Mathematics and Statistics 12(3): 229-233, 2024
DOI: 10.13189/ms.2024.120302 http://www.hrpub.org

On Questions Concerning Finite Prime Distance Graphs

Ram Dayal[1], **A. Parthiban**[2,*], **P. Selvaraju**[3]

[1]Department of Mathematics, Lovely Professional University, Phagwara 144 411, Punjab, India
[2]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, India
[3]Department of Mathematics, Rajalakshmi Institute of Technology, Chennai, India

Received January 1, 2023; Revised April 5, 2024; Accepted April 16, 2024

Cite This Paper in the Following Citation Styles

***(a)**: [1] Ram Dayal, A. Parthiban, P. Selvaraju, "On Questions Concerning Finite Prime Distance Graphs," Mathematics and Statistics, Vol.12, No.3, pp. 229-233, 2024. DOI: 10.13189/ms.2024.120302*

***(b)**: Ram Dayal, A. Parthiban, P. Selvaraju (2024). On Questions Concerning Finite Prime Distance Graphs. Mathematics and Statistics, 12(3), 229-233. DOI: 10.13189/ms.2024.120302*

Abstract Graph labeling is an allocation of labels (mostly integers) to the nodes/lines or both of a graph G_α subject to a few conditions. The field of graph theory, specifically graph labeling, plays a vital role in various fields. To name a few, graph labeling is utilized in coding, $x-$ray crystallography, radar, astronomy, circuit design, communication network addressing, and data base management. It can also be applied to network security, network addressing, channel assignment process, and social networks. A graph G_β is a prime distance graph (PDG) if its nodes can be assigned with distinct integers such that for any two adjacent nodes, the positive difference of their labels is a prime number. A complete characterization of prime distance graphs is an open problem of high interest. This paper contributes partially towards the same. More specifically, Laison et al. raised the following questions. (1) Is there a family of graphs which are PDGs if and only if Goldbach's Conjecture is true? (2) What other families of graphs are PDGs? In this paper, these questions are answered partially and also show certain families of graphs that admit prime distance labeling (PDL) if and only if the Twin Prime Conjecture holds, besides establishing PDL of some special graphs.

Keywords Prime Distance Labeling, Prime Distance Graph, Goldbach's Conjecture, Twin Prime Conjecture

1 Introduction

The graphs studied in the article are "simple, finite, undirected, and connected". Let Z and P denote the set of all integers and primes, respectively. The concept of PDG was proposed by Eggleton et al. in 1985 [1, 2]. "For any set D of positive integers, they defined the distance graph $Z(D)$ as the graph with node set Z and a line between integers s and t if and only if $|s-t| \in D$. The prime distance graph $Z(P)$ is the distance graph with $D = P$". One can observe that the PDG is infinite. This paper deals with finite subgraphs of $Z(P)$. Let $N(v)$ and $N[v]$ represent the open and closed neighborhoods of the node v of a graph G, respectively. Also, let WLG denote "without loss of generality" throughout this paper. For graph theoretic concepts and notations, refer to [3] and for number theoretic concepts and results, refer to [4].

A graph G_α is a PDG if there exists a one-to-one labeling of its nodes $t : V(G_\alpha) \to Z$ such that for any two adjacent nodes x_1 and x_2, the integer $|t(x_1) - t(x_2)|$ is a prime. It is important to note that PDL are not unique [5].

Laison et al. [5] also raised the following questions. (1) Is there a family of graphs which are PDGs if and only if Goldbach's Conjecture is true? (2) What other families of graphs are PDGs?. In this article we answer these two questions partially. We also show certain families of graphs such as extended jewel and triangular book graphs admit PDL if and only if the Twin Prime Conjecture is true.

2 Certain Known Results

This section is dedicated to recalling some results concerning prime distance labeling which are relevant for the study undertaken.

Theorem 2.1 *[5] Every subgraph of a PDG is PDG.*

Conjecture 2.1 ***(Goldbach's)****[5, 6] Every even number $2k > 2$ can be expressed as a sum of two prime numbers.*

Conjecture 2.2 ***("Twin Prime Conjecture (TPC)")****[5, 6] Infinitely many pairs of primes exist whose difference is 2.*

Definition 2.1 *[7] Duplication of a node v_s in a graph H_α by a line $e = v'_s v''_s$ is constructed by inserting new nodes v'_s, v''_s such that $N(v'_s) = v_s, v''_s$ and $N(v''_s) = v_s, v'_s$.*

Definition 2.2 *[7] Duplication of a line $e = st$ in a graph H_β by a node z is formed by adding a new node z to H_β such that $N(z) = \{s, t\}$.*

Definition 2.3 *[8] Duplication of a node $v_i; 1 \leq i \leq n$ of G_β by a node is formed by inserting a new node $v'_i; 1 \leq i \leq n$ to G_β and adding new lines so that $N(v'_i) = N(v_i)$.*

Theorem 2.2 *[9] The graph formed by performing duplication of a node by a line at all the nodes in any PDG H_β admits PDL if the TPC is true.*

Proposition 2.1 *[9] If S is a subgraph of H with no PDL, then H cannot have PDL.*

3 Main Results

In this section, derivation of PDL of some new classes of graphs is done.

Question (1): "Is there a family of graphs which are prime distance graphs if and only if Goldbach's Conjecture is true? [5]"

A. Parthiban et al. answered this question partially.

Theorem 3.1 *[9] Let $C_k = \{u_1, u_2, \ldots, u_k, u_1\}$ be a cycle with $k \geq 6$. If G is obtained from C_k by duplicating an arbitrary node by a node, then G admits PDL if Goldbach's Conjecture holds.*

In this paper, we give the complete asnwer to Question (1).

Theorem 3.2 *Let G_n be the graph obtained from the n cycle C_n by duplicating an arbitrary node by a node. Then G_n admits prime distance labeling for all $n \geq 6$ if Goldbach's Conjecture holds.*

Proof 1 *Let C_n be the given cycle with $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $n \geq 6$. Obtain G_n by performing duplication of node v_n by a node v'_n as given in Figure 1. So, $V(G_n) = V(C_n) \cup \{v'_n\}$ where $N(v'_n) = N(v_n) = \{v_{n-1}, v_1\}$. First suppose that G_n has a PDL for any $n \geq 6, n \in Z^+$, and consider one such PDL of G_n. W.L.G., assuming that $v_l = 2(l-1)$ for $1 \leq l \leq n-1$. Note that the remaining nodes v_n and v'_n cannot be of even labels (If v_n or v'_n is assigned -2, then it gives a contradiction). Moreover, one cannot give an arbitrary odd number to v_n or v'_n as they must be of odd primes only. So the labels of v_n and v'_n must be odd primes whose sum is equal to the label of v_{n-1}. Therefore, if all $G_n, n \geq 6$ are PDG, then the Goldbach's conjecture holds.*

Conversely, if the Goldbach's Conjecture holds, then define $g : V(G_n) \to Z^+$ as follows: $g(v_i) = 2(i-1)$ for $1 \leq i \leq n-1$. Now $g(v_{n-1})$ can be expressed as the sum of two primes and so $g(v_{n-1}) = p_1 + p_2$. Now lett $g(v_n) = p_1$ and $g(v'_n) = p_2$ imply that g is a PDL of G_n.

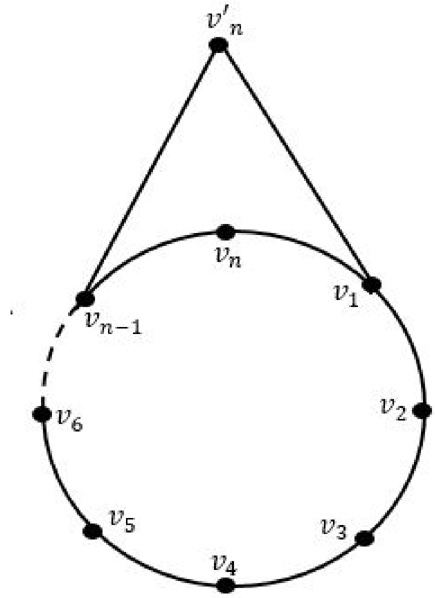

Figure 1. The graph resulted by performing duplication of v_n by a node v'_n in C_n

Question (2): What other families of graphs are prime distance graphs? [5]

We give partial answer to this question. More specifically, we establish the PDL of triangular book graphs and extend jewel graphs using the famous Twin prime conjecture.

Definition 3.1 *[10] The triangular book graph $B_n^{(3)}$ for $n \geq 1$ is a planar graph with $n+2$ nodes $u, v, v_1, v_2, \ldots v_n$ and $2n+1$ lines formed by n times K_3's sharing a common line (v, u). (See Figure 2).*

Definition 3.2 *The triangular book with book marks is $B_n^{(3)}$ with a finite number of pendant lines attached at any one of the end nodes of the spine.*

Theorem 3.3 *$B_n^{(3)}$ admits a PDL for all $n \in Z^+$ if and only if the TPC holds.*

Proof 2 *First suppose that $B_n^{(3)}$ has a PDL for an arbitrary positive integer n, and consider one such PDL of $B_n^{(3)}$. W.L.G., suppose that u and v are labeled with 1 and -1, respectively. Note that the remaining nodes cannot be of odd labels. In each of these C_3, since both odd-labeled nodes are connected to $v_i; 1 \leq i \leq n$, their labels must be even. Since their difference is a prime number, so each v_i is labeled with an even number, so that their difference with u and v gives twin primes. i.e., if $B_n^{(3)}$ is a PDG, there are exactly n twin primes. Hence, if all Triangular Book graphs are PDG, then the TPC holds.*

Conversely, if the TPC holds, then the labeling u with 1, v with -1, and v_i of each triangle with an even number (which are in between any twin primes) is a PDL of $B_n^{(3)}$.

Corollary 3.1 *The triangular book graph $B_n^{(3)}$ with any finite number of book marks admits PDL if the TPC holds.*

Proof 3 *The labeling for the triangular book graph is done as it is given in Theorem 3.3. W.L.G., the book mark nodes, say $w_i : i \geq 1$ is attached with u. The nodes w_i can be labeled with unused sufficiently large even numbers, say $2t_i$ where $t_i \in N$ such that $2t_i - 1$ is a prime. Hence the proof.*

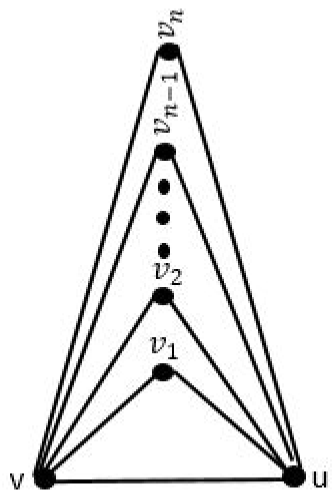

Figure 2. $B_n^{(3)}$

Definition 3.3 *[11] The jewel J_n is defined with $V(J_n) = \{u, v, s, t, v_i : 1 \leq i \leq n\}$ and $E(J_n) = \{uv_i, vv_i, us, vs, vt, ut, st : 1 \leq i \leq n\}$. (See Figure 3)*

Theorem 3.4 *The jewel graph J_n admits prime distance labeling if and only if the TPC holds.*

Proof 4 *Let J_n be on $2n + 4$ nodes with $V(J_n) = \{u, v, s, t, v_i : 1 \leq i \leq 2n\}$. First suppose that J_n has a PDL for any $n \in Z^+$, and consider one such PDL of J_n. W.L.G., suppose that u, v, s, and t are labeled with $1, -1, 6$, and 4, respectively. Note that the remaining nodes cannot be of odd labels. In each of these C_4, since u and v are already labeled with consecutive odd labels $v_i; 1 \leq i \leq 2n$, their labels must be even. Since their difference is prime, so each v_i is labled with an even number, so that their difference with u and v gives twin primes. i.e., if J_n is a PDG, there are exactly n twin primes. Hence, if all J_n are PDG, then the TPC holds.*

Conversely, if the TPC holds, then define $g : V(J_n) \to Z$ as follows: W.L.G., let $g(u) = 1$, $g(v) = -1$, $g(s) = g(u) + 3$, and $g(t) = g(u) + 5$. Now $g(v_i) = 2r : 1 \leq i \leq n, r \in N, 2r > g(t)$ in such away that $|g(u) - g(v_i)| = p_{k+1}$ and $|g(v) - g(v_i)| = p_{k+2}; 1 \leq i \leq n$, where p_{k+1} and p_{k+2} are twin primes. Similarly, $g(v_i) = -g(v_i) : n + 1 \leq i \leq 2n$. Clearly, g is a PDL of J_n.

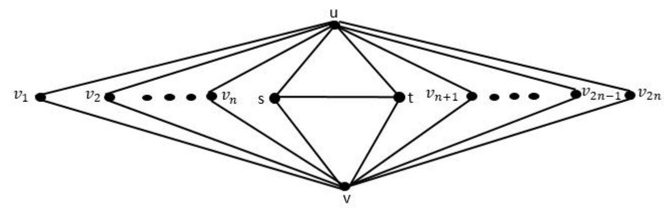

Figure 3. J_n

Definition 3.4 *[12] Every line which is incident with the center node of W_n is called spoke. A pizza graph on $2n+1$ nodes, denoted by Pz_n, is obtained from a subdivision of W_n in each of its spokes.*

Theorem 3.5 *The pizza graph Pz_n permits prime distance labeling $\forall n \geq 3$ if the TPC holds.*

Proof 5 *Let Pz_n be defined with $V(Pz_n) = u, v_i, w_i; 1 \leq i \leq n$. Define $v_{n+1} = v_1$, $w_{n+1} = w_1$ and $E(Pz_n) = uv_i, v_iw_i, w_i, w_{i+1}; 1 \leq i \leq n$, so $|V(Pz_n)| = 2n + 1$. Define a 1-1 function $f : V(Pz_n) \to Z$ as follows; W.L.G., let $f(u) = 0$, $f(w_i) = 2i; 1 \leq i \leq n - 1$. Now choose sufficiently large prime p_α and twin primes (p_1, p_2) (with $p_1 < p_2$) so that $f(w_n) = f(w_{n-1}) + p_\alpha = p_2$ with a condition that $|f(w_{n-1}) - f(w_n)|$ is a prime. Then $f(v_n) = p_1$. Also, let $f(v_i) = f(w_i) + p_i$, where p_i's are sufficiently large prime numbers. One can check that Pz_n permits PDL $\forall n \geq 3$. (See Figure 4)*

Definition 3.5 *[13] The generalized Jahangir graph $J_{m,k}$, for $m \geq 3$, $k \geq 1$, is a graph on $m(k + 1) + 1$ nodes i.e., a graph consisting of $C_{m(k+1)}$ with one additional node which is adjacent to m nodes of $C_{m(k+1)}$ at distance $k + 1$ to each other on $C_{m(k+1)}$.*

Theorem 3.6 $J_{m,k}$ *permits PDL* $\forall m \geq 3, k \geq 1$.

Proof 6 *Let $J_{m,k}$ be the Jahangir graph with $V(J_{m,k}) = \{u\} \cup \{u_1, \ldots, u_m\} \cup \{v_1^i, v_2^i, \ldots, v_{k-1}^i; i = 1, ..., m\}$. Here arises two cases:*
Case 1:- When k is even
Here $J_{m,k}$ is bipartite and the proof follows.
Case 2:- When k is odd
Define an injection $g : V(J_{m,k}) \to Z$ as follows: W.L.G., let $g(u) = 0$, $g(u_1) = p_1$, $g(v_1^1) = g(u_1) + 2$ and $g(v_j^1) = g(v_{j-1}^1) + 2$, where $2 \leq j \leq k - 1$ and p_1 is a prime. Next, $g(u_2) = p_2$ such that $|g(v_{k-1}^1) - g(u_2)| = p$, where

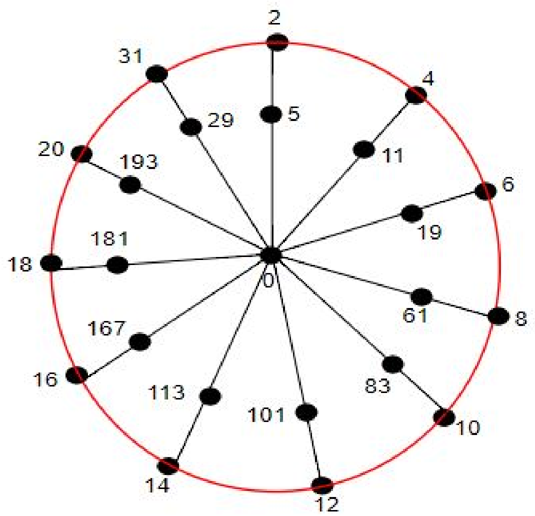

Figure 4. Prime distance labeling of Pz_{11}

.

$p \in P$. Again, $g(u_2) = p_2$, $g(v_1^2) = g(u_2) + 2$ and $g(v_j^2) = g(v_{j-1}^2) + 2$, where $2 \leq j \leq k-1$ and p_2 is a prime. Next, $g(u_3) = p_3$ such that $|g(v_{k-1}^2) - g(u_3)| = p$, where $p \in P$. Thus continuing the process up to the node u_m, one can verify that g is the required prime distance labeling of $J_{m,k}$. (See Figure 5)

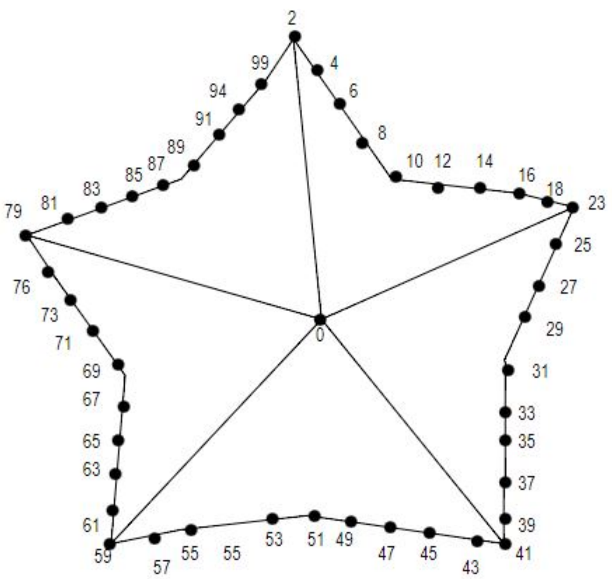

Figure 5. PDL of $J_{5,9}$

.

4 Conclusions

In this article, the questions raised by Laison et al. [5] are solved partially. A few interesting results have also been derived using the famous Twin Prime Conjecture. These results may serve as a path either in finding complete solution for those questions or completely characterizing prime distance graphs.

Acknowledgments

Authors are thankful to the anonymous referees for their valuable comments to improve presentation of the paper. The third author acknowllines that this research is carried out with the financial grant and support of NBHM-DAE, Government of India under the grant sanction no. 02011/14/2020/NBHM(R.P)/R&D II/7830.

REFERENCES

[1] Eggleton R.B., Erdos P., and Skilton D.K., "Colouring Prime Distance Graphs", Graphs and Combinatorics, vol. 6, pp. 17-32, 1990.

[2] Eggleton R.B., Erdos P., and Skilton D.K., "Colouring the Real Line", J. Combin. Theory Ser. B, vol. 39, pp. 86-100, 1985.

[3] West D.B., "Introduction to Graph Theory", Prentice Hall, 2001.

[4] Burton D.M., "Elementary Number Theory", McGraw Hill LLC, 2010.

[5] Laison J.D., Starr C., and Walker A, "Finite Prime Distance Graphs and 2-odd Graphs", Discrete Mathematics, vol. 313, pp. 2281-2291, 2013.

[6] Kenneth R.H., "Discrete Mathematics and Its Applications", CRC Press, 1999.

[7] Yegnanarayanan V., and Parthiban A, "On Graph Theory Application-I", Applied Research and Development Institute Journal, vol. 3, pp. 28-42, 2012.

[8] Ajaz Ahmad Pir, Tabasum Mushtaq, A. Parthiban, "2-Odd Labeling of Graphs Using Certain Number Theoretic Concepts and Graph Operations", Mathematics and Statistics, vol. 10(4), pp. 875-883, 2023. DOI: 10.13189/ms.2022.100419

[9] Parthiban A., David N.G, "On Prime Distance Labeling of Graphs", In: S. Arumugam, J. Bagga, L. Beineke, B. Panda, (eds) Theoretical Computer Science and Discrete Mathematics. ICTCSDM 2016. Lecture Notes in Computer Science, vol. 10398, 2017.

[10] Daoud S.N., and Elsawy A.N, "Edge Even Graceful Labeling of Some Book Graphs", Journal of Taibah University For Science, vol. 12, pp. 315-330, 2018.

[11] Jeba Jesinath J, Vinodhini N.K., and Munavar Hussain S, "Odd Graceful Labeling for the Jewel Graph and the Extended Jewel Graph Without the Prime Edge", Bulletin of Pure and Applied Sciences Section-E- Mathematics & Statistics, vol. 39, no. E2, pp. 212-217, 2020.

[12] Nabila S., Salman A.N.M, "The Rainbow Connection Number of Origami Graphs and Pizza", Procedia Computer Science, vol. 74, pp. 162-167, 2015.

[13] Jia-Bao Liu, Agha Kashif, Tabasam Rashid and Muhammad Javaid, "Fractional Metric Dimension of Generalized Jahangir Graph", Mathematics, vol. 7, pp. 1-10, 2019.