean square of study variable.
- $C_y=\frac{S_y}{\bar{Y}}$ denotes the coefficient of variation of study variable.
- Similarly, $\bar{X}, S_x^2$ and C_x denote the population mean, mean square error and coefficient of variation respectively of the auxiliary variable.

A sample of size n is drawn using simple random sampling without replacement(SRSWOR) such that n_1 and $n-n_1=n_2$ units of the sample belong to A and $A^{'}$ respectively. Let (y_{1i},x_{1i}) and y_{2j} be the observations corresponding to the i^{th} and j^{th} units of parts of sample belonging to A and $A^{'}$ respectively.

- $w_1=\frac{n_1}{n}$ and $w_2=\frac{n-n_1}{n}=1-w_1$ are proportions of units belonging to A and $A^{'}$ respectively.
- $\bar{y}_1=\frac{1}{n_1}\sum_{i=1}^{n_1}y_{1i}$ and $\bar{x}_1=\frac{1}{n_1}\sum_{i=1}^{n_1}x_{1i}$ are sample means of study and auxiliary variables for those units belong to A.
- $\bar{y}_2=\frac{1}{n_2}\sum_{j=1}^{n_2}y_{2j}$ is the sample mean of study variable for those units which belong to $A^{'}$.
- $\bar{y}=\frac{1}{n}\sum_{h=1}^{2}\sum_{i=1}^{n_h}y_{hi}$ is the sample mean of study variable.

For more details one can see [[17], [18]].

3 Existing Estimators

In the absence of auxiliary information, the mean per unit estimator ($\bar{y}$) is commonly employed. In such situations, this estimator is unbiased, with variance given as:

$$V(\bar{y})=\left(\frac{1}{n}-\frac{1}{N}\right)\bar{Y}^2C_y^2. \tag{1}$$

Bahl[7] developed the product type exponential estimator of population mean as

$$\bar{y}_{pe}=\bar{y}\ exp\left[\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right].$$

The expression for mean square is given as

$$MSE\left(\bar{y}_{pe}\right)=\left(\frac{1}{n}-\frac{1}{N}\right)\bar{Y}^2\left(C_y^2+\frac{1}{4}C_x^2+\rho C_yC_x\right).$$

This approach is applicable when auxiliary information is available (or relevant) for every unit within the population. However, challenges may arise in situations where such information is not accessible or relevant for each unit in the entire population; though, it might only be available for a subset or particular sub-population. In such cases, the previously mentioned method becomes impractical.

Brar[16] introduced partial product estimators for the cases where partial auxiliary information is available and is negatively correlated with the study variable within sub-population as

Case-1: *When W_1 and W_2 are known and no auxiliary information available.*
Under these circumstances, a post-stratified approach is used to formulate an estimator as follows:

$$\bar{y}_{st}^{\prime}=W_1\bar{y}_1+W_2\bar{y}_2.$$

It is an unbiased estimator of population mean with approximate variance

$$V\left(\bar{y}_{st}^{'}\right)=\frac{1}{n}\left(W_1\bar{Y}_1^2C_{1y}^2+W_2\bar{Y}_2^2C_{2y}^2\right). \tag{2}$$

Case-2: *When W_1 and W_2 are known and partial auxiliary information available.*
Brar[16] suggested the estimator of population mean as

$$\bar{y}_{pp}=W_1\bar{y}_{p1}+W_2\bar{y}_2,$$

where $\bar{y}_{p1} = \bar{y}_1 \frac{\bar{x}_1}{\bar{X}_1}$.

For large N, the expressions for the variances are as follows:

$$V(\bar{y}_{pp}) = \frac{1}{n}\left[W_1\bar{Y}_1^2\left(C_{1y}^2 + C_{1x}^2 + 2\rho_1 C_{1x}C_{1y}\right) + W_2\bar{Y}_2^2 C_{2y}^2\right]. \quad (3)$$

Here, $\bar{y}_{pp}$ perform better than $\bar{y}'_{st}$ under certain conditions.

Case-3: *When W_1 and W_2 are unknown and partial auxiliary information available.*

In this case, the existing estimator is

$$\bar{y}'_{pp} = w_1\bar{y}_{p1} + w_2\bar{y}_2.$$

For large N,

$$V\left(\bar{y}'_{pp}\right) = \frac{1}{n}\left[W_1\bar{Y}_1^2\left(C_{1y}^2 + C_{1x}^2 + 2\rho_1 C_{1x}C_{1y}\right) + W_2\bar{Y}_2^2 C_{2y}^2\right] + \frac{W_1W_2}{n}\left(\bar{Y}_1 - \bar{Y}_2\right)^2. \quad (4)$$

Moreover, $\bar{y}'_{pp}$ is more efficient than $\bar{y}$ under certain conditions.

4 Proposed Estimators

In this section, a partial product exponential estimator is introduced for the estimation of the population mean in situations where only partial auxiliary information is accessible or relevant. Here, two specific cases are to be addressed: firstly, when the weights of classes A and A' are predetermined, and secondly, when these weights are unknown.

4.1 Case I: When W_1 and W_2 are Known

In this case, the partial product exponential estimator for the population mean can be defined as follows:

$$\bar{y}_{ppe} = W_1\bar{y}_{1pe} + W_2\bar{y}_2, \quad (5)$$

where $\bar{y}_{1pe} = \bar{y}_1\, exp\left[\frac{\bar{x}_1 - \bar{X}_1}{\bar{x}_1 + \bar{X}_1}\right]$.

4.1.1 Approximate Expression for $Bias(\bar{y}_{ppe})$

By using the property of conditional expectation and from the additive property of expectation, the approximate expression for bias can be derived as

$$E(\bar{y}_{ppe}) = E_1\left[E_2\left(\bar{y}_{ppe}|n_1\right)\right] = W_1E_1\left[E_2\left(\bar{y}_{1pe}|n_1\right)\right] + W_2\bar{Y}_2,$$

such that after simplification, the expression becomes

$$E(\bar{y}_{ppe}) = W_1E_1\left[\bar{Y}_1 + \bar{Y}_1\left(\frac{1}{n_1} - \frac{1}{N_1}\right) \times \left(\rho_1\frac{C_{1y}C_{1x}}{2} - \frac{C_{1x}^2}{8}\right)\right] + W_2\bar{Y}_2$$

$$Bias(\bar{y}_{ppe}) = W_1\bar{Y}_1\left(E_1\left(\frac{1}{n_1}\right) - \frac{1}{N_1}\right) \times \left(\rho_1\frac{C_{1y}C_{1x}}{2} - \frac{C_{1x}^2}{8}\right). \quad (6)$$

An exact expression for $E\left(\frac{1}{n_1}\right)$ in the form of an infinite series has been obtained by Stephan[19]. Here, an approximate expression is utilized, as

$$E\left(\frac{1}{n_1}\right) \approx \frac{1}{nW_1}\left(1 + \frac{1 - W_1}{nW_1}\right). \quad (7)$$

After using (7) in (6), we have

$$Bias(\bar{y}_{ppe}) = \left(\frac{1}{n} - \frac{1}{N}\right)\bar{Y}_1\left(\rho_1\frac{C_{1y}C_{1x}}{2} - \frac{C_{1x}^2}{8}\right) + \frac{1 - W_1}{n^2W_1}\left(\rho_1\frac{C_{1y}C_{1x}}{2} - \frac{C_{1x}^2}{8}\right).$$

Approximate expression upto first order is

$$Bias(\bar{y}_{ppe}) = \left(\frac{1}{n} - \frac{1}{N}\right)\bar{Y}_1\left(\rho_1\frac{C_{1y}C_{1x}}{2} - \frac{C_{1x}^2}{8}\right).$$

For large N, the bias becomes

$$Bias(\bar{y}_{ppe}) = \frac{\bar{Y}_1}{n}\left(\rho_1\frac{C_{1y}C_{1x}}{2} - \frac{C_{1x}^2}{8}\right), \quad (8)$$

which is the required approximate expression for bias of $(\bar{y}_{ppe})$.

4.1.2 Approximate Expression for $V(\bar{y}_{ppe})$

The approximate expression for variance can be derived as

$$V(\bar{y}_{ppe}) = E_1\left[V_2\left(\bar{y}_{ppe}|n_1\right)\right] + V_1\left[E_2\left(\bar{y}_{ppe}|n_1\right)\right]. \quad (9)$$

After using the basic theory of simple random sampling, we have

$$V(\bar{y}_{ppe}) = \left(\frac{1}{n} - \frac{1}{N}\right)\left[W_1\bar{Y}_1^2\left(C_{1y}^2 + \frac{1}{4}C_{1x}^2 + \rho_1C_{1y}C_{1x}\right) + W_2\bar{Y}_2^2C_{2y}^2\right] + \frac{1 - W_2}{n^2W_2}\left[W_2\bar{Y}_1^2\left(C_{1y}^2 + \frac{1}{4}C_{1x}^2 + \rho_1C_{1y}C_{1x}\right) + W_1\bar{Y}_2^2C_{2y}^2\right] + \bar{Y}_1^2\left(\rho_1\frac{C_{1y}C_{1x}}{2} - \frac{C_{1x}^2}{8}\right)^2 V_1\left(\frac{1}{n_1}\right).$$

Upto order n^{-1},

$$V(\bar{y}_{ppe}) = \left(\frac{1}{n} - \frac{1}{N}\right)\left[W_1\bar{Y}_1^2\left(C_{1y}^2 + \frac{1}{4}C_{1x}^2 + \rho_1C_{1y}C_{1x}\right) + W_2\bar{Y}_2^2C_{2y}^2\right]. \quad (10)$$

For large N, the expression for variance is

$$V(\bar{y}_{ppe}) = \frac{1}{n}\left[W_1\bar{Y}_1^2\left(C_{1y}^2 + \frac{1}{4}C_{1x}^2 + \rho_1C_{1y}C_{1x}\right) + W_2\bar{Y}_2^2C_{2y}^2\right], \quad (11)$$

which is the required expression for approximate variance when W_1 and W_2 are known.

4.2 Case II: When W_1 and W_2 are Unknown

By using the estimated values of W_1 and W_2, the estimator of population mean can be defined as follows

$$\bar{y}_{ppe}^{'} = w_1 \bar{y}_{1pe} + w_2 \bar{y}_2,$$

where $\bar{y}_{1pe} = \bar{y}_1 \, exp\left[\frac{\bar{x}_1 - \bar{X}_1}{\bar{x}_1 + \bar{X}_1}\right]$.

4.2.1 Approximate Expression for $Bias\left(\bar{y}_{ppe}^{'}\right)$

Proceeding in similar manner as in the Case-I, the approximate expression for bias upto first order can be obtained as

$$Bias\left(\bar{y}_{ppe}^{'}\right) = \left(\frac{1}{n} - \frac{1}{N}\right) \bar{Y}_1 \left(\rho_1 \frac{C_{1y} C_{1x}}{2} - \frac{C_{1x}^2}{8}\right).$$

After ignoring finite population correction, the expression becomes

$$Bias\left(\bar{y}_{ppe}^{'}\right) = \frac{\bar{Y}_1}{n} \left(\rho_1 \frac{C_{1y} C_{1x}}{2} - \frac{C_{1x}^2}{8}\right), \tag{12}$$

which is the required approximate expression for bias of $\bar{y}_{ppe}^{'}$.

4.2.2 Approximate Expression for $V\left(\bar{y}_{ppe}^{'}\right)$

Similarly,

$$\begin{aligned} V\left(\bar{y}_{ppe}^{'}\right) &= \left(\frac{1}{n} - \frac{1}{N}\right) \left[W_1 \bar{Y}_1^2 \left(C_{1y}^2 + \frac{1}{4} C_{1x}^2 + \rho_1 C_{1y} C_{1x}\right) \right. \\ &\left. + W_2 \bar{Y}_2^2 C_{2y}^2\right] + \frac{N-n}{nN(N-1)} \\ &\times \left[W_2 \bar{Y}_1^2 \left(C_{1y}^2 + \frac{1}{4} C_{1x}^2 + \rho_1 C_{1y} C_{1x}\right) + W_1 \bar{Y}_2^2 C_{2y}^2\right] \\ &+ \frac{N-n}{n(N-1)} W_1 W_2 \left(\bar{Y}_1 - \bar{Y}_2\right)^2 \\ &+ \frac{N-n}{nN(N-1)} W_2 \left(\rho_1 \frac{C_{1y} C_{1x}}{2} - \frac{C_{1x}^2}{8}\right) \left[2\bar{Y}_1^2 + 2\bar{Y}_1 \bar{Y}_2 \right. \\ &\left. + \bar{Y}_1^2 \frac{1}{N W_1} \left(\rho_1 \frac{C_{1y} C_{1x}}{2} - \frac{C_{1x}^2}{8}\right)\right]. \end{aligned}$$

For large N, the expression becomes

$$\begin{aligned} V\left(\bar{y}_{ppe}^{'}\right) &= \frac{1}{n} \left[W_1 \bar{Y}_1^2 \left(C_{1y}^2 + \frac{1}{4} C_{1x}^2 + \rho_1 C_{1y} C_{1x}\right) \right. \\ &\left. + W_2 \bar{Y}_2^2 C_{2y}^2\right] + \frac{1}{n} W_1 W_2 (\bar{Y}_1 - \bar{Y}_2)^2. \end{aligned} \tag{13}$$

which is the required approximate expression for $V\left(\bar{y}_{ppe}^{'}\right)$.

5 Comparison

With the aim of establishing the superiority of the suggested estimators over the existing mean per unit estimator, post stratified estimator and partial product estimator, the theoretical comparisons are as follows:

Theorem 5.1 *Upto the first order approximation, the proposed estimator when weights are known $\bar{y}_{ppe}$ is better than mean per unit estimator $\bar{y}$ if the condition*

$$\rho_1 < -\frac{1}{4} \frac{C_{1x}}{C_{1y}} + \frac{W_2}{C_{1y} C_{1x}} \left(\frac{\bar{Y}_1 - \bar{Y}_2}{\bar{Y}_1}\right)^2 \tag{14}$$

holds.

Proof: After comparing equations (1) and (10), it can be seen that

$$V(\bar{y}) - V(\bar{y}_{ppe}) > 0,$$

if

$$W_2 \left(\bar{Y}_1 - \bar{Y}_2\right)^2 - \frac{1}{4} \bar{Y}_1^2 C_{1x}^2 - \rho_1 \bar{Y}_1^2 C_{1y} C_{1x} > 0,$$

or

$$\frac{W_2}{C_{1y} C_{1x}} \left(\frac{\bar{Y}_1 - \bar{Y}_2}{\bar{Y}_1}\right)^2 - \frac{1}{4} \frac{C_{1x}}{C_{1y}} > \rho_1.$$

Hence, the proof.

Theorem 5.2 *Upto the first order approximate, the proposed estimator $\bar{y}_{ppe}'$ is better than mean per unit estimator $\bar{y}$ if condition*

$$\rho_1 < -\frac{1}{4} \frac{C_{1x}}{C_{1y}} \tag{15}$$

holds.

Proof: After comparing equations (1) and (13), it can be seen that

$$V(\bar{y}) - V\left(\bar{y}_{ppe}^{'}\right) > 0,$$

if

$$-\frac{1}{4} \bar{Y}_1^2 C_{1x}^2 - \rho_1 \bar{Y}_1^2 C_{1y} C_{1x} > 0,$$

or

$$\rho_1 < -\frac{1}{4} \frac{C_{1x}}{C_{1y}}.$$

Hence, the proof.

Theorem 5.3 *Upto the first order approximation, the proposed estimator $\bar{y}_{ppe}$ is better than post stratified estimator $\bar{y}_{st}'$ if condition*

$$\rho_1 < -\frac{1}{4} \frac{C_{1x}}{C_{1y}} \tag{16}$$

holds.

Proof: After comparing equations (2) and (10), it can be seen that

$$V\left(\bar{y}_{st}^{'}\right) - V(\bar{y}_{ppe}) > 0,$$

if

$$-\frac{1}{4} \bar{Y}_1^2 C_{1x}^2 - \rho_1 \bar{Y}_1^2 C_{1y} C_{1x} > 0,$$

or

$$\rho_1 < -\frac{1}{4} \frac{C_{1x}}{C_{1y}}.$$

Hence, the proof.

Theorem 5.4 *Upto first order approximation, the proposed estimator $\bar{y}_{ppe}$ and $\bar{y}^{'}_{ppe}$ are better than partial product estimator $\bar{y}_{pp}$ and $\bar{y}^{'}_{pp}$ respectively if condition*

$$\rho_1 > -\frac{3}{4}\frac{C_{1x}}{C_{1y}} \tag{17}$$

holds.

Proof: After comparing equation (3) with equation (11) and (4) with equation (13), it is found that

$$V\left(\bar{y}_{pp}\right) - V\left(\bar{y}_{ppe}\right) = V\left(\bar{y}^{'}_{pp}\right) - V\left(\bar{y}^{'}_{ppe}\right) > 0,$$

if

$$\frac{1}{n}\left(\frac{3}{4}W_1\bar{Y}_1^2C_{1x}^2 + \rho_1 W_1\bar{Y}_1^2C_{1y}C_{1x}\right) > 0,$$

or

$$\frac{3}{4}C_{1x} + \rho_1 C_{1y} > 0,$$

or

$$\rho_1 > -\frac{3}{4}\frac{C_{1x}}{C_{1y}}.$$

Hence, the proof.

Remark 5.5 *From equations* (14)*,* (15)*,* (16)*, and* (17) *it is concluded that the partial product type exponential estimator outperforms $\bar{y}$, $\bar{y}^{'}_{st}$, $\bar{y}_{pp}$, and $\bar{y}^{'}_{pp}$ in terms of efficiency when the correlation coefficient ρ_1 falls within the range of:*

$$-\frac{3}{4}\frac{C_{1x}}{C_{1y}} < \rho_1 < -\frac{1}{4}\frac{C_{1x}}{C_{1y}} \tag{18}$$

Within this specific range of ρ_1 values, the partial product type exponential estimator demonstrates superior efficiency compared to the corresponding existing estimators.

6 Simulation Study

To validate the theoretical findings, a simulation study is conducted using the R software under two different populations.

Generated Population-I

A population consisting of size 500 units, generated from a normal distribution, is presented in Table 1.

From the generated population, 10,000 simple random samples are selected for every ρ_1 value across sample sizes of 20, 30, and 40. Table-2 presents the biases of various estimators. Furthermore, mean square errors and percentage gains in the precision(PGP) of proposed estimators $\bar{y}_{ppe}$ and $\bar{y}'_{ppe}$ over $\bar{y}$, $\bar{y}'_{st}$, $\bar{y}_{pp}$ and $\bar{y}'_{pp}$ are shown in Table-3 and Table-4, respectively. Efficiency and PGP of an estimator $\hat{\theta}$ can be defined as:
When W_1 and W_2 are known,

$$\begin{aligned} eff(\hat{\theta}) &= \frac{MSE(\bar{y}'_{st})}{MSE(\hat{\theta})} \times 100 \\ PGP(\hat{\theta}) &= \frac{MSE(\bar{y}'_{st}) - MSE(\hat{\theta})}{MSE(\hat{\theta})} \times 100 \end{aligned}$$

When W_1 and W_2 are unknown,

$$\begin{aligned} eff(\hat{\theta}) &= \frac{MSE(\bar{y})}{MSE(\hat{\theta})} \times 100 \\ PGP(\hat{\theta}) &= \frac{MSE(\bar{y}) - MSE(\hat{\theta})}{MSE(\hat{\theta})} \times 100. \end{aligned}$$

Generated Population-II

Another non-normal population consisting of size 500 units, generated from a exponential distribution, is presented in Table 5. For the generated population, a simulation study has been carried out. Further, mean square errors and percentage gains in the precision(PGP) of proposed estimators $\bar{y}_{ppe}$ and $\bar{y}'_{ppe}$ over $\bar{y}$, $\bar{y}'_{st}$, $\bar{y}_{pp}$ and $\bar{y}'_{pp}$ are shown in Table-7 and Table-8

6.1 Results of simulation study

- Referring to Table-2 and Table-6, it is evident that all the estimators are approximately unbiased for the specified sample sizes within the provided population.

- The suggested estimators outperform the mean per unit estimator under conditions (14) and (16), their efficiency improves with an increase in the correlation coefficient magnitude or the sample size.

- Table-3 and Table-7 reveal that when ρ_1 takes values from $-\frac{3}{4}\frac{C_{1x}}{C_{1y}}$ to $-\frac{1}{4}\frac{C_{1x}}{C_{1y}}$, the proposed estimator $\bar{y}_{ppe}$ is always more efficient than the $\bar{y}'_{st}$ and $\bar{y}_{pp}$. So, the newly developed estimator performs better than the corresponding existing estimators under the condition (18).

- Likewise, Table-4 and Table-8 reveal that when ρ_1 takes values from $-\frac{3}{4}\frac{C_{1x}}{C_{1y}}$ to $-\frac{1}{4}\frac{C_{1x}}{C_{1y}}$, the proposed estimator $\bar{y}'_{ppe}$ is always more efficient than the corresponding existing estimators $\bar{y}$ and $\bar{y}'_{pp}$.

- When the values of W_1 and W_2 are known, there is a significantly higher improvement in precision compared to the case where W_1 and W_2 are unknown.

Table 1: **Details of the generated population-I**

Sub-Population-1			
$N_1 = 300$ $\bar{Y}_1 \approx 60.1506515$ $S^2_{1y} \approx 25.4900048$			
ρ_1	-0.30	-0.57	-0.91
X_1	70.0494092	70.0165198	69.951065
S^2_{1x}	8.3661634	8.7399601	9.272685
$\frac{-1}{4}\frac{C_{1x}}{C_{1y}}$	-0.1229855	-0.125762	-0.1296593
$\frac{-3}{4}\frac{C_{1x}}{C_{1y}}$	-0.3689566	-0.3772861	-0.388978
Sub-Population-2			
$N_2 = 200$ $\bar{Y}_2 \approx 90.39518$ $S^2_{2y} = 24.64653$			
Population			
$N = 500$ $\bar{Y} = 72.24846$ $S^2_2 = 245.07809$			

Table 2: (Population-I) Biases of $\bar{y}, \bar{y}'_{st}, \bar{y}_{pp}, \bar{y}'_{pp}, \bar{y}_{ppe}, \bar{y}'_{ppe}$

ρ_1	$Bias\,(\$\bar{y})$	$Bias\left(\bar{y}'_{st}\right)$	$Bias\,(\bar{y}_{pp})$	$Bias\left(\bar{y}'_{pp}\right)$	$Bias\,(\bar{y}_{ppe})$	$Bias\left(\bar{y}'_{ppe}\right)$
$n = 20$						
-0.3	0.0443	0.0088	0.0109	0.0453	0.0093	0.0442
-0.57	0.0443	0.0088	0.0075	0.0419	0.0075	0.0425
-0.91	0.0443	0.0088	0.0015	0.0359	0.0045	0.0395
$n = 30$						
-0.3	0.0246	0.0110	0.0118	0.0251	0.01104108	0.0244
-0.57	0.0246	0.0110	0.0098	0.0231	0.0100	0.0234
-0.91	0.0246	0.0110	0.0064	0.0197	0.0083	0.0217
$n = 40$						
-0.3	0.0423	0.0062	0.0099	0.0457	0.0078	0.0437
-0.57	0.0423	0.0062	0.0087	0.0444	0.0072	0.0431
-0.91	0.0423	0.0062	0.0057	0.0416	0.0057	0.0417

Table 3: (Population-I) MSEs, Efficiencies and PGPs of the estimators when W_1 and W_2 are known

ρ_1	MSE $(\bar{y}'_{st})$	MSE $(\bar{y}_{pp})$	MSE $(\bar{y}_{ppe})$	eff $(\bar{y}_{pp})$	eff $(\bar{y}_{ppe})$	PGP $(\bar{y}_{pp})$	PGP $(\bar{y}_{ppe})$
$n = 20$							
-0.3	1.2800	1.2454	1.2176	102.7734	105.1219	2.7734	5.1219
-0.57	1.2800	1.0462	1.1160	122.3438	114.6894	22.3438	14.6894
-0.91	1.2800	0.7722	0.9759	165.7508	131.1527	65.7508	31.1527
$n = 30$							
-0.3	0.8417	0.8175	0.7999	102.9578	105.2179	2.9578	5.2179
-0.57	0.8417	0.6836	0.7316	123.1272	115.0364	23.1272	15.0364
-0.91	0.8417	0.4996	0.6374	168.4631	132.0381	68.4631	32.0381
$n = 40$							
-0.3	0.5980	0.5798	0.5676	103.1416	105.3687	3.1416	5.3687
-0.57	0.5980	0.4836	0.5185	123.6639	115.3454	23.6639	15.3454
-0.91	0.5980	0.3518	0.4509	170.0126	132.6388	70.0126	32.6388

Table 4: (Population-I) MSEs, Efficiencies and PGPs of the estimators when W_1 and W_2 are unknown

ρ_1	MSE $(\bar{y})$	MSE $(\bar{y}'_{pp})$	MSE $(\bar{y}'_{ppe})$	eff $(\bar{y}'_{pp})$	eff $(\bar{y}'_{ppe})$	PGP $(\bar{y}'_{pp})$	PGP $(\bar{y}'_{ppe})$
$n = 20$							
-0.3	11.6474	11.6488	11.6049	99.9881	100.3659	-0.0119	0.3659
-0.57	11.6474	11.4613	11.5092	101.6238	101.2005	1.6238	1.2005
-0.91	11.6474	11.1971	11.3738	104.0220	102.4057	4.0220	2.4057
$n = 30$							
-0.3	7.6321	7.6244	7.5995	100.1003	100.4293	0.1003	0.4293
-0.57	7.6321	7.4964	7.5342	101.8098	101.2991	1.8098	1.2991
-0.91	7.6321	7.3174	7.4426	104.3008	102.5465	4.3008	2.5465
$n = 40$							
-0.3	5.5854	5.5796	5.5616	100.1041	100.4266	0.1041	0.4266
-0.57	5.5854	5.4855	5.5136	101.8206	101.3009	1.8206	1.3009
-0.91	5.5854	5.3537	5.4461	104.3273	102.5579	4.3273	2.5579

Table 5: **Details of the generated population-II**

Sub-Population-1			
$N_1 = 300$ $\bar{Y}_1 \approx 24.00$ $S^2_{1y} \approx 7.00$			
ρ_1	-0.176	-0.526	-0.696
$\bar{X}_1$	45.00	45.00	45.00
S^2_{1x}	9.283	12.440	17.439
$\frac{-1}{4}\frac{C_{1x}}{C_{1y}}$	-0.1535495	-0.177743	-0.2104478
$\frac{-3}{4}\frac{C_{1x}}{C_{1y}}$	-0.4606485	-0.5332291	-0.6313433
Sub-Population-2			
$N_2 = 200$ $\bar{Y}_2 = 89.8852$ $S^2_{2y} = 22.4706$			
Population			
$N = 500$ $\bar{Y} = 50.35408$ $S^2_2 = 1057.0495$			

Table 6: (Population-II) Bias of $\bar{y}, \bar{y}'_{st}, \bar{y}_{pp}, \bar{y}'_{pp}, \bar{y}_{ppe}, \bar{y}'_{ppe}$

ρ_1	$Bias\,(\bar{y})$	$Bias\left(\bar{y}'_{st}\right)$	$Bias\,(\bar{y}_{pp})$	$Bias\left(\bar{y}'_{pp}\right)$	$Bias\,(\bar{y}_{ppe})$	$Bias\left(\bar{y}'_{ppe}\right)$
$n = 20$						
-0.176	-0.0052	-0.0040	-0.0063	-0.0071	-0.0058	-0.0068
-0.526	-0.0052	-0.0040	-0.0091	-0.0097	-0.0075	-0.0083
-0.696	-0.0052	-0.0040	-0.0113	-0.0117	-0.0090	-0.0097
$n = 30$						
-0.176	0.0127	0.0017	0.0004	0.0117	0.0006	0.0118
-0.526	0.0127	0.0017	-0.0026	0.0088	-0.0010	0.0102
-0.696	0.0127	0.0017	-0.0050	0.0064	-0.0025	0.0087
$n = 40$						
-0.176	0.0005	-0.0038	-0.0035	0.0007	-0.0040	0.0003
-0.526	0.0005	-0.0038	-0.0059	-0.0017	-0.0053	-0.0010
-0.696	0.0005	-0.0038	-0.0077	-0.0035	-0.0064	-0.0021

Table 7: (Population-II) MSEs, Efficiencies and PGPs of the estimators when W_1 and W_2 are known

ρ_1	MSE $(\bar{y}'_{st})$	MSE $(\bar{y}_{pp})$	MSE $(\bar{y}_{ppe})$	eff $(\bar{y}_{pp})$	eff $(\bar{y}_{ppe})$	PGP $(\bar{y}_{pp})$	PGP $(\bar{y}_{ppe})$
$n = 20$							
-0.176	0.6888	0.7212	0.6854	95.4960	100.4829	-4.5040	0.4829
-0.526	0.6888	0.6347	0.6353	108.5086	108.4176	8.5086	8.4176
-0.696	0.6888	0.5881	0.6009	117.1145	114.6245	17.1145	14.6245
$n = 30$							
-0.176	0.4387	0.4595	0.4363	95.4596	100.5509	-4.5404	0.5509
-0.526	0.4387	0.4034	0.4037	108.7275	108.6684	8.7275	8.6684
-0.696	0.4387	0.3732	0.3813	117.5373	115.0358	17.5373	15.0358
$n = 40$							
-0.176	0.3209	0.3360	0.3190	95.4879	100.5795	-4.51205	0.5795
-0.526	0.3209	0.2955	0.2955	108.5940	108.5940	8.5940	8.5940
-0.696	0.3209	0.2734	0.2793	117.3440	114.8839	17.3441	14.8839

Table 8: (Population-II) MSEs, Efficiencies and PGPs of the estimators when W_1 and W_2 are unknown

ρ_1	MSE $(\bar{y})$	MSE $(\bar{y}'_{pp})$	MSE $(\bar{y}'_{ppe})$	eff $(\bar{y}'_{pp})$	eff $(\bar{y}'_{ppe})$	PGP $(\bar{y}'_{pp})$	PGP $(\bar{y}'_{ppe})$
$n = 20$							
-0.176	50.2714	50.2954	50.2646	99.9523	100.0136	-0.0477	0.0136
-0.526	50.2714	50.2013	50.2109	100.1397	100.1206	0.1397	0.1206
-0.696	50.2714	50.1478	50.1735	100.2466	100.1953	0.2466	0.1953
$n = 30$							
-0.176	33.3815	33.3750	33.3657	100.0196	100.0474	0.0196	0.0474
-0.526	33.3815	33.3169	33.3322	100.1938	100.1478	0.1938	0.1478
-0.696	33.3815	33.2847	33.3090	100.2908	100.2175	0.2908	0.2175
$n = 40$							
-0.176	24.8583	24.8733	24.8565	99.9398	100.0072	-0.0602	0.0072
-0.526	24.8583	24.8383	24.8358	100.0808	100.0907	0.0808	0.0907
-0.696	24.8583	24.8205	24.8219	100.1526	100.1470	0.1526	0.1470

7 Conclusions

The suggested partial product-type exponential estimators for population mean in simple random sampling without replacement, incorporating partial auxiliary information, demonstrate greater efficiency compared to existing methods such as the mean per unit estimator, partial product estimators, and post-stratified estimator, under specific conditions. Notably, when W_1 and W_2 are known, the proposed estimator $(\bar{y}_{ppe})$ outperforms the post-stratified weighted estimator and corresponding partial product estimator under certain conditions. Similarly, in situations where W_1 and W_2 are unknown, the proposed estimator $(\bar{y}'_{ppe})$ is more efficient than the mean per unit estimator and corresponding partial product estimator under specific conditions. In the practical cases, the use of the proposed estimators has been proved to be more useful than the corresponding existing estimators.

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