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Emerging Frameworks: 2-Multiplicative Metric and Normed Linear Spaces

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Abstract This new study helps us understand 2-multiplicative or product metric spaces and normed linear spaces (NDLS) better than before, going beyond what we already know. Seeing a gap in existing research, our main aim is to thoroughly explore the natural properties of 2-multiplicative NDLS. Using a careful approach that looks at continuity, compactness, and convergence properties, our research finds results that point out the special features of these spaces and show the connections between their algebraic and topological sides. The importance of our findings goes beyond just theory, affecting practical uses and encouraging collaboration across different fields. Our research builds a strong base in mathematical analysis, giving useful insights for making nuanced decisions. Acknowledging some limitations in our study opens the door for future improvements, creating promising paths for further exploration. In real-world terms, what we learn from this thorough study not only informs but also changes how we make decisions in mathematical analysis. In research community, our work makes people appreciate the connection between algebraic and topological spaces more deeply, sparking curiosity and inspiring future research. In essence, this research acts as a guiding light, showcasing the unique features of 2-multiplicative NDLS and paving the way for a deeper understanding of mathematical structures and their flexible uses in both theory and practice. Furthermore, our exploration motivates future researchers to dive into the details of 2-multiplicative NDLS, expanding their knowledge and looking into broader implications in the field of mathematical analysis.

Keywords 2-multiplicative, 2-multiplicative Convergent, 2-multiplicative Open Ball, 2-multiplicative Continuous

1 Introduction

In 2008, a significant event unfolded in the realm of mathematical analysis. Bashirov and his team introduced a groundbreaking concept known as multiplicative calculus, causing a stir. This pivotal work not only provided fresh insights into the structure of mathematical functions but also captured the attention of researchers in the analysis field. They discovered that multiplicative calculus proves highly beneficial in scenarios involving growth, scaling, and proportionality areas where conventional differentiation and integration methods fall short. Additionally, Bashirov and his team introduced a novel mathematical space called a multiplicative metric space, stemming from multiplicative calculus. This space signaled the commencement of a new era in mathematical analysis. Instead of the usual addition operations, they employed multiplication, offering a distinct perspective on comprehending mathematical phenomena. The inclusion of multiplicative metric spaces not only expanded the scope of traditional analysis but also laid the groundwork for novel mathematical structures and their functionalities. In 2017 another significant event unfolded when Renu Chungh and her team [1] applied a mathematical concept called multiplicative calculus to linear algebra. This marked a noteworthy development, bringing multiplicative calculus into a new realm of study. Their work introduced the notion of "multiplicative NDLS," signifying a crucial shift in our understanding of mathematics. Renu Chungh and her team [1] scrutinized the distinctive properties of these NDLS through careful exploration, revealing the connection between multiplicative calculus and traditional metric spaces, offering new avenues for mathematical comprehension. Bipolar soft limit points, the solvability and reliability of such control problems strongly depend on the topological structure that has been dealt by Saleh et al [2] and Byszewski et al [3].

Multiplicative calculus proves potent in handling scenarios involving rapid growth, scaling, and proportionality. Traditional calculus, the usual mathematical approach, often faces challenges when describing phenomena influenced by multiplication and proportionality. In contrast, multiplicative calculus provides a natural method to address such situations, enhancing our precision in mathematical understanding. This type of mathematics is particularly beneficial in economics, finance, physics, engineering, and control theory, where rapid growth and proportional relationships are prevalent. It finds practical application in situations such as compound interest, population growth, investment portfolios, and systems with multiplicative interactions.

In simple terms, traditional calculus involves understanding changes and accumulation using addition and subtraction. In contrast, multiplicative calculus, like the mentioned 2-multiplicative metric spaces and NDLS, examines mathematical spaces where multiplication, not just addition and subtraction, is the key. This exploration extends beyond basic calculus, providing insights into unique space properties and their connections with algebra and topology.

The concept behind 2-multiplicative metric spaces and 2-NDLS is to investigate spaces where multiplication is crucial in defining distances and sizes of vectors in two dimensions. Unlike regular metric spaces and normed linear spaces that use addition for these purposes, the "2-" signifies a special focus on multiplication. The aim of this study is to comprehend how these spaces function when multiplication plays a significant role, exploring new possibilities beyond what traditional mathematical theories have provided so far. This paper undertakes a thorough investigation of multiplicative calculus and its applications, with the goal of better understanding its distinct advantages and potential implications. Using a systematic approach, we strive to reveal the intricate connections between multiplicative calculus, metric spaces (MCS), and NDLS. Our aim is to contribute to the continuously evolving field of mathematical analysis, offering insights that could be relevant in various scientific and engineering areas.

Multiplicative calculus is a way of using mathematics that focuses on how multiplication works. For instance, in biology and medicine, it helps us understand processes like cell division and population growth, which follow multiplication patterns. In economic and financial modeling, where things like compound interest happen, it's a helpful tool. In physics, especially when things grow or change fast, multiplicative calculus comes into play. For example, in quantum mechanics and population dynamics, it makes models more accurate. In engineering, it helps analyze complex systems, making them work better. Also, in signal processing, when we need to change the volume of signals, multiplicative calculus refines models and makes communication systems more efficient. In short, multiplicative calculus is handy in many areas, providing a good way to understand and deal with things that grow or change rapidly in different scientific and practical applications.

2 Preliminaries

Definition 2.1 [4]: A mapping $d : X \times X \to R$ is called the metric on the non empty set X if the succeeding conditions hold.
(i) $d(\vartheta_1,\ \vartheta_2) \geq 0\ \forall\ \vartheta_1, \vartheta_2 \in X$
(ii) $d(\vartheta_1, \vartheta_2) = 0$ if $\vartheta_1 = \vartheta_2, \quad \forall\, \vartheta_1, \vartheta_2 \in X$
(iii) $d(\vartheta_1, \vartheta_2) = d(\vartheta_2, \vartheta_1)\ \forall\, \vartheta_1, \vartheta_2 \in X$
(iv) $d(\vartheta_1, \vartheta_2) \leq d(\vartheta_1, \vartheta_3) + d(\vartheta_3, \vartheta_2), \forall\, \vartheta_1, \vartheta_2, \vartheta_3 \in X.$
The pair (X, d) is known as a metric space (MCS).

Definition 2.2 [4,5]: Consider the linear space X over the field K. A function $\|.\| : X \to R$ is called a norm on X if the succeeding conditions hold.
(i) $\|\vartheta\| \geq 0\ \forall\, \vartheta \in X$
(ii) $\|\vartheta\| = 0$ if $\vartheta = 0$
(iii) $\|\vartheta_1 + \vartheta_2\| \leq \|\vartheta_1\| + \|\vartheta_2\|\ \forall\ \vartheta_1, \vartheta_2 \in X$ (triangle inequality)
(iv) $\|\alpha\vartheta\| = |\alpha|\ \|\vartheta\|, \forall\, \vartheta \in X$, and $\forall\, \alpha \in K$ (homogeneity of norm).
A norm $\|.\|$ on X is called NDLS. It is symbolized by $(X, \|.\|)$.

Definition 2.3 [6]: A mapping $d : X \times X \times X \to R$ is called a 2-metric on X if the succeeding conditions hold.
(i) For distinct points $\vartheta_1, \vartheta_2 \in X$, there is $\vartheta_3 \in X$ such that $d(\vartheta_1, \vartheta_2, \vartheta_3) \neq 0$
(ii) $d(\vartheta_1, \vartheta_2, \vartheta_3) = 0$ when two of the three elements $\vartheta_1, \vartheta_2, \vartheta_3 \in X$ are equal
(iii) $d(\vartheta_1, \vartheta_2, \vartheta_3) = d(\vartheta_1, \vartheta_3, \vartheta_2) = d(\vartheta_2, \vartheta_1, \vartheta_3) = d(\vartheta_2, \vartheta_3, \vartheta_1) = ...$ (symmetry in all three variables)
(iv)$d(\vartheta_1, \vartheta_2, \vartheta_3) \leq d(\vartheta_1, \vartheta_2, a) + d(\vartheta_1, a, \vartheta_3) + d(a, \vartheta_2, \vartheta_3)\ \forall\, \vartheta_1, \vartheta_2, \vartheta_3, a \in X$ (rectangle inequality).
The set X fortified with such a 2-metric is known as 2-MCS.

Definition 2.4 [7]: Consider the function $\|., .\|$ on $X \times X$ satisfying the following conditions:
i) $\|\vartheta_1, \vartheta_2\| \geq 0\ \forall\, \vartheta_1, \vartheta_2 \in X$ and $\|\vartheta_1, \vartheta_2\| = 0$ if ϑ_1, ϑ_2 are linearly dependent
(ii) $\|\vartheta_1, \vartheta_2\| = \|\vartheta_2, \vartheta_1\|\ \forall\, \vartheta_1, \vartheta_2 \in X$
(iii) $\|\alpha\vartheta_1, \vartheta_2\| = |\alpha|.\|\vartheta_1, \vartheta_2\|\ \forall\, \vartheta_1, \vartheta_2 \in X$ and $\alpha \in R$
(iv) $\|\vartheta_1, \vartheta_2 + \vartheta_3\| \leq \|\vartheta_1, \vartheta_2\| + \|\vartheta_1, \vartheta_3\|\ \forall\, \vartheta_1, \vartheta_2, \vartheta_3 \in X.$
Then $\|., .\|$ and the pair $(X, \|., .\|)$ are said to be a 2-norm and 2-NDLS respectively.

Definition 2.5 [8]: A multiplicative or product metric is a mapping $\hbar : X \times X \to R^+$ that satisfies the conditions:
i) $\hbar(\vartheta_1, \vartheta_2) \geq 1\ \forall\, \vartheta_1, \vartheta_2 \in X$ and $\hbar(\vartheta_1, \vartheta_2) = 1$ if $\vartheta_1 = \vartheta_2$
(ii) $\hbar(\vartheta_1, \vartheta_2) = \hbar(\vartheta_2, \vartheta_1)\ \forall\, \vartheta_1, \vartheta_2 \in X$
(iii) $\hbar(\vartheta_1, \vartheta_2) \leq \hbar(\vartheta_1, \vartheta_3).\hbar(\vartheta_3, \vartheta_2)\ \forall\, \vartheta_1, \vartheta_2, \vartheta_3 \in X.$
The pair $(X, \hbar)$ is known as a multiplicative or product MCS.

Definition 2.6 [1]: A mapping $\|.\| : X \to R^+$ is called the product norm on X if the succeeding conditions hold.
(i) $\|\vartheta\| \geq 1\ \forall\, \vartheta \in X$
(ii) $\|\vartheta\| = 1$ if $\vartheta = 0$
(iii)$\|\alpha\vartheta\| = \|\vartheta\|^{|\alpha|}\ \forall\, \vartheta \in X$ and α is any scalar
(iv) $\|\vartheta_1 + \vartheta_2\| \leq \|\vartheta_1\| . \|\vartheta_2\|\ \forall\, \vartheta_1, \vartheta_2 \in X.$
The pair $(X, \|.\|)$ is named as a product NDLS.

3 Main Results

In this part, we present the conception of 2-multiplicative MCS and 2-multiplicative NDLS and investigate topological properties in 2-multiplicative NDLS.

Definition 3.1: A product 2-metric on X is a mapping $\hbar_2 : X \times X \times X \to R^+$ that satisfies the following conditions.
(i) $\hbar_2(\vartheta_1,\ \vartheta_2,\ \vartheta_3) \geq 1,\ \forall\ \vartheta_1,\ \vartheta_2,\ \vartheta_3 \in X$ and $\hbar_2(\vartheta_1,\ \vartheta_2,\ \vartheta_3) = 1$ when two of the three elements $\vartheta_1,\ \vartheta_2,\ \vartheta_3 \in X$ are equal
(ii) $\hbar_2(\vartheta_1,\ \vartheta_2,\ \vartheta_3) = \hbar_2(\vartheta_1,\ \vartheta_3,\ \vartheta_2)$
$= \hbar_2(\vartheta_2,\ \vartheta_1,\ \vartheta_3) = \ldots\ \ \forall\ \vartheta_1,\ \vartheta_2,\ \vartheta_3 \in X$
(iii)$\hbar_2(\vartheta_1,\ \vartheta_2,\ \vartheta_3) \leq$
$\hbar_2(\vartheta_1,\ \vartheta_2, a).\hbar_2(\vartheta_1,\ a, \nu_3).\hbar_2(a, \vartheta_2,\ \vartheta_3)$
$\forall\ \vartheta_1,\ \vartheta_2,\ \vartheta_3, a \in X.$
The pair $(X, \hbar_2)$ is known as a product 2-MCS.

Example 3.2: Consider the linear space $X = R$ and $\hbar_2 : X \times X \times X \to R^+$ defined by $\hbar_2(\vartheta_1,\ \vartheta_2,\ \vartheta_3) = a^{min\{|\vartheta_1-\vartheta_2|,|\vartheta_2-\vartheta_3|,|\vartheta_3-\vartheta_1|\}}\ \forall\ \vartheta_1,\ \vartheta_2,\ \vartheta_3 \in X$, where $a > 1 \in R$. Then $\hbar_2$ is a product 2-metric on X and $(X, \hbar_2)$ is a product 2-MCS.

Definition 3.3: Consider the linear space X of dimension > 1 over R (or C). A mapping $\|.,.\|_{\hbar_2} : X \times X \to R^+$ is called the product 2-norm for X if the succeeding conditions hold.
(i) $\|\vartheta_1,\ \vartheta_2\|_{\hbar_2} \geq 1\ \ \ \forall\ \vartheta_1,\ \vartheta_2 \in X$ and $\|\vartheta_1,\ \vartheta_2\|_{\hbar_2} = 1$ if ϑ_1, ϑ_2 are linearly dependent
(ii) $\|\vartheta_1,\ \vartheta_2\|_{\hbar_2} = \|\vartheta_2,\ \vartheta_1\|_{\hbar_2}\ \ \forall\ \vartheta_1,\ \vartheta_2 \in X$
(iii) $\|\alpha\vartheta_1,\ \vartheta_2\|_{\hbar_2} = \|\vartheta_1,\ \vartheta_2\|_{\hbar_2}^{|\alpha|}\ \ \forall\ \vartheta_1,\ \vartheta_2 \in X$ and α is any scalar
(iv) $\|\vartheta_1,\ \vartheta_2 + \vartheta_3\|_{\hbar_2} \leq \|\vartheta_1,\ \vartheta_2\|_{\hbar_2}.\|\vartheta_1,\ \vartheta_3\|_{\hbar_2}$
$\forall\ \vartheta_1,\ \vartheta_2, \vartheta_3 \in X.$
The pair $(X, \|.,.\|_{\hbar_2})$ is known as a product 2-NDLS. It is denoted by $2 - MNS$.

Example 3.4: Let $X = R^3$ be the linear space. Let $\|.,.\|_{\hbar_2} : X \times X \to R^+$ defined by $\|\vartheta_1, \vartheta_2\|_{\hbar_2} = a^{|\vartheta_1 \times \vartheta_2|}\ \ \forall\ \vartheta_1, \vartheta_2 \in X$, where $a > 1$ is a fixed real number and
$|\vartheta_1 \times \vartheta_2| = |b_1c_2 - b_2c_1| + |a_1c_2 - a_2c_1| + |a_1b_2 - a_2b_1|$, for $\vartheta_1 = (a_1, b_1, c_1),\ \ \vartheta_2 = (a_2, b_2, c_2)$.
Then $(X, \|.,.\|_{\hbar_2})$ is a $2 - MNS$.

Example 3.5: Let $X = R^3$ be the Euclidean 3-dimensional linear space.
Let $\|.,.\|_{\hbar_2} : X \times X \to R^+$ be defined by $\|\vartheta_1, \vartheta_2\|_{\hbar_2} = a^{|\vartheta_1\times\vartheta_2|}\ \ \forall\ \vartheta_1, \vartheta_2 \in X$, where $a > 1 \in R$ and

$$|\vartheta_1 \times \vartheta_2| = abs \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

for $\vartheta_1 = a_1i + a_2j + a_3k$ and $\vartheta_2 = b_1i + b_2j + b_3k$.
Then $(X, \|.,.\|_{\hbar_2})$ is a $2 - MNS$.

Theorem 3.6: Let $(X, \|.,.\|_{\hbar_2})$ be $2 - MNS$. For all $\nu_1, \nu_2, \nu_3 \in X$ and any $\alpha \in K$, the follwing results are true.
(1) $\|\vartheta_1, \vartheta_2\|_{\hbar_2} = \|\vartheta_1, \vartheta_2 + \alpha\vartheta_1\|_{\hbar_2}$
(2) $\|\vartheta_1 - \vartheta_3, \vartheta_1 - \vartheta_2\|_{\hbar_2} = \|\vartheta_1 - \vartheta_3, \vartheta_2 - \vartheta_3\|_{\hbar_2}$
(3) $\|\vartheta_1 - \vartheta_3, \vartheta_2 - \vartheta_3\|_{\hbar_2} \leq$
$\|\vartheta_1, \vartheta_2\|_{\hbar_2}.\|\vartheta_2, \vartheta_3\|_{\hbar_2}.\|\vartheta_3, \vartheta_1|_{\hbar_2}.$
Proof. $\|\vartheta_1, \vartheta_2 + \alpha\vartheta_1\|_{\hbar_2} \leq \|\vartheta_1, \vartheta_2\|_{\hbar_2}.\|\vartheta_1, \alpha\vartheta_1\|_{\hbar_2} = \|\vartheta_1, \vartheta_2\|_2.\|\vartheta_1, \vartheta_1\|_{\hbar_2}^{|\alpha|}$. Hence $\|\vartheta_1, \vartheta_2 + \alpha\vartheta_1\|_{\hbar_2} = \|\vartheta_1, \vartheta_2\|_{\hbar_2}$.

Remark 3.7: Every $2 - MNS\ (X, \|.,.\|_{\hbar_2})$ is a multiplicative 2-MCS under the multiplicative 2-metric given by $\hbar_2(\vartheta_1, \vartheta_2, \vartheta_3) = \|\vartheta_1 - \vartheta_3, \vartheta_2 - \vartheta_3\|_{\hbar_2}\ \forall\ \vartheta_1, \vartheta_2, \vartheta_3 \in X$.
Proof. For any $\vartheta_1, \vartheta_2, \vartheta_3 \in X$, we have
(1) $\hbar_2(\vartheta_1, \vartheta_2, \vartheta_3) = \|\vartheta_1 - \vartheta_3, \vartheta_2 - \vartheta_3\|_{\hbar_2} \geq 1$ which implies that $\hbar_2(\vartheta_1, \vartheta_2, \vartheta_3) \geq 1$ and
$\hbar_2(\vartheta_1, \vartheta_2, \vartheta_3) = 1$ if $\|\vartheta_1 - \vartheta_3, \vartheta_2 - \vartheta_3\|_{\hbar_2} = 1$ if $\vartheta_1, \vartheta_2, \vartheta_3$ are linearly dependent, i.e., when two of the three elements $\vartheta_1, \vartheta_2, \vartheta_3 \in X$ are equal
(2) $\hbar_2(\vartheta_1, \vartheta_2, \vartheta_3) = \hbar_2(\vartheta_1, \vartheta_3, \vartheta_2) = ...,$
$\forall\ \vartheta_1, \vartheta_2, \vartheta_3 \in X$
(3) $\hbar_2(\vartheta_1, \vartheta_2, \vartheta_3) = \|\vartheta_1 - \vartheta_3, \vartheta_2 - \vartheta_3\|_{\hbar_2}\leq$
$\|\vartheta_1 - t, \vartheta_2 - t\|_{\hbar_2}.\|\vartheta_1 - \vartheta_3, t - \vartheta_3\|_{\hbar_2}.$
$\|t - \vartheta_3, \vartheta_2 - \vartheta_3\|_{\hbar_2}$
Hence $\hbar_2(\vartheta_1, \vartheta_2, \vartheta_3)\leq\hbar_2(\vartheta_1, \vartheta_2, t).\hbar_2(\vartheta_1, t, \vartheta_3).$
$\hbar_2(t, \vartheta_2, \vartheta_3)\ \ \forall\ \vartheta_1, \vartheta_2, \vartheta_3, t \in X.$

Definition 3.8: Let $(X, \|.,.\|_{\hbar_2})$ be a $2 - MNS$. For given $\vartheta_0 \in X$, $a \in X$ and $r > 1$, we define 2-product open ball $B_a(\vartheta_0, r)$ to be a subset of X given by $B_a(\vartheta_0, r) = \{v \in X : \|v - \vartheta_0, a\|_{\hbar_2} < r\}$ and 2-product closed ball $B_a[\vartheta_0, r]$ in X as $B_a[\vartheta_0, r] = \{v \in X : \|v - \vartheta_0, a\|_{\hbar_2} \leq r\}$ putting $v = ru + \vartheta_0$ we get $B_a(\vartheta_0, r) = \{ru + \vartheta_0 \in X : \|ru, a\|_{\hbar_2} < r\} = \vartheta_0 + r\{u \in X : \|u, a\|_{\hbar_2}^r < r\} = \vartheta_0 + r\{u \in X : \|u, a\|_{\hbar_2} < \sqrt[r]{r}\} = \vartheta_0 + rB_a(0, r_1)$, where $r_1 = \sqrt[r]{r}$.

Definition 3.9: Let $(X, \|.,.\|_{\hbar_2})$ be a $2 - MNS$. A set $G \subset X$ is called the 2-product open set if $\forall\vartheta \in X$, $\exists\ \epsilon > 1$ such that $B_a(\vartheta_0, r) \subset G$.

Definition 3.10: Let $(X, \|.,.\|_{\hbar_2})$ be a $2 - MNS$. A set $F \subset X$ is called the 2-product closed set if $F^C = X - F$ is 2-product open.

Definition 3.11: A sequence $\{\vartheta_n\}$ in a $2 - MNS$ $(X, \|.,.\|_{\hbar_2})$ is called the 2-product convergent to $\vartheta \in X$ if for any $\epsilon > 1$, $\exists\ \ n_0 \in N$ such that $\|\vartheta_n - \vartheta, a\|_{\hbar_2} < \epsilon$ for all $n \geq n_0$, for all $a \in X$ or $\vartheta_n \to_{**} \vartheta$ as $n \to \infty$ if $\lim_{n\to\infty} \|\vartheta_n - \vartheta, a\|_{\hbar_2} = 1$.

Definition 3.12: A sequence $\{\vartheta_n\}$ in a $2 - MNS$ $(X, \|.,.\|_{\hbar_2})$ is called the 2-product Cauchy sequence if $\forall\epsilon > 1, \exists\ \ n_0 \in N$ such that $\|\vartheta_n - \vartheta_l, a\|_{\hbar_2} < \epsilon$ for all $n, l \geq n_0$, for all $a \in X$. It is denoted by $\|\vartheta_n - \vartheta_l, a\|_{\hbar_2} \to_{**} 1$ as $n, l \to \infty$.

Definition 3.13: Let $(X, \|.,.\|_{\hbar_2})$ be a $2 - MNS$. A set $S \subset X$ is called the bounded if $\exists$ a constant C such that $\|\vartheta, a\|_{\hbar_2} \leq C$ for all $\vartheta, a \in S$.

Lemma 3.14: Let $(X, \|.,.\|_{\hbar_2})$ be a $2-MNS$.
(i) Every 2-product convergent sequence is 2-product Cauchy sequence in X
(ii) If $\vartheta_n \rightarrow_{**} \vartheta$ & $\vartheta_n \rightarrow_{**} \vartheta_1$ as $n \rightarrow \infty$ then $\vartheta = \vartheta_1$ i.e., 2-product limits are unique
(iii) Every 2-product Cauchy sequence in X is bounded.
Proof. (i) Let $\vartheta \in X$ such that $\vartheta_n \rightarrow_{**} \vartheta$ then $\forall\ \epsilon > 1$, $\exists\ n_0 \in N$ such that $\|\vartheta_n - \vartheta, a\|_{\hbar_2} < \sqrt{\epsilon}$ and $\|\vartheta_l - \vartheta, a\|_{\hbar_2} < \sqrt{\epsilon}$ for all $n, l \geq n_0$, for all $a \in X$. Now $\|\vartheta_n - \vartheta_l, a\|_{\hbar_2} = \|\vartheta_n - \vartheta + \vartheta - \vartheta_l, a\|_{\hbar_2} \leq \|\vartheta_n - \vartheta, a\|.\|\vartheta_l - \vartheta, a\|_{\hbar_2} < \epsilon\ \forall\ n, l \geq n_0, \forall a \in X$ which implies that ϑ_n is 2-product Cauchy sequence.
(ii) $\|\vartheta - \vartheta_1, a\|_{\hbar_2} = \|\vartheta - \vartheta_n + \vartheta_n - \vartheta_1, a\|_{\hbar_2} \leq \|\vartheta_n - \vartheta, a\|_{\hbar_2}.\|\vartheta_n - \vartheta_1, a\|_{\hbar_2} \rightarrow_{**} 1$ as $n \rightarrow \infty$ which implies that $\|\vartheta - \vartheta_1, a\|_{\hbar_2} = 1$ i.e, $\vartheta = \vartheta_1$.
(iii) Let ϑ_n be a 2-product Cauchy sequence in X. Take $\epsilon = 2$, then by definition of 2-product Cauchy sequence $\exists n_0 \in N$ such that $\|\vartheta_n - \vartheta_l, a\|_{\hbar_2} < 2\ \forall\ n, l \geq n_0, \forall\ a \in X$. In particular $\|\vartheta_n - \vartheta_{n_0}, a\|_{\hbar_2} < 2$ which implies, for all $n \geq n_0$.
$\|\vartheta_n, a\|_{\hbar_2} = \|\vartheta_n - \vartheta_{n_0} + \vartheta_{n_0}, a\|_{\hbar_2}$
$\leq \|\vartheta_n - \vartheta_{n_0}, a\|_{\hbar_2}.\|\vartheta_{n_0}, a\|_{\hbar_2} < 2\|\vartheta_{n_0}, a\|_{\hbar_2}$. Then
$\|\vartheta_{n_0}, a\|_{\hbar_2} \leq Max(\|\vartheta_1, a\|_{\hbar_2}, \|\vartheta_2, a\|_{\hbar_2}, ..., \|\vartheta_{n_0-1}, a\|_{\hbar_2}, 2\|\vartheta_{n_0}, a\|_{\hbar_2})$ for all $n \in N$, for all $a \in X$. Hence the sequence $\{\nu_n\}$ is bounded.

Definition 3.15: Let $(X, \|.,.\|_{\hbar_2})$ be a $2-MNS$ and let $A \subseteq X$.
(i)A point $\vartheta \in X$ is said to be 2-product limit points of A if $\exists$ a sequence $\vartheta_n \in A$ with $\vartheta_n \neq \vartheta$ such that $\vartheta_n \rightarrow_{**} \vartheta$. The collection of all 2-product limit points of A is denoted by $A^{'}$.
(ii) The set A is 2-product closed if it contains all its 2-product limit points. i.e., A is 2-product closed, whenever $\{\vartheta_n\}$ is a sequence of elements of A and $\vartheta_n \rightarrow_{**} \vartheta \in X$ then ϑ must be the element of A.
(iii) The set $\overline{A} = A \cup A^{'}$ is said to be 2-product closure of A, where $A^{'}$ is the collection of all 2-product limit points of A.
(iv) The set A is said to be 2-product dense in X if $\overline{A} = A$.

Definition 3.16: A $2-MNS$ $(X, \|.,.\|_{\hbar_2})$ is called 2-product complete if every 2-product Cauchy sequence in X is 2-product convergent to a limit in X. A complete $2-MNS$ is called a 2-product Banach space.

Definition 3.17: A linear function f from a $2-MNS$ $(X, \|.,.\|_{\hbar_{2X}})$ into $2-MNS$ $(Y, \|.,.\|_{\hbar_{2Y}})$ is said to be 2-product bounded if $\exists\ K > 1$ such that $\|f(\vartheta_1), f(\vartheta_2)\|_{\hbar_{2Y}} \leq K\|\vartheta_1, \vartheta_2\|_{\hbar_{2X}}$ $\forall\ \vartheta_1, \vartheta_2 \in X$.

Definition 3.18: Let $(X, \|.,.\|_{\hbar_{2X}})$ and $(Y, \|.,.\|_{\hbar_{2Y}})$ be two $2-MNS$ and $f : X \rightarrow Y$ be a function then f is called the 2-product continuous at $\vartheta_0 \in X$ if for any given $\epsilon > 1\ \exists\ \delta > 1$ such that $\|\vartheta - \vartheta_0, a\|_{\hbar_{2X}} < \delta$ implies $\|f(\vartheta) - f(\vartheta_0), f(a)\|_{m_{2Y}} < \epsilon\ \ \forall\ \vartheta, a \in X$.

Theorem 3.19: Consider two 2-MNS $(X, \|.,.\|_{\hbar_{2X}})$ and $(Y, \|.,.\|_{\hbar_{2Y}})$. A mapping $f : X \rightarrow Y$ is 2-product continuous at $\vartheta_0 \in X$ if $\vartheta_n \rightarrow_{**} \vartheta_0$ in $X \Rightarrow f(\vartheta_n) \rightarrow_{**} f(\vartheta_0)$ in Y.
Proof. Consider f is a continuous function at ϑ_0. Then for a given $\epsilon > 1\ \exists\ \delta > 1$ such that $\|\vartheta - \vartheta_0, a\|_{m_{2X}} < \delta \Rightarrow \|f(\vartheta) - f(\vartheta_0), f(a)\|_{\hbar_{2Y}} < \epsilon$.
Now let $\vartheta_n \rightarrow_{**} \vartheta_0$ in X then by definition of 2-product convergent sequence $\exists n_0 \in N$ such that $\|\vartheta - \vartheta_0, a\|_{\hbar_{2X}} < \delta$ for all $n \geq n_0$ which implies that $\|f(\vartheta) - f(\vartheta_0), f(a)\|_{\hbar_{2Y}} < \epsilon$ for all $n \geq n_0$.
Therefore, $f(\vartheta_n) \rightarrow_{**} f(\vartheta_0)$ in Y.
Conversely, we assume that $\vartheta_n \rightarrow_{**} \vartheta_0$ in $X \Rightarrow f(\vartheta_n) \rightarrow_{**} f(\vartheta_0)$ in Y. We shall prove, f is continuous at ϑ_0. Assume this is not true then $\exists\ \epsilon > 1$ such that $\forall \delta > 1$, we have ϑ in X other than ϑ_0 satisfying $\|\vartheta - \vartheta_0, a\|_{\hbar_{2X}} < \delta$ but $\|f(\vartheta) - f(\vartheta_0), f(a)\|_{\hbar_{2Y}} \geq \epsilon$.
In precise for $\delta_n = 1 + \frac{1}{n}$, we have x_n in X satisfying $\|\vartheta - \vartheta_0, a\|_{\hbar_{2X}} < 1 + \frac{1}{n}$ but $\|f(\vartheta) - f(\vartheta_0), f(a)\|_{\hbar_{2Y}} \geq \epsilon$ which implies that $\vartheta_n \rightarrow_{**} \vartheta_0$ but $\{f(\vartheta_n)\}$ does not 2-product converge to $f(\vartheta_0)$ which is a contradiction.
Hence f is continuous at ϑ_0.

Proposition 3.20: Let $(X, \|.,.\|_{\hbar_{2X}})$ and $(Y, \|.,.\|_{\hbar_{2Y}})$ be two $2-MNS$. Then the cartesian product $X \times Y = \{(\vartheta_1, \vartheta_2) : \vartheta_1 \in X, \vartheta_2 \in Y\}$ is also $2-MNS$ with the following norms
(1) $\|(\vartheta_1, \vartheta_2), (\lambda, \mu)\|_{\hbar_2} = \|\vartheta_1, a\|_{\hbar_{2X}}.\|\vartheta_2, b\|_{\hbar_{2Y}}$
(2) $\|(\vartheta_1, \vartheta_2), (\lambda, \mu)\|_{\hbar_2} = \max\{\|\vartheta_1, \lambda\|_{\hbar_{2X}}, \|\vartheta_2, \mu\|_{\hbar_{2Y}}\}$
Proof. (1) We shall prove that $X \times Y$ is a $2-MNS$ under the 2-product norm $\|(\vartheta_1, \vartheta_2), (\lambda, \mu)\|_{\hbar_2} = \|\vartheta_1, \lambda\|_{\hbar_{2X}}.\|\vartheta_2, \mu\|_{\hbar_{2Y}}$.
(i) $\|(\vartheta_1, \vartheta_2), (\lambda, \mu)\|_{\hbar_2} = \|\vartheta_1, \lambda\|_{\hbar_{2X}}.\|\vartheta_2, \mu\|_{\hbar_{2Y}} \geq 1$, since $\|\vartheta_1, \lambda\|_{\hbar_{2X}} \geq 1$, $\|\vartheta_2, \mu\|_{\hbar_{2Y}} \geq 1$ for all $(\vartheta_1, \vartheta_2),\ (\lambda, \mu) \in X \times Y$
(ii) $\|(\vartheta_1, \vartheta_2),\ , (\lambda, \mu)\|_{\hbar_2} = 1$ if $\|\vartheta_1, \lambda\|_{\hbar_{2X}}.\|\vartheta_2, \mu\|_{\hbar_{2Y}} = 1$ if $\|\vartheta_1, \lambda\|_{\hbar_{2X}} = 1 = \|\vartheta_2, \mu\|_{\hbar_{2Y}}$ if $(\vartheta_1, \vartheta_2), (\lambda, \mu)$ are linearly dependent in $X \times Y$.
(iii) $\|\alpha(\vartheta_1, \vartheta_2), (\lambda, \mu)\|_{\hbar_2} = \|(\alpha\vartheta_1, \alpha\vartheta_2), (\lambda, \mu)\|_{\hbar_2}$
$= \|\alpha\vartheta_1, \lambda\|_{\hbar_{2X}}.\|\alpha\vartheta_2, \mu\|_{\hbar_{2Y}} = \|\vartheta_1, \lambda\|^{|\alpha|}_{\hbar_{2X}}.\|\vartheta_2, \mu\|^{|\alpha|}_{\hbar_{2Y}}$
$= \{\|\vartheta_1, \lambda\|_{\hbar_{2X}}.\|\vartheta_2, \mu\|_{\hbar_{2Y}}\}^{|\alpha|} = \|(\vartheta_1, \vartheta_2), (\lambda, \mu)\|^{|\alpha|}_{\hbar_2}$
(iv) Let $(\vartheta_1, \rho_1), (\vartheta_2, \rho_2), (\lambda, \mu) \in X \times Y$, then $\|(\vartheta_1, \rho_1) + (\vartheta_2, \rho_2), (\lambda, \mu)\|_{\hbar_2}$
$= \|(\vartheta_1 + \vartheta_2, \rho_1 + \rho_2), (\lambda, \mu)\|_{\hbar_2}$
$= \|\vartheta_1 + \vartheta_2, \lambda\|_{\hbar_{2X}}.\|\rho_1 + \rho_2, \mu\|_{\hbar_{2Y}}$
$\leq \|\vartheta_1, \lambda\|_{\hbar_{2X}}.\|\vartheta_2, \mu\|_{\hbar_{2X}}.\|\rho_1, \mu\|_{\hbar_{2Y}}.\|\rho_2, \mu\|_{\hbar_{2Y}}$
$= \|\vartheta_1, \lambda\|_{\hbar_{2X}}.\|\rho_1, \mu\|_{\hbar_{2Y}}.\|\vartheta_2, \lambda\|_{\hbar_{2X}}.\|\rho_2, \mu\|_{\hbar_{2Y}}$
$= \|(\vartheta_1, \rho_1), (\lambda, \mu)\|_{\hbar_{2X}}.\|(\vartheta_2, \rho_2), (\lambda, \mu)\|_{\hbar_{2Y}}$.
(2) We shall prove that $X \times Y$ is a $2-MNS$ under the 2-product norm $\|(\vartheta, \rho), (\lambda, \mu)\|_{\hbar_2} = \|\vartheta, \lambda\|_{\hbar_{2X}}.\|\rho, \mu\|_{\hbar_{2Y}}$.
(i) $\|(\vartheta, \rho), (\lambda, \mu)\|_{\hbar_2} = max\{\|\vartheta, \lambda\|_{\hbar_{2X}}, \|\rho, \mu\|_{\hbar_{2Y}}\} \geq 1$, since $\|\vartheta, \lambda\|_{\hbar_{2X}} \geq 1$, $\|\rho, \mu\|_{\hbar_{2Y}} \geq 1$ for all $(\vartheta, \rho), (\lambda, \mu) \in X \times Y$.
(ii) $\|(\vartheta, \rho), (\lambda, \mu)\|_{\hbar_2} = 1$ if $max\{\|\vartheta, \lambda\|_{\hbar_{2X}}, \|\rho, \mu\|_{\hbar_{2Y}}\} = 1$ if $\|\vartheta, \lambda\|_{\hbar_{2X}} = 1 = \|\rho, \mu\|_{\hbar_{2Y}}$ if $(\vartheta, \rho),\ (\lambda, \mu)$ are linearly dependent in $X \times Y$.
(iii) $\|\alpha(\vartheta, \rho),\ (\lambda, \mu)\|_{\hbar_2} = \|(\alpha\vartheta, \alpha\rho), (\lambda, \mu)\|_{\hbar_2}$
$= max\{\|\alpha\vartheta, \lambda\|_{\hbar_{2X}}, \|\alpha\rho, \mu\|_{\hbar_{2Y}}\}$
$= max\{\|\vartheta, \lambda\|^{|\alpha|}_{\hbar_{2X}}, \|\rho, \mu\|^{|\alpha|}_{\hbar_{2Y}}\}$
$= max\{\{\|\vartheta, \lambda\|_{\hbar_{2X}}.\|\rho, \mu\|_{\hbar_{2Y}}\}\}^{|\alpha|}$
$= \|(\vartheta, \rho), (\lambda, \mu)\|^{|\alpha|}_{\hbar_2}$.

(iv) Let $(\vartheta_1, \rho_1), (\vartheta_2, \rho_2), (\lambda, \mu \in X \times Y$, then
$\|(\vartheta_1, \rho_1) + (\vartheta_2, \rho_2), (\lambda, \mu)\|_{\hbar_2}$
$= \|(\vartheta_1 + \vartheta_2, \rho_1 + \rho_2), (\lambda, \mu)\|_{\hbar_2}$
$= max\{\|\vartheta_1 + \vartheta_2, \lambda\|_{\hbar_{2X}}, \|\rho_1 + \rho_2, \mu\|_{\hbar_{2Y}}\}$
$\leq max\{\|\vartheta_1, \lambda\|_{\hbar_{2X}}, \|\vartheta_2, \lambda\|_{\hbar_{2X}}, \|\rho_1, \mu\|_{\hbar_{2Y}}, \|\rho_2, \mu\|_{\hbar_{2Y}}\}$
$\leq max\{\|\vartheta_1, \lambda\|_{\hbar_{2X}}, \|\rho_1, \mu\|_{\hbar_{2Y}}\}$
$max\{\|\vartheta_2, \lambda\|_{\hbar_{2X}}.\|\rho_2, \mu\|_{\hbar_{2Y}}\}$
$= \|(\vartheta_1, \rho_1), (\lambda, \mu)\|_{\hbar_{2X}} \|(\vartheta_2, \rho_2), (\lambda, \mu)\|_{\hbar_{2Y}}$.
Hence $\|(\vartheta_1, \rho_1) + (\vartheta_2, \rho_2), (\lambda, \mu)\|_{\hbar_2}$
$\leq \|(\vartheta_1, \rho_1), (\lambda, \mu)\|_{\hbar_{2X}}.\|(\nu_2, \rho_2), (\lambda, \mu)\|_{\hbar_{2Y}}$.

Theorem 3.21: Let $(X, \|., .\|_{\hbar_2})$ be $2-MNS$ over the field F, then
(i) $(\alpha, \vartheta) \to \alpha\nu$ from $f: F \times X \to X$ is 2-product continuous
(ii) $(\vartheta, \rho) \to \vartheta + \rho$ from $f: X \times X \to X$ is 2-product continuous
(iii) $\vartheta \to \|\vartheta\|$ from $f: X \to R$ is 2-product continuous that is, 2-product norm is 2-product continuous.
Proof.(i) Let $\alpha_n \to \alpha$ in F & $\vartheta_n \to_{**} \vartheta$ in X. To prove that $\alpha_n\vartheta_n \to_{**} \alpha\vartheta$ as $n \to \infty$.
Consider $\|\alpha_n\vartheta_n - \alpha\vartheta, \lambda\|_{\hbar_2}$
$= \|\alpha_n\vartheta_n - \alpha_n\vartheta + \alpha_n\vartheta - \alpha\vartheta, \lambda\|_{\hbar_2}$
$\leq \|\alpha_n(\vartheta_n - \vartheta), \lambda\|_{\hbar_2}.\|(\alpha_n - \alpha)\vartheta, \ \lambda\|_{\hbar_2}$
$= \|\vartheta_n - \vartheta, \ \lambda\|_{\hbar_2}^{|\alpha_n|}.\|\vartheta, \ \lambda\|_{\hbar_2}^{|\alpha_n - \alpha|}$,
since $|\alpha_n - \alpha| \to 0$ and $\|\vartheta_n - \vartheta, \lambda\|_{\hbar_2} \to_{**} 1$ as $n \to \infty$ we obtain $\|\alpha_n\vartheta_n - \alpha\vartheta, \lambda\|_{\hbar_2} \to_{**} 1$ as $n \to \infty$. i.e., $\alpha_n\vartheta_n \to_{**} \alpha\vartheta$ as $n \to \infty$.
(ii) Let $\vartheta_n \to_{**} \vartheta$ & $\rho_n \to_{**} \rho$ as $n \to \infty$ then $\|\vartheta_n - \vartheta, \lambda\|_{\hbar_2} \to_{**} 1$ and $\|\rho_n - \rho, \lambda\|_{\hbar_2} \to_{**} 1$ as $n \to \infty$.
Consider $\|(\vartheta_n + \rho_n) - (\vartheta + \rho), \lambda\|_{\hbar_2}$
$= \|(\vartheta_n - \vartheta) + (\rho_n - \rho), \lambda\|_{\hbar_2}$
$\leq \|\vartheta_n - \vartheta, \lambda\|_{\hbar_2}.\|\rho_n - \rho, \lambda\|_{\hbar_2} \to_{**} 1$ as $n \to \infty$
i.e., $(\vartheta_n + \rho_n) \to_{**} (\vartheta + \rho)$ as $n \to \infty$.
(iii) we first prove the inequality
$\frac{\|\rho, \lambda\|_{\hbar_2}}{\|\vartheta - \rho, \lambda\|_{\hbar_2}} \leq \|\vartheta, \lambda\|_{\hbar_2} \leq \|\vartheta - \rho, \lambda\|_{\hbar_2}.\|\rho, \lambda\|_{\hbar_2}$. Now

$$\|\vartheta, \lambda\|_{\hbar_2} = \|\vartheta - \rho + \rho, \lambda\|_{\hbar_2} \leq \|\vartheta - \rho, \lambda\|_{\hbar_2}.\|\rho, \lambda\|_{\hbar_2} \quad (1)$$

interchanging the role of ϑ and ρ, we obtain

$$\|\rho, \lambda\|_{\hbar_2} = \|\rho - \vartheta + \vartheta, \ \lambda\|_{\hbar_2} \leq \|\rho - \vartheta, \ \lambda\|_{\hbar_2}.\|\vartheta, \ \lambda\|_{\hbar_2} \quad (2)$$

using (1) and (2), we get

$$\frac{\|\rho, \lambda\|_{\hbar_2}}{\|\vartheta - \rho, \lambda\|_{\hbar_2}} \leq \|\vartheta, \lambda\|_{\hbar_2} \leq \|\vartheta - \rho, \lambda\|_{\hbar_2}.\|\rho, \lambda\|_{\hbar_2} \quad (3)$$

put $\vartheta = \vartheta_n$ and $\rho = \vartheta$ in (3), we get

$$\frac{\|\vartheta, \lambda\|_{\hbar_2}}{\|\vartheta_n - \vartheta, \lambda\|_{\hbar_2}} \leq \|\vartheta_n, \lambda\|_{\hbar_2} \leq \|\vartheta_n - \vartheta, \lambda\|_{\hbar_2}.\|\vartheta, \lambda\|_{\hbar_2} \quad (4)$$

Now let $\vartheta_n \to_{**} \vartheta$, then by using (4), we get $\|\vartheta_n, \lambda\| \to_{**} \|\vartheta, \lambda\|$. Thus the product norm is 2-product continuous function.

4 Conclusion and future scope

Multiplicative calculus changes how we grasp things that grow significantly, like in economics, physics, engineering, and finance. It precisely handles growth and proportions. This mathematical tool proves to be increasingly useful in various fields, venturing into new areas like dealing with fractional or complex parts. It could prove valuable in studying detailed patterns in complex systems and wavelet theory. Researchers and experts are just beginning to explore this engaging mathematical realm, with the potential to solve vital problems and reshape our understanding of complex aspects in our world.

In summary, multiplicative calculus and its spaces hold significance in the realm of mathematics. By exploring and utilizing it, we can deepen our comprehension of theories and achieve noteworthy advancements. The journey into multiplicative calculus has just started, promising to alter how we perceive the world.

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