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Coneighbor Graphs and Related Topologies

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Abstract The primary aim of this paper is to establish and analyze certain topological structures linked with a specified graph $\mathcal{G}$. In a graph $\mathcal{G}$, a vertex u is considered a neighbor of another vertex v if there exists an edge uv in $\mathcal{G}$. Furthermore, we define two vertices (or edges) in $\mathcal{G}$ as coneighbors if they share identical sets of neighboring vertices (or edges). The topology under consideration arises from the collections of vertices that are coneighbor and the collections of edges that are coneighbor within the graph. It is proved that the coneighbor topology of every non-coneighbor graph is homeomorphic to the included point topology while this space is quasi-discrete if and only if the graph contains at least one coneighbor set of vertices and some examples of coneighbor topologies of special graphs are presented to be quasi-discrete spaces such as (a path, a cycle and a bipartite) graphs. Moreover, several topological properties of the coneighbor space are presented. We proved that the coneighbor topological space associated with a graph $\mathcal{G}$ always has dimension one and satisfies the $T_{1/2}$ axiom. Also, the family of θ-open sets is determined in this spaces and it is proved that this space is almost compact whenever the family of coneighbor sets is finite. Finally, we looked at some graphs in which the coneighbor space fulfills other topological concepts such as connectedness, compactness and countable compactness.

Keywords Coneighbor Graph, Coneighbor Set, Coneighbor Topological Space

1 Introduction

Graph theory and topology are two important branches of mathematics. These branches have been applied to numerous problems in mathematics, computer science, optimization, operational research, as well as other scientific areas including chemistry, sociology, and genetics. Over the past two decades, several investigations have introduced innovative topological configurations formed by sets of vertices and edges in graph theory and defined concepts in topological space via graphs. Majumdar [1] constructed a new topology induced by undirected graph and presented some examples to illustrate this type of topology. Also, he discussed the connections between some special functions in topological space. Kilicman and Abdulkalek [2] introduced the incidence topology and presented some results of this type of topology and showed that this topology satisfies the property of Alexandroff. Novel types of topological structures were explored, and an algorithm to create these structures from various graphs was investigated. Additionally, applications in medicine and geography were presented in [3]. Sarı and Kopuzlu [4] created a topological space by using undirected simple graph and presented some of its properties. Moreover, he examined the topologies generated by using standard graphs, and presented a condition in which the topological space is generated by two different graphs to be homeomorphic. Jasim and Awad [5] introduced some concepts of topological spaces generalized to graph and the relations of these concepts were studied and explained by some examples. A new category of topological structures was introduced utilizing graphs, using the notion of homeomorphism between topological graphs to demonstrate the isomorphism of two graphs by Kozae et al [6] and a computer program was developed to generate both graphs and their corresponding topological graphs. Several examples of diverse graphs and their associated topological graphs were provided, and their topological and algebraic properties were examined. Ibrahim and Khalaf [7] formulated a new topological structure, known as the maximal block topological space based on connected simple undirected graphs. Some results of this newly defined topology were investigated in detail. Furthermore, they demon-

strated that a maximal block topological space becomes a $T_{1/2}$-space when G is an acyclic graph. Additionally, they introduced the concepts of irreducibility and topological independence in maximal block topological spaces.
The objective of this article is to introduce a novel type of topological spaces produced by sets of vertices and edges in undirected graphs. Furthermore, the paper aims to explore various results associated with a newly defined type of topological space on graphs, which is called coneighbor topological spaces.

2 Preliminaries

This section provides an introduction to several definitions and results in both topological spaces and graph theory.

Definition 2.1. [8] Let $X \neq \phi$ and τ be a family of subsets of X. We say that τ is a topology on X if it satisfies the following conditions:

1. $\phi, X \in \tau$
2. If $\mathcal{U}, \mathcal{V} \in \tau$, then $\mathcal{U} \cap \mathcal{V} \in \tau$.
3. If $\{\mathcal{U}_\alpha \in \tau\}$, $\alpha \in \Delta$, then $\cup_{\alpha\in\Delta}\mathcal{U}_\alpha \in \tau$.
   The ordered pair (X, τ) is called a topological space. And members of τ are open sets, while their complements are called closed sets.

Definition 2.2. [8] For any non-empty set X, the family $\tau_{ind} = \{\phi, X\}$ is called the indiscrete topology and $\tau_d = P(X)$, where $P(X)$ is the collection of all subsets of X and it is called the discrete topology. A topological space (X, τ) is called quasi-discrete if $\mathcal{U}$ and $\mathcal{V}$ are open sets in τ, then $\mathcal{U} \cap \mathcal{V} = \phi$ and $\mathcal{U} \cup \mathcal{V} = X$.

Definition 2.3. [8] Let (X, τ) be a topological space and $\mathcal{U} \subseteq X$. Then

1. The closure of $\mathcal{U}$ denoted by $Cl(\mathcal{U})$ is defined as the smallest closed set that includes $\mathcal{U}$.
2. The interior of $\mathcal{U}$ denoted by $Int(\mathcal{U})$ is defined as the largest open set contained in $\mathcal{U}$.

Definition 2.4. [8] A space (X, τ) is called T_0, if for any distinct elements $u, v \in X$, there exists an open set $\mathcal{H}$ such that $u \in \mathcal{H}$ and $v \notin \mathcal{H}$ or $u \notin \mathcal{H}$ and $v \in \mathcal{H}$.

Definition 2.5. [9] A space (X, τ) is called $T_{1/2}$, if either Int({x})={x} or Cl({x})={x} for every element $x \in X$.

Definition 2.6. [8] A space (X, τ) is called disconnected topology if X can be written as a disjoint union of two non-empty open sets in (X, τ). Otherwise is said to be connected topology.

Definition 2.7. [8] A (X, τ) is called:

1. separable if X contains a countable dense subset.
2. extremely disconnected if the closure of every open set is open.
3. locally connected if every point has a connected neighborhood.

Definition 2.8. [10, 11] A graph $\mathcal{G}$ is an ordered pair comprising a non-empty set of vertices, denoted as $\mathcal{V}(\mathcal{G})$, and a set of edges, denoted as $E(G)$. The elements within $E(G)$ are two-element subsets from $\mathcal{V}(\mathcal{G})$.

Definition 2.9. [12] Let $\mathcal{V}(C_1), \mathcal{V}(C_2), \cdots, \mathcal{V}(C_p)$ be the sets of coneighbor vertices in $\mathcal{V}(\mathcal{G})$ and $E(K_1), E(K_2), \cdots, E(K_q)$ be the sets of coneighbor edges in $E(G)$. Then we have the following:

1. $\mathcal{G}$ is called coneighbor graph if it includes $\mathcal{V}(C_i)$ for $i \in \{1, 2, ..., p\}$, while, a graph $\mathcal{G}$ is called edge coneighbor if it includes $E(K_j)$ for $j \in \{1, 2, ..., q\}$.
2. $\mathcal{G}$ is called complete coneighbor graph if $\bigcup_{i=1}^{p} \mathcal{V}(C_i) = \mathcal{V}(\mathcal{G})$.
3. $\mathcal{G}$ is called complete edge coneighbor graph if $\bigcup_{m=1}^{q} E(K_m) = E(\mathcal{G})$.
4. $\mathcal{G}$ is called completely coneighbor graph if $\bigcup_{i=1}^{p} \mathcal{V}(C_i) = \mathcal{V}(\mathcal{G})$ and $\bigcup_{m=1}^{q} E(K_m) = E(\mathcal{G})$.

3 Coneighbor Topological Space

This section introduces novel types of topological spaces derived from coneighbor sets of vertices and coneighbor sets of edges in a graph $\mathcal{G}$, along with an exploration of some of their properties. By $\mathcal{N}(\mathcal{V}, v)$, we denote all subsets of V containing v.

Lemma 3.1. 1. Suppose that $\mathcal{V}(C_1), \mathcal{V}(C_2), \cdots$ are the sets of coneighbor vertices in $\mathcal{V}(\mathcal{G})$. Let $\mathcal{V}(C_0) = \mathcal{V}(\mathcal{G}) \setminus \bigcup_{i=1}^{\infty} \mathcal{V}(C_i)$, then $\{\mathcal{V}(C_i) : i \geq 0\}$ is a collection of pairwise disjoint sets of vertices.

2. Suppose that $E(K_1), E(K_2), \cdots$ are the sets of coneighbor edges in $E(\mathcal{G})$. Let $E(K_0) = E(\mathcal{G}) \setminus \bigcup_{m=1}^{\infty} E(K_m)$, then $\{E(K_m) : m \geq 0\}$ is a collection of pairwise disjoint sets of edges.

Proof. 1. If $v \in \mathcal{V}(C_i) \cap \mathcal{V}(C_j)$, then $N(\{v\}) = N(\mathcal{V}(C_i)) = N(\mathcal{V}(C_j))$ which is contradiction. Hence $\{\mathcal{V}(C_i) : i \geq 0\}$ is a collection of pairwise disjoint sets of vertices.

2. Similar to the proof of 1.

□

Definition 3.2. Suppose that $\mathcal{V}(C_i) = \{v_1^i, v_2^i, \cdots\}$ and $E(K_m) = \{e_1^m, e_2^m, \cdots\}$ for all $i, m \geq 0$. Fixing a vertex $v_j^i \in \mathcal{V}(C_i)$ and $e_n^m \in E(K_m)$ for some $j \geq 1$ and some $n \geq 1$. Then we define the following collections:

$$\mathcal{B}_C(\mathcal{G}) = \{\mathcal{N}(\mathcal{V}(C_i), v_j^i) : i \geq 0\}$$

and

$$\mathcal{B}_K(\mathcal{G}) = \{\mathcal{N}(E(K_m), e_n^m) : m \geq 0\}$$

Theorem 3.3. If $\mathcal{G}$ is any graph (infinite), then the family

$$\mathcal{B}_C(\mathcal{G}) = \{\mathcal{N}(\mathcal{V}(C_i), v_j^i) : i \geq 0\}$$

is a base for a topology on $\mathcal{V}(\mathcal{G})$.

Proof. Clearly,

$$\mathcal{V}(\mathcal{G}) = \bigcup \{\mathcal{N}(\mathcal{V}(C_i), v_j^i) : i \geq 0\}$$

If $B_1, B_2 \in \mathcal{B}_C(\mathcal{G})$ and $v \in B_1 \cap B_2$, so obviously, B_1, B_2 are in the same $\mathcal{V}(C_i)$ and hence, by Definition 3.2, there exists v_j^i such that
$v \in \{v, v_j^i\} \subseteq B_1 \cap B_2$. This completes the proof. □

Similarly, we obtain the subsequent result.

Theorem 3.4. If $\mathcal{G}$ is any graph (infinite), then the family

$$\mathcal{B}_K(\mathcal{G}) = \{\mathcal{N}(E(K_m), e_n^m) : m \geq 0\}$$

is a base for a topology on $E(\mathcal{G})$.

The topologies induced by $\mathcal{B}_C(\mathcal{G})$ and $\mathcal{B}_K(\mathcal{G})$ are denoted by $\tau_C(\mathcal{G})$ and $\tau_K(\mathcal{G})$, respectively, the spaces $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ and $(E(\mathcal{G}), \tau_K(\mathcal{G}))$ are called coneighbor and edge coneighbor topological spaces, respectively.
It is clear that for each $v_j^i \in \mathcal{V}(C_i)$, there is a topology which is induced by $\mathcal{B}_K(\mathcal{G})$ for $j \geq 1$ and hence there are several topologies on $\mathcal{V}(\mathcal{G})$ by taking different fixed vertices in each $\mathcal{V}(C_i)$. Hence, if $u, v \in \mathcal{V}(C_i)$, then the topology on $\mathcal{V}(\mathcal{G})$ induced by fixing u is homeomorphic to the topology on $\mathcal{V}(\mathcal{G})$ induced by fixing v because the map $f : \mathcal{V}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$ given by $f(u) = v$, $f(v) = u$ and $f(w) = w$ otherwise is a homeomorphism. Therefore, all these topologies are homeomorphic.
Similarly, we get homeomorphic topologies on $E(\mathcal{G})$.

Remark 3.5. All members of $\mathcal{B}_C(\mathcal{G})$ are minimal open sets in $\tau_C(\mathcal{G})$ and each member of $\mathcal{B}_K(\mathcal{G})$ is a minimal open set in $\tau_K(\mathcal{G})$.

Example 3.6. Consider the graph $\mathcal{G}_1$as shown below.

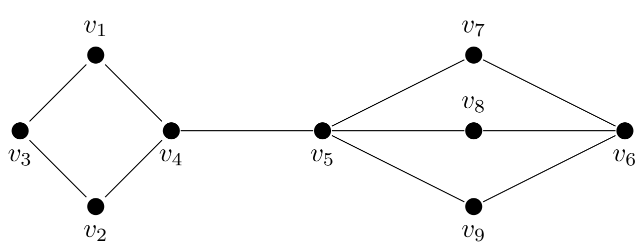

Figure 1. Coneighbor graph $\mathcal{G}_1$.

We have $\mathcal{V}(\mathcal{G}_1) = \{v_1, v_2, \cdots, v_9\}$, by Definition 3.2, the coneighbor base of $\mathcal{G}_1$ is given by $\mathcal{B}_C(\mathcal{G}_1) = \{\{v_1\}, \{v_1, v_2\}, \{v_7\}, \{v_7, v_8\}, \{v_7, v_9\}, \{v_7, v_8, v_9\}, \{v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_3, v_4, v_5\}, \{v_3, v_4, v_6\}, \{v_3, v_5, v_6\}, \{v_3, v_4, v_5, v_6\}\}$. The coneighbor topology of $\mathcal{G}_1$ is obtained by taking the union of members of $\mathcal{B}_C(\mathcal{G}_1)$. Here, the fixed vertices are v_1, v_3 and v_7. Other basis can be obtained by fixing different vertices.

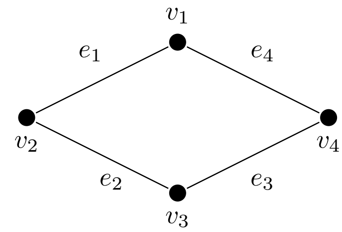

Figure 2. Completely coneighbor graph $\mathcal{G}_2$.

Now consider the graph $\mathcal{G}_2$ as given below.

If we fix v_1 and v_2 in $\mathcal{V}(\mathcal{G}_2) = \{v_1, v_2, v_3, v_4\}$, then the coneighbor base of $\mathcal{G}_2$ is given by:
$\mathcal{B}_C(\mathcal{G}_2) = \{\{v_1\}, \{v_2\}, \{v_1, v_3\}, \{v_2, v_4\}\}$.
The coneighbor topology of $\mathcal{G}_2$ is given by:
$\tau_C(\mathcal{G}_2) = \{\phi, \{v_1\}, \{v_2\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \mathcal{V}(\mathcal{G}_2)\}$.
The homeomorphic topologies on $\mathcal{V}(\mathcal{G}_2)$ are :
$\tau_1(\mathcal{G}_2) = \{\phi, \{v_1\}, \{v_4\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \mathcal{V}(\mathcal{G}_2)\}$ when we fix v_1 and v_4.
$\tau_2(\mathcal{G}_2) = \{\phi, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}, \mathcal{V}(\mathcal{G}_2)\}$ when we fix v_3 and v_4.
$\tau_3(\mathcal{G}_2) = \{\phi, \{v_2\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \mathcal{V}(\mathcal{G}_2)\}$ when we fix v_2 and v_3.
By similar way, we get the base for $E(\mathcal{G}_2) = \{e_1, e_2, e_3, e_4\}$.

Proposition 3.7. 1. For any graph $\mathcal{G} \neq K_1$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is neither discrete nor indiscrete topology.

2. For any graph G not containing only one edge, the space $(E(\mathcal{G}), \tau_K(\mathcal{G}))$ is neither discrete nor indiscrete topology.

Proof. 1. If $\mathcal{G} \neq K_1$, then there exist two vertices $v_1, v_2 \in \mathcal{V}(\mathcal{G})$ such that $\{v_1\} \in \tau_C(\mathcal{G})$ and $\{v_2\} \notin \tau_C(\mathcal{G})$. Hence, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is neither discrete nor indiscrete topology.

2. If $\mathcal{G}$ not containing only one edge, then there exist two edges $e_1, e_2 \in E(\mathcal{G})$ such that $\{e_1\} \in \tau_K(\mathcal{G})$ and $\{e_2\} \notin \tau_K(\mathcal{G})$. Hence, the space $(E(\mathcal{G}), \tau_K(\mathcal{G}))$ is neither discrete nor indiscrete topology.

□

Theorem 3.8. If G is a graph, the coneighbor topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is quasi-discrete if and only if G contains at least one set of coneighbor vertices.

Proof. Suppose that $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is quasi-discrete, then there exist at least two proper open subgraphs $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U} \cap \mathcal{V} = \phi$ and $\mathcal{G} = \mathcal{U} \cup \mathcal{V}$. Hence, there is at least one set of coneighbor vertices $\mathcal{U}$ and the other open subgraph is $\mathcal{V} = \mathcal{U}^c$. Conversely, suppose that $\mathcal{G}$ contains at least one set $\mathcal{U}$ of coneighbor vertices, then $\tau_C(\mathcal{G}))$ contains $\mathcal{U}$ and $\mathcal{V}(\mathcal{G}) \setminus \mathcal{U}$. Hence, $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is quasi-discrete. □

Proposition 3.9. For any non-coneighbor graph $\mathcal{G}$, $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is homeomorphic to the included point topological space.

Proof. Since $\mathcal{G}$ is a non-coneighbor graph, so by definition, we have $\mathcal{V}(C_0) = \mathcal{V}(\mathcal{G})$ and hence

$$\mathcal{B}_C(\mathcal{G}) = \{\mathcal{N}(\mathcal{V}(\mathcal{G}), v) : v \in \mathcal{G}\}.$$

Therefore, $\tau_C(\mathcal{G})$ is the included point topology. □

In the following examples, we introduce coneighbor topological space and edge coneighbor topological space of some standard graphs.

Example 3.10.
1. $(\mathcal{V}(P_p), \tau_C(P_p))$ is the included point topology, when $p \neq 3$. If $p = 3$, then $(\mathcal{V}(P_3), \tau_C(P_3))$ contains one set of coneighbor vertices and hence, it is a quasi-discrete topology.
2. $(\mathcal{V}(C_p), \tau_C(C_p))$ is the included point topology, when $p \neq 4$. If $p = 4$, then $(\mathcal{V}(C_4), \tau_C(C_4))$ contains two sets of coneighbor vertices and hence, it is a quasi-discrete topology.
3. $(\mathcal{V}(K_p), \tau_C(K_p))$ is the included point topology because $\mathcal{V}(K_p)$ is non-coneighbor.
4. $(\mathcal{V}(K_{p_1,p_2}), \tau_C(K_{p_1,p_2}))$, p_1 or $p_2 \geq 2$ and $(\mathcal{V}(S_p), \tau_C(S_p))$, $p \geq 3$ are quasi-discrete topologies because they contains sets of coneighbor vertices.

Example 3.11.
1. $(E(P_p), \tau_K(P_p))$ is the included point topology, when $p \neq 4$. If $p = 4$, then $(E(P_4), \tau_K(P_4))$ is quasi-discrete topology.
2. $(E(C_p), \tau_K(C_p))$ is the included point topology, when $p \neq 4$. If $p = 4$, then $(E(C_4), \tau_K(C_4))$ is quasi-discrete topology.
3. $(E(K_p), \tau_K(K_p))$ and $(E(S_p), \tau_K(S_p))$ are included point topologies.
4. $(E(K_{p_1,p_2}), \tau_K(K_{p_1,p_2}))$ is quasi-discrete topology if $p_1 = p_2 = 2$.

Proofs of the following corollaries follows directly from their definitions.

Corollary 3.12. In general, $|\tau_K(\mathcal{G})| \leq |\tau_C(\mathcal{G})|$ and equality holds if $\mathcal{V}(\mathcal{G}) = E(\mathcal{G})$ and $\mathcal{G}$ is completely coneighbor graph.

Corollary 3.13. If $\mathcal{G}$ is a completely coneighbor graph, then $\tau_K(\mathcal{G}) = \tau_C(\mathcal{G})$.

Proposition 3.14. If $\mathcal{G}$ is a non-coneighbor graph, then $|\tau_C(\mathcal{G})| = 2^{|\mathcal{V}(\mathcal{G})|-1} + 1$ and $|\tau_K(G)| = 2^{|E(\mathcal{G})|-1} + 1$.

Proof. If $\mathcal{G}$ is a non-coneighbor graph, then we have only one fixed vertex, say v, so by Definition 3.2, we have $\mathcal{B}_C(\mathcal{G}) = \{\{v\} \cup \{\mathcal{H}\}\}$, where $\mathcal{H}$ is any subgraph in $\mathcal{G}$, so $\tau_C(\mathcal{G}) = \{\{v\} \cup \{\mathcal{H}\}\} \cup \{\phi\}$, hence $|\tau_C(\mathcal{G})| = 2^{|\mathcal{V}(\mathcal{G})|-1} + 1$. Similarly, we obtain that $|\tau_K(\mathcal{G})| = 2^{|E(\mathcal{G})|-1} + 1$. □

Proposition 3.15. In the topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$. If $\mathcal{H} \subseteq \mathcal{V}(\mathcal{G})$, then $Int_C(\mathcal{H}) = \{v \in \mathcal{H} : u \in U_v \Rightarrow u \in \mathcal{H}\}$.

Proof. If for $u \in \mathcal{H}$ for all $u \in U_v$, this implies $U_v \subseteq \mathcal{H}$. Hence, $v \in Int_C(\mathcal{H})$. Conversely, if $v \in Int_C(\mathcal{H})$, then $U_v \subseteq \mathcal{H}$, so every $u \in U_v$ implies that $u \in \mathcal{H}$. □

Proposition 3.16. In the topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$. If $\mathcal{H} \subseteq \mathcal{V}(\mathcal{G})$, then $Cl_C(\mathcal{H}) = \{v \in \mathcal{V}(\mathcal{G}) : u \in U_v\}$ for some $u \in \mathcal{H}$.

Proof. The proof follows from that fact that $v \in Cl_C(\mathcal{H})$ if and only if $\mathcal{H} \cap O \neq \phi$ for each open subgraph O, implies that $\mathcal{H} \cap U_v \neq \phi$. Hence, there is $u \in \mathcal{H}$ such that $u \in U_v$. □

Proposition 3.17. In the topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$. If $v \in \mathcal{V}(\mathcal{G})$ and $v \in \mathcal{V}(C)$ for some coneighbor vertex set $\mathcal{V}(C)$, then:

1. $Cl_C\{v\}$ is either $\{v\}$ or $\mathcal{V}(C)$.
2. $Int_C\{v\}$ is either $\{v\}$ or ϕ.

Proof. For any $v \in \mathcal{V}(\mathcal{G})$ if it is a fixed vertex in some $\mathcal{V}(C)$, so $\{v\}$ is open otherwise it is closed, hence the proof. □

Proposition 3.18. In the topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ the following statements are true:

1. If $\mathcal{G}$ is a complete coneighbor graph with an isolated vertex v, then $Cl_C(\{v\}) = Int_C(\{v\}) = \{v\}$.
2. If $\mathcal{G}$ contains one set of coneighbor vertices having one neighborhood vertex v, then $Cl_C(\{v\}) = Int_C(\{v\}) = \{v\}$.

Proof.
1. Since $\mathcal{G}$ is a complete coneighbor graph and let $\mathcal{V}(C_1), \mathcal{V}(C_2), \cdots$ be the sets of coneighbor vertices and v is an isolated vertex in $\mathcal{G}$, then by Definition 3.2, the base $\mathcal{B}_C(\mathcal{G})$ that contains a minimal open subgraph containing v is $\{v\}$. Then, $\{v\} \in \tau_C(\mathcal{G})$ and $\{v\} \in [\tau_C(\mathcal{G})]^c$. Hence, $Cl_C(\{v\}) = Int_C(\{v\}) = \{v\}$.
2. Let $\mathcal{V}(C)$ be the only set of coneighbor vertices in $\mathcal{G}$ and $N(\mathcal{V}(C)) = \{v\}$, then by Definition 3.2, the base $\mathcal{B}_C(\mathcal{G})$ contains a minimal open subgraph containing v is $\{v\}$. Then, $\{v\} \in \tau_C(\mathcal{G})$ and $\{v\} \in [\tau_C(\mathcal{G})]^c$. Hence, $Cl_C(\{v\}) = Int_C(\{v\}) = \{v\}$.

□

The converse of the above proposition is not true in general. For case (1), consider the graph $\mathcal{G} = P_3 = \{v_1, v_2, v_3\}$ as shown in Figure 3.

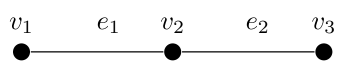

Figure 3. Path graph P_3 of order 3.

If we fix v_1, then $\mathcal{B}_C(P_3) = \{\{v_1\}, \{v_2\}, \{v_1, v_3\}\}$ and hence $\tau_C(P_3) = \{\phi, \{v_1\}, \{v_2\}, \{v_1, v_2\}, \{v_1, v_3\}, V(P_3)\}$. Also,
$[\tau_C(P_3)]^c = \{\phi, \{v_2\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}, \mathcal{V}(P_3)\}$. Thus, we have

$Cl_C(\{v_2\}) = Int_C(\{v_2\}) = \{v_2\}$, but v_2 is not an isolated vertex in $\mathcal{G}$.
For case (2), take $\mathcal{G} = C_4 \cup v$, then $Cl_C(\{v\}) = Int_C(\{v\}) = \{v\}$, but $\mathcal{G}$ is a complete coneighbor graph with one non-coneighbor vertex v.

Remark 3.19. If $\mathcal{H}$ is any subgraph of a finite graph $\mathcal{G}$, then $Cl_C(\mathcal{H}) = \bigcup_{v\in\mathcal{H}} Cl_C(\{v\})$ and this follows from the fact that $Cl_C(\bigcup_{i\in M} \mathcal{H}_i) = \bigcup_{i\in M} Cl_C(\mathcal{H}_i)$ where M is any finite index set.

Proposition 3.20. In the topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$. If $\mathcal{H}$ is any subgraph of $\mathcal{V}(\mathcal{G})$, then $Cl_C(Int_C(\mathcal{H}))$ is the set of $v \in \mathcal{V}(\mathcal{G})$ such that there is a fixed vertex $u \in \mathcal{H}$ with $u \in U_v$.

Proof. If $\mathcal{H}$ does not contains fixed vertices, then $Int_C(\mathcal{H}) = \phi$ and hence, $Cl_C(Int_C(\mathcal{H})) = \phi$. Now from Proposition 3.16, $Cl_C(Int_C(\mathcal{H})) = \{v \in \mathcal{V}(\mathcal{G}) : z \in U_v\}$ for some $z \in Int_C(\mathcal{H})$. Hence, $U_z \subseteq \mathcal{H}$. Therefore, there is a fixed vertex $u \in \mathcal{H}$ such that $u \in U_z$ and hence $u \in U_v$. □

Proposition 3.21. In the topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$. If $\mathcal{H}$ is any subgraph of $\mathcal{V}(\mathcal{G})$, then $Int_C(Cl_C(\mathcal{H}))$ is the set of $v \in \mathcal{V}(\mathcal{G})$ such that every fixed vertex u with $u \in U_v$ belongs to $\mathcal{H}$.

Proof. From Propositions 3.15 and 3.16, we have
$Int_C(Cl_C(\mathcal{H})) = \{v \in Cl_C(\mathcal{H}) : u \in U_v$ implies $u \in Cl_C(\mathcal{H})\} = \{v \in Cl_C(\mathcal{H}) : u \in U_v$ implies that there is $z \in \mathcal{H}, z \in U_u\} = \{v \in Cl_C(\mathcal{H}) :$ all fixed vertices $u \in U_v$
must belong to $\mathcal{H}\}$

To prove the equality, if u is fixed, then $u \in \mathcal{H}$ since there is no other z in $\mathcal{H}$. Hence $u \in Int_C(Cl_C(\mathcal{H}))$. On the other hand, if all vertices in $\mathcal{H}$ are fixed and $u \in U_v$, then there is a fixed vertex $z \in U_u$ with $z \in \mathcal{H}$. □

Theorem 3.22. In the topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$. If $\mathcal{H}$ is any subgraph of $\mathcal{V}(\mathcal{G})$, then $Int_C(Cl_C(\mathcal{H})) = Cl_C(Int_C(\mathcal{H}))$.

Proof. If $\mathcal{H}$ does not contains any fixed vertices, so by definition $Int_C(\mathcal{H}) = \phi$ and $Cl_C(\mathcal{H}) = \mathcal{H}$. Hence, $Int_C(Cl_C(\mathcal{H})) = Cl_C(Int_C(\mathcal{H})) = \phi$. Now, suppose that $\mathcal{H}$ contains fixed vertices $v_1, v_2, \cdots, v_n$ where each $v_i \in \mathcal{V}(C_i)$. Hence, $Cl_C(\mathcal{H}) = \cup Cl_C(\{v_i\}) = \cup\mathcal{V}(C_i)$ and since $\cup\mathcal{V}(C_i)$ is open subgraph, so $Int_C(Cl_C(\mathcal{H})) = \cup\mathcal{V}(C_i)$. On the other hand, since $v_1, v_2, \cdots, v_n \in \mathcal{H}$, so $v_1, v_2, \cdots, v_n \in Int_C(\mathcal{H})$. Therefore, $\cup Cl_C(\{v_i\}) = \cup\mathcal{V}(C_i) \subseteq Cl_C(Int_C(\mathcal{H}))$ implies that $\cup Cl_C(\{v_i\}) = \cup\mathcal{V}(C_i) = Cl_C(Int_C(\mathcal{H}))$. Thus, $Int_C(Cl_C(\mathcal{H})) = Cl_C(Int_C(\mathcal{H}))$. □

Proposition 3.23. Suppose that H consists of all fixed vertices in $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$. Then $\mathcal{H}$ is dense.

Proof. Since every vertex in $\mathcal{H}$ is a fixed vertex, so for each $v_i \in \mathcal{H}$, we have $v_i \in \mathcal{V}(C_i)$ which is a fixed vertex. Hence, by Proposition 3.17, $Cl_C(\{v_i\}) = \mathcal{V}(C_i)$ for every $i \geq 0$. Therefore, by Remark 3.19, $Cl_C(\mathcal{H}) = \bigcup Cl_C(\{v_i\}) = \bigcup \mathcal{V}(C_i) = \mathcal{V}(\mathcal{G})$. Hence, $\mathcal{H}$ is dense. □

Proposition 3.24. For any graph $\mathcal{G}$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is a $T_{1/2}$-space.

Proof. If $v \in \mathcal{V}(\mathcal{G})$ is one of the fixed vertices in $\mathcal{V}(C_i)$, so by Definition 3.2, $\{v\}$ is open subgraph. Suppose that $v \in \mathcal{V}(C_k)$ for some $k \geq 0$ and v is not the fixed vertex, so $\mathcal{V}(C_k) \setminus \{v\}$ is an open subgraph and hence $\bigcup_{i=0\neq k}^{\infty} \mathcal{V}(C_i) \bigcup (\mathcal{V}(C_k) \setminus \{v\})$ is open subgraph in $\tau_C(\mathcal{G})$ impling that $\{v\}$ is a closed subgraph. Hence, by Definition 2.5, $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is a $T_{1/2}$-space. □

From Proposition 3.24, and the fact that every $T_{1/2}$-space is a T_0-space, we obtain the subsequent corollary.

Corollary 3.25. For any graph $\mathcal{G}$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is a T_0-space.

Remark 3.26. For any infinite non-coneighbor graph $\mathcal{G}$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is not compact because, we have only one fixed vertex v and hence the family $\{\{v, u\} : u \in \mathcal{V}(\mathcal{G})\}$ is an open cover for $\mathcal{V}(\mathcal{G})$ which has no finite subcover.

Definition 3.27. [13] A subset A of a space X is called θ-open, if for every $x \in A$ there exists an open set $\mathcal{H}$ such that $x \in \mathcal{H} \subseteq Cl(\mathcal{H}) \subseteq A$. The family of all θ-open sets of X is denoted by $\theta O(X)$ and it is clear that $\theta O(X)$ forms a topology on X.

From the above definition, we deduce that the family of θ-open sets of the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is the family of all coneighbor sets of $\mathcal{V}(\mathcal{G})$. Hence, for any non-coneighbor graph $\mathcal{G}$, we have $\theta O(\mathcal{G}) = \{\phi, \mathcal{V}(\mathcal{G})\}$. Also, we obtain the subsequent result.

Proposition 3.28. For any infinite graph $\mathcal{G}$, the topological space $(\mathcal{V}(\mathcal{G}), \theta O(\mathcal{G}))$ is compact whenever $\mathcal{G}$ has a finite number of coneighbor sets of vertices.

Proposition 3.29. For any infinite non-coneighbor graph $\mathcal{G}$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is almost compact, connected and submaximal.

Proof. Since all open sets contain the fixed vertex, so every open set is dense and there is no clopen sets in this space. Moreover, the closure of every open set covers $\mathcal{V}(\mathcal{G})$. Therefore, $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is almost compact, connected and submaximal. □

Proposition 3.30. For any graph $\mathcal{G}$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ contains a topologically isolated vertex.

Proof. The proof is clear because from Definition 3.2, the fixed vertices are topologically isolated vertices in $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$. □

Proposition 3.31. If $\mathcal{G}$ is any graph such that the collection $\{\mathcal{V}(C_i)\}$ is countable. Then $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is separable.

Proof. Since $\{\mathcal{V}(C_i)\}$ is countable, so the subgraph $\mathcal{H}$ which consists of all fixed vertices in $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is countable. By Proposition 3.23, $\mathcal{H}$ is dense. Hence, $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is separable. □

Corollary 3.32. If $\mathcal{G}$ is any non-coneighbor graph, then $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is separable.

Proof. In this space, the set of fixed vertices consists only one (countable) vertex which is dense. □

Proposition 3.33. For any graph $\mathcal{G}$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is extremely disconnected.

Proof. If $\mathcal{G}$ is a non-coneighbor graph, then the closure of each open graph is $\mathcal{V}(\mathcal{G})$ which is open. If $\mathcal{G}$ contains coneighbor sets, then the closure of each open graph is a union of some coneighbor sets which is open. Hence, $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is extremely disconnected. □

Definition 3.34. ([8], p.195) The small inductive dimension of a space X is denoted by $indX$ and is defined as follows. $indX = -1$ if and only if $X = \phi$. Let n be a positive integer and $indX \leq k$ be defined for each $k \leq n-1$. Then $indX \leq n$ if X has a base β such that $ind\partial B \leq n-1$ for all $B \in \beta$, where ∂B is the boundary of B.

It is obvious that the discrete space X has $indX = 0$ because the boundary of each member of its base is empty.
For the topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$, we have the subsequent result.

Theorem 3.35. For every topological space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$, $ind\mathcal{V}(\mathcal{G}) = 1$ for any non-empty graph $\mathcal{V}$.

Proof. Since for any non-empty graph $\mathcal{G}$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is not discrete, so $ind\mathcal{V}(\mathcal{G}) \geq 0$. Since the boundary of each open set in $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ has a discrete topology, so by Definition 3.34, $ind\partial B \leq 0$ for all $B \in \beta$. But the boundary of each open set is non-empty, hence $ind\partial B = 0$ for all $B \in \beta$. Therefore, we get that $ind\mathcal{V}(\mathcal{G}) \leq 1$ and greater than 0. Thus $ind\mathcal{V}(\mathcal{G}) = 1$. □

Proposition 3.36. For any non-empty graph $\mathcal{G}$, the space $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is always locally connected.

Proof. Since for each vertex $v \in \mathcal{V}(\mathcal{G})$, there exists an open subgraph $\mathcal{H}$ containing v and the closure of $\mathcal{H}$ is one of $\mathcal{V}(C)$ or it is $\mathcal{V}(\mathcal{G})$ when $\mathcal{G}$ is a non-coneighbor graph and in both cases $\mathcal{V}(C)$ and $\mathcal{V}(\mathcal{G})$ are connected. Hence, $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is locally connected. □

Proposition 3.37. For any non-empty graph $\mathcal{G}$. If $|\mathcal{V}(C_i)|$ is finite for each $i \geq 0$, then $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is locally compact.

Proof. Since each vertex in $\mathcal{V}(\mathcal{G})$ has an open graph whose closure is one of the $\mathcal{V}(C_i)$s and since $|\mathcal{V}(C_i)|$ is finite for each $i \geq 0$. Hence, each $\mathcal{V}(C_i)$ is compact implying that $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is locally compact. □

Theorem 3.38. If $\mathcal{V}$ is any non-empty graph, then $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is countably compact.

Proof. Suppose that $\mathcal{V}(\mathcal{H})$ is an infinite set of vertices of $\mathcal{V}(\mathcal{G})$. If $\mathcal{V}(\mathcal{H}) \subseteq \mathcal{V}(C_i)$ for some i, then we have two cases either $\mathcal{V}(\mathcal{H})$ contains the fixed vertex, so all other vertices in $\mathcal{V}(C_i)$ are limit points of it. If $\mathcal{V}(\mathcal{H})$ does not contains the fixed vertex, so vertices in $\mathcal{V}(\mathcal{H})$ are its limit points. If $\mathcal{V}(\mathcal{H})$ is an infinite set with non-empty intersection of all $\mathcal{V}(C_i)$, then every non-fixed vertex of $\mathcal{V}(\mathcal{H})$ is its limit point. Lastly, if $\mathcal{V}(\mathcal{H})$ contains only fixed vertices, the all vertices in the same coneighbor set are limit points of $\mathcal{V}(\mathcal{H})$. Hence, $(\mathcal{V}(\mathcal{G}), \tau_C(\mathcal{G}))$ is countably compact. □

Remark 3.39. Similar results are valid for the edge coneighbor topological space $(E(\mathcal{G}), \tau_K(\mathcal{G}))$.

4 Conclusions

This paper focuses on constructing a topological space derived from the sets of coneighbor vertices and edges within a graph. Various properties and findings concerning these types of topologies are elucidated. We proved that these spaces are $T_{1/2}$, separable, connected, extremelly disconnected and countably compact spaces.

REFERENCES

[1] Majumdar, T. K., ”A Uniform Topological Structure on a Graph and its Properties”, PhD Thesis, University of Calcutta, India , 1983. http://hdl.handle.net/10603/158380

[2] Kilicman A. and Abdulkalek K., ”Topological spaces associated with simple graphs”. Journal of Mathematical Analysis, vol. 9, no. 4, pp. 44-52, 2018. http://ilirias.com/jma/vol_9_issue_4.html

[3] Nada S., El Atik A.E.F. and Atef, M., ”New types of topological structures via graphs”, Mathematical methods in the applied sciences, vol. 41, no. 15, pp. 5801-5810, 2018. https://doi.org/10.1002/mma.4726

[4] Sarı H.K. and Kopuzlu A., ”On topological spaces generated by simple undirected graphs”, Aims Mathematics, vol. 5, no. 6, pp. 5541-5550, 2020. http://dx.doi.org/10.3934/math.2020355

[5] Jasim T.H. and Awad A.I., ”Some topological concepts via graph theory” Tikrit Journal of pure science, vol. 25, no. 4, pp. 117-122, 2020. http://dx.doi.org/10.25130/tjps.25.2020.078

[6] Kozae A. M., El Atik A. A., Elrokh A. and Atef M., ”New types of graphs induced by topological spaces”, Journal of Intelligent and Fuzzy Systems, vol. 36, no. 6, pp. 5125-5134, 2019. http://dx.doi.org/10.3233/JIFS-171561

[7] Ibrahim N. B. and Khalaf A. B., ”On Maximal Block Topological Spaces”, Journal of Intelligent and Fuzzy Systems, vol. 45, no. 5, pp. 8541-8551, 2023. http://dx.doi.org/10.3233/JIFS-223749

[8] Engelking R., ”General Topology”, Sigma series in pure mathematics, 6, 1989. https://cir.nii.ac.jp/crid/1574231874534327936

[9] Levine N., "Generalized closed sets in topology", Rendiconti del Circolo Matematico di Palermo, vol. 19, pp. 89-96, 1970. https://link.springer.com/article/10.1007/BF02843888

[10] Chartrand G., Lesniak L. and Zhang P., "Textbooks in Mathematics (Graphs and Digraphs)". Taylor and Francis Group, LLC, 2015. https://doi.org/10.1201/b19731

[11] Ray S. S. , Graph Theory with Algorithms and its Applications. In Applied Sciences and Technology, Springer, New Delhi, 2013. https://link.springer.com/book/10.1007/978-81-322-0750-4

[12] Ibrahim N. B., "On the nullity of some sequential element identified, element introduced graphs", M.Sc. Thesis, University of Zakho, Iraq, 2013. https://scholar.google.com/citations?user=Qi4eeT0AAAAJ&hl=en&oi=sra

[13] Velicko N. V., "H-closed topological spaces", Mat. Sb., vol. 70, no. 112, pp. 98–112, 1966. English transl., Amer. Math. Soc. Transl. vol. 2, no. 78, pp. 103-118, 1969. https://doi.org/10.1090/trans2/078/05