Mathematics and Statistics 12(3): 234-239, 2024 http://www.hrpub.org
DOI: 10.13189/ms.2024.120303

Sharp Bounds on Vertex N-magic Total Labeling Graphs

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Received September 25, 2023; Revised March 27, 2024; Accepted April 22, 2024

Cite This Paper in the Following Citation Styles

***(a)**: [1] R. Nishanthini, R. Jeyabalan, "Sharp Bounds on Vertex N-magic Total Labeling Graphs," Mathematics and Statistics, Vol.12, No.3, pp. 234-239, 2024. DOI: 10.13189/ms.2024.120303*

***(b)**: R. Nishanthini, R. Jeyabalan (2024). Sharp Bounds on Vertex N-magic Total Labeling Graphs. Mathematics and Statistics, 12(3), 234-239. DOI: 10.13189/ms.2024.120303*

Abstract A vertex N-magic total labeling is a bijective function that maps the vertices and edges of a graph G onto the successive integers from 1 to $p+q$. The labeling exhibits two distinct properties: First, the count of unique magic constants k_i for i belonging to the set $\{1, 2, ..., N\}$ is equivalent to the cardinality of N; secondly, the magic constants k_i must be arranged in a strictly ascending order. In the present context, the constant N is employed to represent different degrees of vertices. The term "magic constant values k_i" for $i \in \{1, 2, ..., N\}$ refers to specific numbers that exhibit unique and interesting properties and are employed in the context of this investigation. By adding up the weights of each vertex in $V(G)$, we might receive a magical constant number k_i for $i \in \{1, 2, ..., N\}$. Within the scope of this study, we discuss the sharp bounds of vertex N-magic total labeling graphs. In terms of magic constants k_i for $i \in \{1, 2, ..., N\}$, we also found the requirement for vertex N-magic total labeling of trees. We investigated the potential for vertex N-magic total labeling at vertices in graphs with varying vertex degrees.

Keywords Vertex N- magic Total, Sharp Bounds, Sun Graph

1 Introduction

Labels, which are often lists of numbers, are used to represent the vertices p and edges q of an undirected, simple finite graph G. J.A.Gallian [1] found an excellent graph labeling survey that compiled the findings of numerous authors who had analysed the efficacy of various labels. The concept of graph magic was initially introduced by Sedl$\acute{a}$cek [2] in 1963, whereas the discovery of vertex-magic total labeling was made by MacDougall. The concept of super vertex-magic total labeling was introduced by MacDougall in [3]. In reference [4], Marimuthu presented the notion of N-magic total labeling for vertices.

What has been discussed thus far is merely a tiny portion of what is known about typical labeling graphs. Here, we'll demonstrate the use of several different kinds of questions in addition to the standard one. In order to consider a novel method of building an N-magic total labeling graph with vertices, we introduce the idea of distinct degrees. In vertex N-magic labeling of G, the weighted sum of every vertex is defined as $wgt_\psi(v) = \psi(v) + \sum_{u \in N(v)} \psi(vu) = k_i(G)$ for some $v \in G$.

When $V(G) \cup E(G)$ is mapped onto the integers 1 through $p+q$, we say that the vertex N-magic total labeling has the properties that there are exactly N distinct magic constants k_i for $i \in \{1, 2, ..., N\}$, and that the magic constants k_i must be in a strictly ascending order. In this context, the constant N is used to denote various degrees of vertices. The magic constant value k_i for i in the range of 1 to N is derived by calculating the weighted sum of each vertex in the graph $V(G)$. A graph that possesses a vertex N-magic total labeling is commonly known as a V_N-magic total graph and the mapping is denoted as ψ. The challenge of non-existence of vertex magic labeling in non-regular graphs can be addressed by employing a vertex N-magic total labeling approach which involves distinct magic constants.

The essential data for regular and non-regular graph vertex magic labeling may be located in the references [5, 6, 7, 8, 9, 10, 11]. For more information on vertex magic trees, see in references [12, 13]. The Result of this evolution is a class of graphs with desirable and potentially fascinating properties and a definition that looks natural. We investigated the possibility of labeling graphs with vertices of different degrees using N-magic totals at the vertices. We first establish that there are numerous kinds of nonregular graphs with bounds.

2 Aspects of Vertex-N-Magic Total Labeling Graphs

The present study focuses on the examination of several aspects pertaining to Vertex-N-Magic total Labeling Graphs. In this study, we examine the requisite condition for vertex N-magic total labeling trees by considering the magic constant k_i, where i belongs to the set $\{1, 2, 3, ..., N\}$. This section examines the characteristics of the vertex N-magic total labeling of graphs. The lower bounds for magic constants are derived by applying the required condition of vertex N-magic total trees. Nevertheless, the task of determining lower bound proves to be challenging in a broad sense. In the subsequent discussion, a comprehensive examination of the intricacies arises when employing novel methodologies to establish lower bound.

Theorem 2.1 *If T is any tree and if T admits vertex N-magic total labeling, then the magic constants k_i for $i \in \{1, 2, ..., N\}$ is always ≥ 5.*

Proof. Suppose $k_i < 5$ for $i \in \{1, 2, ..., N\}$, then there exists a vertex $v_i \in V(T)$ for $i \in \{1, 2, ..., N\}$ such that $\psi(v_i) = p + q$. Clearly,

$$\sum_{u_i \in N(v_i)} \psi(v_i u_i) = k_i - \psi(v_i) < 5 - \psi(v_i) = 6 - 2p.$$

It's impossible for there to be any vertices other than 1 and 2 if $6 - 2p > 0$ gives us $p < 3$. As a result, we have $\sum_{u_i \in N(v_i)} \psi(v_i u_i) \leq 0$. The contradiction emerges from the fact that ψ never acquires null or negative values.

Otherwise, there must be an edge $u_i v_i \in E(T)$ such that $\psi(u_i v_j) = p + q$ for $i \neq 1; i, j = 1, 2, ..., n$, then

$$\begin{aligned} \psi(u_i) &= k_i - \sum_{u_i \in N(v_j)} \psi(u_i v_j) \\ &= k_i - \sum_{u_i \in N(v_j)} \psi(u_i v_j) - (p + q); i \neq 1 \end{aligned}$$

which is ≤ 0.
On the other hand, we can conclude

$$\psi(u_j) = k_i - \sum_{u_j \in N(u_i)} \psi(u_i u_j) \leq 0$$

In both cases, we contradict that ψ never receives either zero or negative values.

3 Sharp Bounds on Vertex N-Magic Total Labeling Graphs

In this section, we will discuss the topic of sharp bounds on vertex N-magic total labeling of graphs. To commence, it is worth noting that the supremum value of the magical parameters $k_i(G)$ for i in the set $\{1, 2, ..., N\}$, where the supremum is determined by considering all potential vertex N-magic total labelings of G, is referred to as the sharp boundaries of a graph G and denoted as $\omega_i(G)$.
(i.e.,) To each i,
$\omega_i(G) = \sup_{1 \leq i \leq N} \{k_i(G)$: G is a vertex N-magic total labeling$\}$ for $i \in \{1, 2, ..., N\}$.

The relationship between the order and size of a graph and the values of $p + q \leq \omega_i(G)$ for i in the set $\{1, 2, ..., N\}$, as well as the uppermost range, is readily apparent.

Theorem 3.1 *If $p \equiv 0(mod\ 4)$ and $q \equiv 3(mod\ 4)$, then the sharp bounds of Bi-star $B_{n,n}$ are $\omega_1 = 4n + 4$ and $\omega_2 = \frac{6n^2+15n+9}{2}$ respectively.*

Proof. Let $V(B_{n,n}) = \{u_1, u_2, u_3 ..., u_n, v_1, v_2, ..., v_{2n}, u, v\}$, where $B_{n,n}$ has $2n + 2$ vertices, and the set of edges of $B_{n,n}$ are $\{uu_i; 1 \leq i \leq n\} \cup uv \cup \{vv_i, 1 \leq i \leq n\}$, where $B_{n,n}$ has $2n + 1$ edges. Let u and v comprise the central vertices of $B_{n,n}$ and uv represent the edge between those vertices as well as for each $1 \leq i \leq n$, u_i and v_i be the pendant vertices connecting with each u and v respectively. We can establish the validity of this function ψ from $V(B_{n,n}) \cup E(B_{n,n})$ onto $\{1, 2, ..., 2(2n) + 3\}$ by employing the aforementioned labelling scheme: $\psi(u) = 2n + 1$, and $\psi(v) = 2n + 2$ and $\psi(uv) = 2n + 3$ as well as the remaining labels for the vertices and edges are

$$\psi(uu_i) = \begin{cases} 4n + 5 - 2i; & i = 1, 2, ..., \frac{n+1}{2} \\ 2(2n - i + 2); & \frac{n+3}{2} \leq i \leq n \end{cases}$$

$$\psi(vv_i) = \begin{cases} 2(2n + 2 - i); & i = 1, 2, ..., \frac{n+1}{2} \\ 4n + 5 - 2i; & \frac{n+3}{2} \leq i \leq n \end{cases}$$

$$\psi(u_i) = \begin{cases} 2i - 1; & i = 1, 2, ..., \frac{n+1}{2} \\ 2i; & \frac{n+3}{2} \leq i \leq n \end{cases}$$

and

$$\psi(v_i) = \begin{cases} 2i; & i = 1, 2, ..., \frac{n+1}{2} \\ 2i - 1; & \frac{n+3}{2} \leq i \leq n \end{cases}$$

with the sharp bounds $4n + 4$ and $\frac{6n^2+15n+9}{2}$.

Theorem 3.2 *Let T be any vertex N-magic tree with even order $p \geq 6$ and $p \equiv 0, 2 \pmod 4$ and $q \equiv 1, 3 \pmod 4$, which has two or $2m$ isolated edges for $m \geq 2$ and m is even, then $k_i(T) = \omega_i(T)$ for $i \in \{1, 2, ..., N\}$.*

Proof. Case(i): Let T be any even order path graph. The labels of the vertices are $\psi(v_1) = 1$ and $\psi(v_2) = n + 1$ for

$n \geq 4$ and $\psi(v_i) = 2i$ for i ranging from 3 to $n-1$, $n \geq 4$ and $\psi(v_n) = 2$ for $n \geq 4$ and the labels of edges for tree in the following manner:

$\psi(v_1v_2) = 4$; and $\psi(v_{2i}v_{2i+1}) = 2n + 1 - 2i$ for $i = 1, 2, ..., \frac{n-2}{2}$, $n \geq 4$ and $\psi(v_{2i+1}v_{2i+2}) = n - i$ for $i = 1, 3, 5, ..., n-5$ & $n \neq 4, n \geq 6$ and $\psi(v_{n-1}v_n) = 3$.

Then ψ is the unique vertex N-magic total labeling with the sharp bounds 5 and $3n+4$ respectively.

Case(ii): Let T be any even order tree with $p \geq 6$ and $p \equiv 2(mod\ 4)$ and $q \equiv 1(mod\ 4)$ and $2m$ isolated edges for $m \geq 2$ and m is even which is Bi-star of even order.

We assert our results by the unique vertex N-magic total labeling those are $\psi(u) = 1$, and $\psi(v) = 2$ and $\psi(uv) = 3$ as well as the remaining labels for the vertices and edges are

$$\psi(uu_i) = \begin{cases} 2i+2; & i = 1, 2, ..., \frac{n}{2} \\ 4n - 2i + 5; & \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

,

$$\psi(u_i) = \begin{cases} 4n - 2i + 5; & i = 1, 2, ..., \frac{n}{2} \\ 2i+2; & \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

,

$$\psi(vv_i) = \begin{cases} 2i+3; & i = 1, 2, ..., \frac{n}{2} \\ 2(2n - i + 2); & \frac{n}{2} + 1 \leq i \leq n-1; n \neq 2 \\ 2n+3 & i = n \end{cases}$$

and

$$\psi(v_i) = \begin{cases} 2(2n - i + 2); & i = 1, 2, ..., \frac{n}{2} \\ 2i+3; & \frac{n}{2} + 1 \leq i \leq n-1; n \neq 2 \\ 2n+4 & i = n \end{cases}$$

respectively. Thus ψ is the unique vertex $N-$magic total labeling with the sharp bounds $4n+7$ and $\frac{6n^2+9n+8}{2} - 4\sum_{i=\frac{n}{2}+1}^{n} i$.

Theorem 3.3 *If for all n and $n \geq 3$, then the sharp bounds of Sun graph C_n^+ are $\omega_1 = 4n+1$ and $\omega_2 = 10n+2$ respectively.*

Proof. A Sun graph C_n^+ is a graph with $2n$ vertices and $2n$ edges whose set of vertices $u_1, u_2, ...u_n, v_1, v_2, ...v_n$ and the edges are $\{u_iv_i; 1 \leq i \leq n\} \cup \{v_iv_{i+1}, 1 \leq i \leq n-1\} \cup v_nv_1$ and is obtained by adding all vertices of the cycle of length n with precisely one pendant vertex.

We can define the bijective function $\psi : V(C_n^+) \cup E(C_n^+) \rightarrow \{1, 2, ..., 4n\}$ and we demonstrate our sharp bounds by accomplishing the following:

Step 1: The vertices of the cycle are labelled as u_i, ranging from 1 to n in a clockwise orientation. The pendant vertices are labelled as v_i, starting from the left and ranging from $n+1$ to $2n$, with the values decreasing from $2n-1$ to $n+2$ for each i, where $1 \leq i \leq n$.

Step 2: Allocate the edges u_iv_i with $4n, 4n-1, ..., 3n+1$ for each $1 \leq i \leq n$ and the edges v_iv_{i+1} beginning with $2n+1, 2n+2, ..., 3n-1$ for each $1 \leq i \leq n-1$ in a clockwise direction. The assignment of the final edge, denoted as v_nv_1, is determined by the value of $3n$.

The lower and upper bounds are given by the expressions $4n+1$ and $10n+2$, respectively.

Theorem 3.4 *If for all n and $n \geq 4$, then the sharp bounds of the Fan graph $F_{1,n}$ are $\omega_1 = 4n+4$, $\omega_2 = 7n+5$ and $\omega_3 = \frac{3n^2+5n+2}{2}$ respectively.*

Proof. A Fan graph $F_{1,n}$ is a graph with $n+1$ vertices and $2n-1$ edges obtained by attaching all the vertices of the path graph to an isolated vertex. Let v be the central vertex and $v_1, v_2, ..., v_n$ be the vertices of the path graph.

The one-one and onto function can be defined as follows $\psi : V(F_{1,n}) \cup E(F_{1,n}) \rightarrow \{1, 2, ..., 3n\}$ in the following way:

The first step involves assigning labels to the vertices in a sequential manner, starting from v_1 and ending with v_n, from left to right. The central vertex is labelled as $n+1$.

In Step 2, the edges vvi are determined for each $i = 1, 2, 3, ..., n-1$. These edges are represented by the numbers $n+3, n+4, ..., 2n+1$. Additionally, the edges v_iv_{i+1} are determined, where i ranges from 1 to $n-1$. These edges are represented by the numbers $3n, 3n-1, ..., 2n+2$. The last vertex of the Fan graph vv_n is determined by the sum of n and 2.

The sharp bounds are $4n+4$ and $7n+5$ and $\frac{3n^2+5n+2}{2}$.

4 Sharp Bounds on Vertex N-Magic Total Labeling Of Disjoint Union Graphs

In this section, We brought up the possibility of the existence of sharp bounds on vertex N- magic total labeling of disjoint union graphs in which vertices have distinct degrees.

Theorem 4.1 *If for all n and $n \geq 4$, then the sharp bounds of nP_3 are $\omega_1 = 4n+1$ and $\omega_2 = 10n+2$ respectively.*

Proof. Let $V(nP_3) = \{u_1, u_2, u_3..., u_n, v_1, v_2, v_3..., v_n, w_1, w_2, ..., w_n\}$ where the order is $3n$ and the size is $2n$, and the set of edges is $\{u_iv_i; 1 \leq i \leq n\} \cup \{v_iw_i; 1 \leq i \leq n\}$. Let $u_1, ..u_n$ and $w_1, ..., w_n$ be the degree one vertices and the $v_1, ..., v_n$ be the central vertices of each copy.

We have a ψ from $V(nP_3) \cup E(nP_3)$ onto the integers $\{1, 2, ..., 5n\}$ with ω_1 and ω_2. The unique vertex N-magic total labeling for the vertices and edges for each $i = 1, 2, ..., n$ is as follows:

Step 1: We begin by labeling the vertices u_i with 1 to n and v_i by $5n, 5n-1, ..., 4n+1$ and w_i by $3n-1, 3n-3, ..., n+1$.

Step 2: The edges u_iv_i are labelled by $4n, 4n-1, ..., 3n+1$ and v_iw_i by $n+2, n+4, ..., 3n$.

According to vertex N-magic labeling, the sharp bounds are $4n+1$ and $10n+2$.

Theorem 4.2 *If n is odd and $n \geq 3$, then the sharp bounds of $nK_{1,n}$ are $\omega_1 = 2n+1$ and $\omega_2 = 4n^2+1+\sum_{m=2i}^{n-1}(2n^2-mn)+\sum_{m=2i}^{n-1}(2n^2-mn+1)$ for $i=1,2,...n$*

Proof. Let us denote the set of vertices as $\{c_i, u_i, v_i, w_i, s_i, r_i, ..., a_i, t_i\}$ and the set of edges as $\{c_iu_i, c_iv_i, c_iw_i, c_is_i, c_ir_i, ..., c_it_i\}$, respectively. For each i=1,2,...,n, c_i represents the central vertices of $nK_{1,n}$ and the first pendant vertex of each copy is represented by u_i, and the succeeding pendant vertices of each copy are named as $v_i, w_i, s_i, r_i, ..., a_i$. The n-th pendant vertex of each copy is denoted by t_i. In the present investigation, our objective is to build a vertex N-magic total labelling for the graph consisting of n^2 edges and n^2+n vertices, namely the graph denoted as $nK_{1,n}$ where each n is odd, and $n \geq 3$.

Choose to have a vertex N-magic total labeling ψ of $nK_{1,n}$ in the integers $\{1,2,...,2n^2+n\}$, with ω_1 and ω_2, and we define a label as follows.

Step 1: Label $2n^2+1$ to the central vertices c_i of $nK_{1,n}$. We acquire the above label as that of the sharp bound for pendant vertices of $nK_{1,n}$.

Step 2: For each i=1,2,...,n, we have the vertex N-magic total labeling for the vertices, $\psi(u_i) = i$, $\psi(v_i) = 2n+i$, $\psi(w_i) = 2n+1-i$, $\psi(s_i) = 4n+i$, $\psi(r_i) = 4n+1-i$, $\psi(a_i) = n(n-1)+i$ and $\psi(t_i) = n(n-1)+1-i$

Step 3: The vertex N-magic total labeling for the edges,

$$\begin{aligned}
\psi(c_iu_i) &= 2n^2+1-i \\
\psi(c_iv_i) &= 2n^2-2n+1-i \\
\psi(c_iw_i) &= 2n^2-2n+i \\
\psi(c_is_i) &= 2n^2-4n-i+1 \\
\psi(c_ir_i) &= 2n^2-4n+i \\
\vdots \quad &\vdots \quad \vdots \\
\vdots \quad &\vdots \quad \vdots \\
\vdots \quad &\vdots \quad \vdots \\
\psi(c_ia_i) &= 2n^2+1-i-n(n-1) \\
\psi(c_it_i) &= 2n^2+i-n(n-1)
\end{aligned}$$

The remaining sharp bound is
$4n^2+1+\sum_{m=2i}^{n-1}(2n^2-mn)+\sum_{m=2i}^{n-1}(2n^2-mn+1)$
for $i=1,2,...,n$ from vertex N-magic labeling.

Corollary 4.3 *If all $n \geq 2$, then the sharp bounds of $nK_{1,3}(n \geq 2)$ with two magic parameters are $\omega_1 = 6n+1$ and $\omega_2 = 20n+2$ respectively.*

Proof. Let $V(nk_{1,3}) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n, w_1, w_2, ..., w_n, s_1, s_2, ..., s_n\}$ and $E(nK_{1,3}) = \{s_iu_i; 1 \leq i \leq n\} \cup \{s_iv_i; 1 \leq i \leq n\} \cup \{s_iw_i; 1 \leq i \leq n\}$ whose order and size are $4n$ and $3n$ respectively. For each i=1,2,...,n, s_i represents the central vertices of $nK_{1,3}$ and the first pendant vertex of each copy is represented by u_i, and the second pendant vertex of each copy is named as v_i, and the third pendant vertex of each copy is named as w_i.

We define a bijective map $\psi : V(nK_{1,3}) \cup E(nK_{1,3}) \to \{1,2,...,7n\}$ for each i=1,2,..n, we have

$$\begin{aligned}
\psi(u_i) &= i \\
\psi(v_i) &= n+2i \\
\psi(w_i) &= 3n-2i+1 \\
\psi(s_i) &= 6n+i \\
\psi(s_iu_i) &= 6n+1-i \\
\psi(s_iv_i) &= 5n-2i+1 \\
\psi(s_iw_i) &= 3n+2i
\end{aligned}$$

The sharp bounds are $6n+1$ and $20n+2$ from the vertex N-magic total.

Corollary 4.4 *If all n and $n \geq 2$, then $2K_{1,n} \cup K_1$ admits vertex N-magic.*

Proof. We can label all the vertices (edges) in $2K_{1,n} \cup K_1$ graph with sequential integers from 1 to n with the same labeling as in theorem 3.1 and theorem 3.2, with the exception of the central edge label. $2n+3$ indicates labeling for the isolated vertex of odd n. Thus the sharp bounds are $2n+3$ and $4n+4$ and $\frac{6n^2+11n+3}{2}$ for odd.

We indicate the labeling for an isolated vertex of even n as 3. Thus the sharp bounds are 3 and $4n+7$ and $\frac{6n^2+9n+2}{2} - 4\sum_{i=\frac{n}{2}+1}^{n} i$ for even.

Finally, We provide open problems on vertex $N-$ magic total labeling for the direction of further research in this area.

Theorem 4.2 that gives the sharp bounds of $nK_{1,n}$ for n is odd and $n \geq 3$. But, it is not known for the case of n is even. So we pose an open problem.

Open Problem:1 *Find all positive integers m and n for $mK_{1,n}$ to be the vertex N-magic total. Also determine the sharp bounds of graph.*

Section 4 gives the vertex $N-$ magic total labeling of disjoint graphs where all are copies from the same family. Here, we propose an unresolved problem with graphs composed entirely of copies from different families.

Open Problem:2 *Determine the sharp bounds for disjoint union of caterpillar trees and lobster trees that admit vertex N-magic total labeling ?. In generally, find an example of vertex N-magic total labeling of graphs for various families of disjoint graphs with sharp bounds*?

The vertex $N-$ magic total labeling of one copy of the sun graph is given in Theorem 3.3. For any integer ≥ 2, we extend the labeling to the disjoint union of non-isomorphic sums. Referring to [11], we conclude with the following unresolved issue.

Open Problem:3 *Find the vertex N-magic total of the disjoint union of sun graphs with sharp bounds*?

5 Application on Vertex N-magic Labeling with Sharp Bounds

The Global Peace Index (GPI) gathered from reputable institutions assesses both internal and external influences using qualitative and quantitative measures. The Earth has achieved global recognition and has become a subject of interest for all nations worldwide. Various nations possess a diverse range of partnerships. Numerous countries maintain a positive and harmonious relationship with each other. Certain nations possess strained diplomatic relations, hence jeopardising global harmony and the attainment of peace. The existence of conflicts among these nations can be attributed to the need to address and rectify the underlying causes of their antagonistic relations. Numerous conflicts, encompassing challenges along various lines, counterterrorism operations, the proliferation of nuclear weapons, domestic disputes, religious tensions, and the actions of foreign nations, provide an enduring menace to global cohesion.

The vertices in the graph represent the nations. Each edge within the context of this study symbolises the most contentious problem existent within the respective nations. A procedural method was developed to outline the process of identifying the most controversial problems among countries. Conduct an analysis to ascertain the nation that exhibits the highest level of livability among the various countries. The utilisation of graph structures in addressing highly contentious topics holds significant potential for enhancing the effectiveness of humanitarian aid organisations and the United Nations in promoting global stability.

Conclusions

To consider a new technique for construct a vertex N-magic total labeling graph, we introduced the idea of distinct degrees. With the help of vertex N-magic total labeling with distinct magic constants, we can resolve the problem of non-existence of vertex magic labeling in non-regular graphs. We found sharp bounds on vertex N-magic total labeling of some families of graphs. By utilising the concept of vertex N-magic and total sharp bounds, a graph structure may be employed to highlight the most controversial problem between two nations within a specified time period. Furthermore, this approach allows for an explanation of the intensity of the issue during that particular time.

Acknowledgement

The article has been written with the joint financial support of RUSA-Phase 2.0 grant sanctioned vide letter No.F 24-51/2014-U, Policy (TN Multi-Gen), Dept. of Edn. Govt. of India, Dt. 09.10.2018, UGC-SAP (DRS-I) vide letter No.F.510/8/DRS-I/2016(SAP-I) Dt. 23.08.2016, DST-PURSE 2nd Phase programme vide letter No. SR/PURSE Phase 2/38 (G) Dt. 21.02.2017 and DST (FST - level I) 657876570 vide letter No.SR/FIST/MS-I/2018/17 Dt. 20.12.2018.

REFERENCES

[1] Gallian, J. A. *A dynamic survey of graph labeling*, Electronic Journal of Combinatorics 1 (Dynamic Surveys), Australian National University, 2018. DOI: 10.37236/11668

[2] Sedlácek, J. *Problem 27. Theory of graphs and its applications* In Proc. Symp. Smolenice. Praha, pp. 163-164, 1963, June.

[3] MacDougall, J. A., Miller, M., & Wallis, W. D. *Vertex-magic total labelings of graphs*, Utilitas Mathematics, Vol. 61, pp. 3-21, 2002. DOI: 10.7151/dmgt.1186

[4] Marimuthu G., Kumar G., *Vertex N-magic total labeling of graphs*, Journal of Graph Label, Vol. 2(2), pp. 123-133, 2016.

[5] Cichacz S., Froncek D., Singgih I.,*Vertex magic total labelings of 2-regular graphs*, Discrete Mathematics, Vol. 340, pp. 3117-3124, 2017. https://doi.org/10.1016/j.disc.2016.06.022.

[6] Gray I.D., MacDougall J.A, *Vertex-magic labeling of non-regular graphs*, Australasian Journal of Combinatorics, Vol. 46, pp. 173-183, 2010. https://api.semanticscholar.org/CorpusID:12496083

[7] Gray I.D., MacDougall J.A., *Vertex-magic labeling of regular graphs: Disjoint unions and assemblages*, Discrete Applied Mathematics, vol. 160, no. 7, pp. 1114-1125, 2012. https://doi.org/10.1016/j.dam.2011.11.025.

[8] Marr A.M., Wallis W.D., *Magic graphs*, Second edition, Birkh$\tilde{A}$ user/Springer, NewYork, January 2013. DOI: 10.1007/978-0-8176-8391-7

[9] MacDougall, J.A., Miller, M., Wallis, W.D. *Vertex-magic total labelings of wheels and related graphs*, Utilitas Mathematica, vol. 62, pp. 175-183, 2002. Retrieved from https://utilitasmathematica.com/index.php /Index/article/view/239.

[10] Rahim M. T., Tomescu, I., & Slamin. *On vertex-magic total labeling of some wheel related graphs*, Utilitas Mathematica, Vol. 73, pp. 97-104, 2007. Retrieved from https://utilitasmathematica.com/index.php /Index/article/view/519.

[11] Tao Ming Wang, Guang Hui Zhang, *On vertex magic total labeling of disjoint union of sun graphs*,Utilitas Mathematica, vol. 103, 2017. Retrieved from https://utilitasmathematica.com/index.php /Index/article/view/1222.

[12] Gray I.D., Macdougall J.A., Wallis W.D.,*Vertex-magic labeling of trees and forests*, Discrete Mathematics, Vol. 261, no. 1, pp. 285-298, 2003. https://doi.org/10.1016/S0012-365X(02)00475-2

[13] Tezer M, *Vertex magic total labeling of selected trees*, AIP Conference Proceedings, vol. 1997, 020034, 2018. https://doi.org/10.1063/1.5049028.