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Integral Graph Spectrum and Energy of Interconnected Balanced Multi-star Graphs

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Abstract Balanced multi-star graph $K_r(n)$ is a specialized type of graph formed by connecting apex vertices of star graphs to create a cohesive structure known as a clique. These graphs comprise r star graphs, where each star graph has an apex vertex connected to n pendant vertices. Balanced multi-star graphs offer benefits in scenarios requiring equal distances between peripheral nodes, such as sensor networks, distributed computing, traffic engineering, telecommunications, supply chain management, and power distribution. The integral graph spectrum derived from the adjacency matrix of balanced multistar graphs holds significance across various domains. It aids in network analysis to understand connectivity patterns, facilitates efficient computation of structural properties through graph algorithms, and enables graph partitioning and community detection. Spectral graph theory assists in identifying connectivity patterns in network visualization, supports modeling biological networks in biomedical research, aids in generating personalized recommendations in recommendation systems and contributes to graph-based segmentation and scene analysis tasks in image processing. This paper aims to characterize the integral graph spectrum of balanced multi-star graphs $K_r(n)$ by focusing on spectral parameters of double-star graphs ($r = 2$), triple-star graphs ($r = 3$), and quadruple-star graphs ($r = 4$). This spectrum serves as an important tool across disciplines, providing insights into graph structure and facilitating tasks ranging from network analysis to computational biology and image processing.

Keywords Multi-star Graph, Balanced Multi-star Graph, Integral Graphs, Spectrum, Graph Energy

1 Introduction

In the realm of graph theory, the study of graph spectral parameters and energy holds significant importance shedding light on the intricate structural characteristics and connectivity patterns within various graph structures [1]. This paper investigates the graph spectral parameters associated with a fascinating class of graphs known as multi-star graphs. These graphs are constructed by interconnecting apex vertices of star graphs to form a cohesive structure known as a clique. The resulting graphs, denoted as $K_r(a_1, a_2, \ldots, a_r)$, consist of r star graphs, where the apex vertex of the i^{th} star graph has a_i pendant vertices. In general, the $a_i's$ need not be the same. When $a_1 = a_2 = a_3 = ... = a_r$; i.e., each star-constituent has the same number of pendent vertices. These graphs exhibit a balanced uniform structure. The balanced multistar graph is denoted by $K_r(n)$ where each $a_i = n$, $1 \leq i \leq r$. The analysis of graph spectral parameters within such balanced multi-star graphs further enriches our understanding of the interplay between graph topology and spectral characteristics. The fundamental goal of this study is to unveil and analyse the graph spectral parameters and energy specific to certain scenarios, namely the double-star graph ($r = 2$), the triple-star graph ($r = 3$) and the quadruple-star graph ($r = 4$). These cases hold particular significance due to their simple yet intricate connectivity, providing valuable insights into the behaviour of graph spectral properties within multi-star configurations. Through this analysis, we aim to contribute to the broader understanding of Spectral Graph Theory while unveiling the unique features embedded within multi-star graph structures. F. Harary and A. J. Schwenk [2] have first investigated about the graphs having integral spectrum followed by some unsolved problems. In this paper, we have found some balanced multi-star graphs having integral spectrum. We delve into the integral graph spectrum and its uses within such balanced multi-star graphs. The exploration of integral spectra provides valuable insights into the algebraic and combinatorial properties of these graphs. It allows us to analyse scenarios where graph eigenvalues are integers, a phenomenon with implications for diverse fields such as network design, coding theory and cryptography.

2 Preliminaries

Definition 1. *A graph G is defined as an ordered triple ($V(G)$, $E(G)$, ψ_G), comprising a non-empty vertex set $V(G)$, an edge set $E(G)$ and an incidence relation ψ_G.*

Definition 2. *A subset of $E(G)$ where no two of its edges shares a common vertex is called a matching. A matching that comprises of k edges is referred to as a k-matching.*

Definition 3. *A matching having maximum number of edges among all other matchings is known as a maximum matching M and its cardinality, denoted $|M|$, is called the size of M.*

Definition 4. *The star graph S_n is a tree graph on n vertices, characterized by a central vertex (hereafter referred to as the apex vertex) with degree of $n-1$, and $n-1$ pendant vertices. Consequently, the star graph S_n is isomorphic to the complete bipartite graph $K_{1,n-1}$.*

Definition 5. *A clique of a graph can be visualized as a cluster of vertices that are mutually adjacent, thus forming a subgraph that is isomorphic to a complete graph. The size of a clique is enumerated by the count of vertices it incorporates.*

Definition 6. *The adjacency matrix $A(G)$, also referred to as the connection matrix, is a square matrix representing the adjacency relationships among the vertices in the graph. This matrix $A(G)$ is constructed by marking the entry a_{ij} as 1 if vertices v_i and v_j are adjacent and 0 if they are not.*

Definition 7. *The characteristic equation of the adjacency matrix $A(G)$, is derived from $P(A;\lambda) = \det(A - \lambda I_n)$, where I_n stands for the identity matrix of order n.*

Definition 8. *The eigenvalues of the adjacency matrix can be calculated by solving the characteristic equation $P(A;\lambda) = 0$.*

Definition 9. *If the characteristic polynomial of the graphs' adjacency matrix has all integer eigenvalues then it is considered an integral graph.*

Definition 10. *The spectrum of a graph $Sp(G)$ is an arrangement of its eigenvalues, each accompanied by its corresponding algebraic multiplicities.*

Definition 11. *The eigenvalues of the adjacency matrix $A(G)$ are typically arranged in a non-increasing order denoted as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Among these eigenvalues, λ_1 holds a particular significance as it is designated as the spectral radius. Additionally, the difference $\lambda_1 - \lambda_2$ is termed the spectral gap.*

Definition 12. *The sum of the absolute values of all the eigenvalues are termed as graph energy $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$.*

Definition 13. *The graph energy is said to be integral graph energy if all the eigenvalues of the graphs' adjacency matrix are integers.*

3 Literature Survey

F.R.K. Chung [3], synthesized 10 lectures delivered during the CBMS workshop on Spectral Graph Theory in June 1994 at Fresno State University. The book explored various topics in Spectral Graph Theory and highlighted the importance of Linear Algebra in the field of Graph Theory. A.E. Brouwer and W.H. Haemers [4] discussed the spectra of graphs, providing insights into their properties and applications. D.A. Spielman [5] offered intuition on the combinatorial significance of eigenvectors and eigenvalues, presenting a survey of their applications. E.R.V. Dam and W.H. Haemers [6] surveyed graphs that could be determined by their spectrum, utilizing adjacency and laplacian matrices. Z. Lin [7] obtained new lower bounds on the $A\alpha$-spectral radius and $A\alpha$-spread. Similar kind of works are done in [8, 9]. F. Celik and I.N. Cangul [10] derived polynomials and recurrence relations for the spectral polynomials of cycles and paths, providing a method to obtain spectra of C_{2n} and P_{2n+1} based on the spectra of C_n and P_n. A method for approximating the total π-electron energy of a conjugated hydrocarbon using spectral moments was proposed by I. Gutman et al. [11]. I. Gutman [12] found that the graph energy exceeded the number of vertices in the graph. I. Gutman and B. Furtula [13] provided an overview of graph energy, its applications and recent trends in research. F. Harary and A.J. Schwenk [2] developed a systematic approach to identify graphs with integral spectra and presented some unsolved problems. Ahmadi et al. [14] showed that only a few graphs have an integral spectrum. G. Indulal and A. Vijayakumar [15] introduced constructions for generating graphs with integral spectra. In a subsequent work, G. Indulal, R. Balakrishnan, and A. Anuradha [16] defined a novel composition operation on an ordered triple of three graphs, leading to the construction of diverse classes of integral graphs. K Balinska et al. [17] has done a survey of results on integral graphs and its corresponding proof techniques. Abdul Hameed et al, [18] has found the Laplacian energy and first Zagreb index of Laplacian integral graphs. Liu et al. [19] has investigated some algebraic properties of the Cayley integral graph. Milan Bašić et al. [20] found maximal value of the diameter of the integral circulant graph. J. W. Sander and T. Sander [21] have presented a method that allows the characterization of all integral circulant graphs with a spectrum, whose multiplicative divisor set possesses a spectrum. S. Mandal [22] has investigated the cospectral as well integral chain graphs for Seidel matrix, a key component to study the structural properties of equiangular lines in space. S.D. Nikolopoulos and I. Rondogiannis [23] formulated a formula for counting spanning trees in double-star ($m = 2$), triple-star ($m = 3$), and quadruple-star ($m = 4$) graphs. W.M. Yan et al. [24] expanded on this work, introducing a novel labeling technique and matrix computations for a more generalized approach. K.L. Chung and W.M. Yan [25] extended the study to determine the number of spanning trees in multi-complete/star-related graphs.

4 Spectral Parameters of Multi-star Graphs

Lemma 1 ([26]). *Let G be a labelled simple graph on n vertices. For an integer i, let N_i denotes the collection of i-vertex subgraphs of G whose components are edges or cycles. Let c_i be the coefficient of λ^{n-i} in the characteristic polynomial of G. Then, the coefficient c_i can be expressed as follows:*

$$c_i = \sum_{N \epsilon N_i} (-1)^{x(N)} 2^{y(N)},$$

where N is a subgraph in the collection N_i, $x(N)$ is the total number of components in the subgraph N and $y(N)$ is the number of components in N that are cycles.

4.1 Graph Spectrum of Double-star graph $K_2(a_1, a_2)$

Consider the double-star graph $K_2(a_1, a_2)$ (Fig. 1). Let $V(K_2(a_1, a_2)) = \{u, u_i,\ 1 \le i \le a_1\} \cup \{v, v_j,\ 1 \le j \le a_2\}$ be its vertex set. Here u and v are the apex vertices and u_i, v_j are pendant vertices. The vertices u_i and v_j are adjacent with u and v respectively. The collection of vertices u_i and v_j form an independant set. Then $E(K_2(a_1, a_2)) = \{uv, uu_i, vv_j,\ 1 \le i \le a_1 \text{ and } 1 \le j \le a_2\}$. Thus $K_2(a_1, a_2)$ is a tree with $a_1 + a_2 + 2$ vertices and $a_1 + a_2 + 1$ edges.

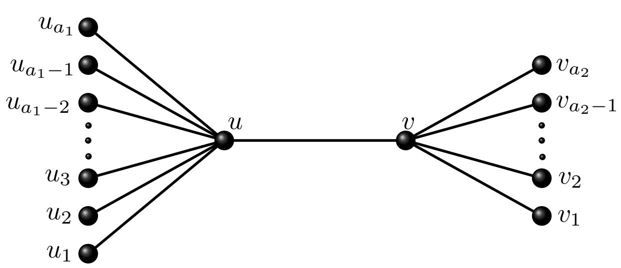

Figure 1. Double-star graph $K_2(a_1, a_2)$

Theorem 1. *The spectrum of $K_2(a_1, a_2)$ is given by,*
$Sp(K_2(a_1, a_2)) =$

$$\left\{ \begin{array}{c} 0^{(a_1+a_2-2)}, \\ \pm\sqrt{\frac{(a_1+a_2+1)+\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}}^{(1)}, \\ \pm\sqrt{\frac{(a_1+a_2+1)-\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}}^{(1)} \end{array} \right\}.$$

Proof. Let the characteristic polynomial of $K_2(a_1, a_2)$ be $P(A; \lambda) = \lambda^{a_1+a_2+2} - c_1\lambda^{a_1+a_2+1} + c_2\lambda^{a_1+a_2} - c_3\lambda^{a_1+a_2-1} + c_4\lambda^{a_1+a_2-2} + \cdots + c_n$. As $K_2(a_1, a_2)$ is a tree (i.e, bipartite and acyclic), $c_i = 0$ for all odd i. The value of c_2 amounts to the sum of two minors which in graph language is the total number of edges in $K_2(a_1, a_2)$. By the arrangement of the edges, it is clear that $K_2(a_1, a_2)$ contains no matching of size greater than 3. Hence $c_i = 0\ \forall\ i \ge 5$. It now remains to compute c_4. c_4 can be calculated by adding the non-zero four minors that are associated to 2-matchings of $K_2(a_1, a_2)$. For each edge uu_i, $1 \le i \le a_1$, any edge of the form vv_j, $1 \le j \le a_2$, forms a 2-matching. Further, there will be no 2-matching containing the edge uv. The number of 2-matchings thus sum upto a_1a_2. All these observations lead to the conclusion that $P(A; \lambda) = \lambda^{a_1+a_2+2} - (a_1 + a_2 + 1)\lambda^{a_1+a_2} + a_1a_2\lambda^{a_1+a_2-2}$. On factorizing $P(A; \lambda)$, we get

$$\begin{aligned} P(A; \lambda) &= 0 \\ \lambda^{a_1+a_2+2} - (a_1 + a_2 + 1)\lambda^{a_1+a_2} + a_1a_2\lambda^{a_1+a_2-2} &= 0 \\ \lambda^{a_1+a_2-2}\left(\lambda^4 - (a_1 + a_2 + 1)\lambda^2 + a_1a_2\right) &= 0 \end{aligned}$$

Thus 0 is an eigenvalue with algebraic multiplicity $(a_1 + a_2 - 2)$. The remaining eigenvalues are roots of the bi-quadratic equation $\lambda^4 - (a_1 + a_2 + 1)\lambda^2 + a_1a_2 = 0$. Thus the spectrum of $K_2(a_1, a_2)$ is given by
$Sp(K_2(a_1, a_2)) =$

$$\left\{ \begin{array}{c} 0^{(a_1+a_2-2)}, \\ \pm\sqrt{\frac{(a_1+a_2+1)+\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}}^{(1)}, \\ \pm\sqrt{\frac{(a_1+a_2+1)-\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}}^{(1)} \end{array} \right\}.$$

□

Theorem 2. *The graph energy of the graph $K_2(a_1, a_2)$,is given by*
$\mathcal{E}(K_2(a_1, a_2)) =$
$$2\left(\begin{array}{c} \sqrt{\frac{(a_1+a_2+1)+\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}} + \\ \sqrt{\frac{(a_1+a_2+1)-\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}} \end{array} \right).$$

Proof. The proof follows by taking the sum of absolute values of the eigenvalues computed in Theorem 1. □

Theorem 3. *If $a_1 = s(s + 1)$ for some positive integer s and $a_2 = a_1$, then the corresponding $K_2(a_1, a_2)$ is integral.*

Proof. Substituting $a_1 = s(s + 1)$ and $a_2 = a_1$ in the eigenvalues computed in Theorem 1, we get $\pm\sqrt{\frac{(a_1+a_2+1)+\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}} = \pm(s + 1)$ and $\pm\sqrt{\frac{(a_1+a_2+1)-\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}} = \pm s$. Hence the spectrum of the corresponding $K_2(n)$ is given by $Sp(K_2(n)) = \left\{0^{(a_1+a_2-2)}, \pm(s + 1)^{(1)}, \pm s^{(1)}\right\}$. This shows that $K_2(a_1, a_2)$ is integral whenever $a_1 = s(s + 1)$ and $a_2 = a_1$. □

Corollary 4. *If $a_1 = s(s + 1)$ for some positive integer s and $a_2 = a_1$, then the corresponding graph energy of $K_2(a_1, a_2)$ is $\mathcal{E}(K_2(a_1, a_2)) = 2(2s + 1)$.*

Proof. On adding the absolute values of the eigenvalues obtained in Theorem 3, the required result is obtained. □

Theorem 5. *If $a_2 = 2(a_1 - 1)$ then the corresponding $K_2(a_1, a_2)$ is integral.*

Proof. Substituting $a_2 = 2(a_1 - 1)$ in the eigenvalues computed in Theorem 1, we get $\pm\sqrt{\frac{(a_1+a_2+1)+\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}} = \pm\sqrt{2a_1}$ and $\pm\sqrt{\frac{(a_1+a_2+1)-\sqrt{(a_1+a_2+1)^2-4a_1a_2}}{2}} = \pm\sqrt{a_1-1}$. Hence the spectrum of the corresponding $K_2(a_1,a_2)$ is given by $Sp\left(K_2(a_1,a_2)\right) = \left\{0^{(a_1+a_2-2)}, \pm\sqrt{2a_1}^{(1)}, \pm\sqrt{a_1-1}^{(1)}\right\}$. If $2a_1$ and $a_1 - 1$ is a perfect square then the corresponding $K_2(a_1,a_2)$ is integral. □

Corollary 6. *If $a_2 = 2(a_1 - 1)$, then the corresponding graph energy of $K_2(a_1,a_2)$ is given by $\mathcal{E}(K_2(a_1,a_2)) = 2\left(\sqrt{2a_1} + \sqrt{a_1-1}\right)$.*

Proof. On adding the absolute values of the eigenvalues obtained in Theorem 5, the required result is obtained. □

Corollary 7. *If $a_1 = 2, 50, 1682$ then the corresponding graph is integral.*

Proof. By substituting $a_1 = 2, 50, 1682$ in the eigenvalues in theorem 5 we get the integral values. □

4.2 Graph Spectrum of Triple-star graph $K_3(a_1,a_2,a_3)$

Consider the triple-star graph $K_3(a_1,a_2,a_3)$ (Fig. 2). Let $V(K_3(a_1,a_2,a_3)) = \{V(K_2(a_1,a_2))\} \cup \{w, w_l,\ 1 \le l \le a_3\}$ be the vertex set. Here w denote the apex vertex of the third star graph S_{a_3}. Then $E(K_3(a_1,a_2,a_3)) = \{E(K_2(a_1,a_2))\} \cup \{uw, vw, ww_l,\ 1 \le l \le a_3\}$. Here $K_3(a_1,a_2,a_3)$ is a unicyclic graph with $|V(K_3(a_1,a_2,a_3))| = |E(K_3(a_1,a_2,a_3))| = a_1 + a_2 + a_3 + 3$. Thus, there are totally $a_1 + a_2 + a_3$ pendant vertices which together form an independent set. For computing the k-matchings of $K_3(a_1,a_2,a_3)$, we partition the edge set into following partitions $P_1=\{uu_1, uu_2, uu_3, \ldots, uu_{a_1}\}$, $P_2=\{vv_1, vv_2, vv_3, \ldots, vv_{a_2}\}$, $P_3=\{ww_1, ww_2, ww_3, \ldots, ww_{a_3}\}$ and $P_4=\{uv, uw, vw\}$. Note that $|P_1| = a_1$, $|P_2| = a_2$, $|P_3| = a_3$, $|P_4| = 3$ and that P_4 forms a clique.

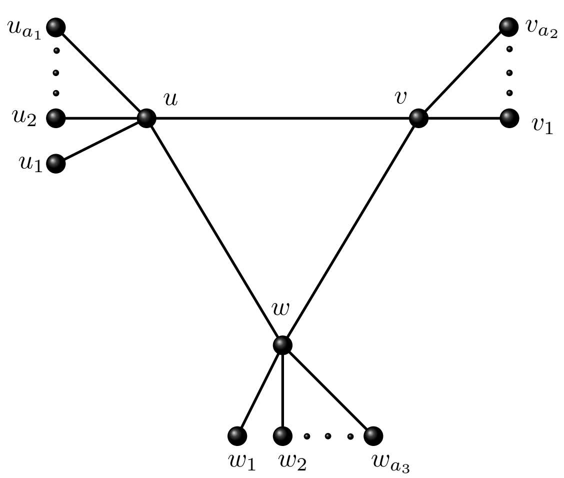

Figure 2. Triple-star graph $K_3(a_1,a_2,a_3)$

4.2.1 Method of computing k-matchings of $K_3(a_1,a_2,a_3)$

From (Fig. 2) we can observe that a maximum matching M of $K_3(a_1,a_2,a_3)$ is of size 3.
Let us find all k-matchings for $K_3(a_1,a_2,a_3)$ where $k \le 3$.

1. $k = 1$: Each edge in a graph is a 1-matching. Hence there are totally $a_1 + a_2 + a_3 + 3$ 1-matchings.

2. $k = 2$: In the Table 1 below, we have listed every possible combination of partitions and set of edges to compute 2-matchings. In the Table 1, the first column entries are of the form (P, Q) which means that each partition contribute an edge to the 2-matching.

Table 1. 2-matchings of triple-star graph $K_3(a_1,a_2,a_3)$

Partitions	**2-matchings**
(P_1, vw) + (P_2, uw) + (P_3, uv)	$\sum_{i=1}^{3} a_i$
(P_1, P_2) + (P_1, P_3) + (P_2, P_3)	$\sum_{1\le i<j\le 3} a_i a_j$
Total number of 2-matchings	$\sum_{i=1}^{3} a_i + \sum_{1\le i<j\le 3} a_i a_j$

3. $k = 3$: From (Fig. 2), we can observe that the only way to form a 3-matching is by selecting exactly an edge from each of P_1, P_2 and P_3. This leads to $a_1a_2a_3$ number of 3-matchings.

Theorem 8. *The characteristic polynomial of $K_3(a_1,a_2,a_3)$ is given by,*
$P(A;\lambda) =$

$$\lambda^m \left(\begin{array}{c} \lambda^6 - (\sum_{i=1}^{3} a_i + 3)\lambda^4 - 2\lambda^3 + \\ (\sum_{i=1}^{3} a_i + \sum_{1\le i<j\le 3} a_i a_j)\lambda^2 - a_1a_2a_3 \end{array} \right)$$

where $m = a_1 + a_2 + a_3 - 3$.

Proof. Let the characteristic polynomial of $K_3(a_1,a_2,a_3)$ be $P(A;\lambda) = \lambda^{a_1+a_2+a_3+3} - c_1\lambda^{a_1+a_2+a_3+2} + c_2\lambda^{a_1+a_2+a_3+1} - c_3\lambda^{a_1+a_2+a_3} + c_4\lambda^{a_1+a_2+a_3-1} + \cdots + c_n$. Let $S = \{u, v, w\}$. As the neighbourhood $N(S)$ of the set S is given by $V(K_3(a_1,a_2,a_3)) - S$, it is clear that $c_i = 0$ for all odd $i \ne 3$. From (Fig. 2) we can find that there exist no k-matching greater than 3. Thus $c_i = 0$ for all even $i \ge 6$. Then by using the Lemma 1 and the k-matchings that are found in section 4.2.1 the theorem follows. □

4.3 Graph Spectrum of Quadruple-star graph $K_4(a_1,a_2,a_3,a_4)$

Consider the quadruple-star graph $K_4(a_1,a_2,a_3,a_4)$ (Fig.3). Let $V(K_4(a_1,a_2,a_3,a_4)) = \{V(K_3(a_1,a_2,a_3))\} \cup \{x, x_m,\ 1 \le m \le a_4\}$ be the vertex set. Here x denotes the apex vertex of the fourth star graph S_{a_4}. Then $E(K_4(a_1,a_2,a_3,a_4)) = \{E(K_3(a_1,a_2,a_3))\} \cup \{ux, vx, wx, xx_m,\ 1 \le m \le a_4\}$. Thus $K_4(a_1,a_2,a_3,a_4)$

has $a_1+a_2+a_3+a_4+4$ vertices and $a_1+a_2+a_3+a_4+6$ edges. Thus, there are totally $a_1+a_2+a_3+a_4$ pendant vertices which together form an independent set. Similar to the partitions of $K_3(a_1,a_2,a_3)$ we make the following edge partitions: $P_1 = \{uu_1, uu_2, uu_3, \ldots, uu_{a_1}\}$, $P_2 = \{vv_1, vv_2, vv_3, \ldots, vv_{a_2}\}$, $P_3 = \{ww_1, ww_2, ww_3, \ldots, ww_{a_3}\}$, $P_4 = \{xx_1, xx_2, xx_3, \ldots, xx_{a_4}\}$ and $P_5 = \{uv, ux, uw, vw, vx, wx\}$. Note that $|P_1| = a_1$, $|P_2| = a_2$, $|P_3| = a_3$, $|P_4| = a_4$, $|P_5| = 6$ and that P_5 forms a clique.

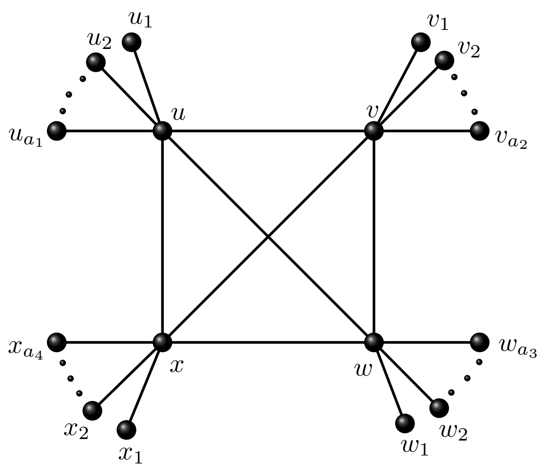

Figure 3. Quadruple-star graph $K_4(a_1,a_2,a_3,a_4)$

4.3.1 Method of computing k-matchings of $K_4(a_1,a_2,a_3,a_4)$

From (Fig. 3) we can observe that a maximum matching M of $K_4(a_1,a_2,a_3,a_4)$ is of size $|M|$ = 4. Let us find all k-matchings for $K_4(a_1,a_2,a_3,a_4)$ where $k \leq 4$.

1. $k = 1$: As in section 4.2.1, there are $a_1+a_2+a_3+a_4+6$ 1-matchings.

2. $k = 2$: In the Table 2 below, we have listed every possible combination of partitions and set of edges to compute 2-matchings. The methodology is similar to that explained in section 4.2.1.

Table 2. 2-matchings of quadruple-star graph $K_4(a_1,a_2,a_3,a_4)$

Partitions	2-matchings
$(P_1, \{vw, vx, xw\})$ + $(P_2, \{ux, uw, wx\})$ + $(P_3, \{uv, ux, vx\})$ + $(P_4, \{uv, uw, vw\})$	$3\sum_{i=1}^{4} a_i$
(P_1,P_2) + (P_1,P_3) + (P_1,P_4) + (P_2,P_3) + $(P_2,P_4)+(P_3,P_4)$	$\sum_{1\leq i<j\leq 4} a_ia_j$
$\{uv, xw\}$+$\{ux, vw\}$+ $\{uw, vx\}$	3
Total number of 2-matchings	$3\sum_{i=1}^{4} a_i + \sum_{1\leq i<j\leq 4} a_ia_j + 3$

3. $k = 3$: In the Table 3 below, we have listed every possible combination of partitions and set of edges to compute 3-matchings.

Table 3. 3-matchings of quadruple-star graph $K_4(a_1,a_2,a_3,a_4)$

Partitions	3-matchings
(P_1, xw, P_2) + (P_1, vx, P_3) + (P_1, vw, P_4) + (P_2, ux, P_3) + $(P_2, uw, P_4) + (P_3, uv, P_4)$	$\sum_{1\leq i<j\leq 4} a_ia_j$
(P_1,P_2,P_3) + (P_1,P_2,P_4) + $(P_1,P_3,P_4)+(P_2,P_3,P_4)$	$\sum_{1\leq i<j<k\leq 4} a_ia_ja_k$
Total number of 3-matchings	$\sum_{1\leq i<j\leq 4} a_ia_j + \sum_{1\leq i<j<k\leq 4} a_ia_ja_k$

4. $k = 4$: From (Fig. 3), we can observe that we can form 4-matchings by selecting an edge from P_1, P_2, P_3 and P_4. This leads to $a_1a_2a_3a_4$ number of 4-matchings.

Theorem 9. *The characteristic polynomial of $K_4(a_1,a_2,a_3,a_4)$ is given by,*

$$P(A;\lambda) = \lambda^m \left(\begin{array}{c} \lambda^8 - (\sum_{i=1}^{4} a_i + 6)\lambda^6 - 8\lambda^5 \\ +(3\sum_{i=1}^{4} a_i + \sum_{1\leq i<j\leq 4} a_ia_j + 3)\lambda^4 + 2\sum_{i=1}^{4} a_i\lambda^3 \\ -(\sum_{1\leq i<j\leq 4} a_ia_j + \sum_{1\leq i<j<k\leq 4} a_ia_ja_k)\lambda^2 \\ +a_1a_2a_3a_4 \end{array} \right)$$

where $m = a_1+a_2+a_3+a_4-4$.

Proof. The characteristic polynomial of $K_4(a_1,a_2,a_3,a_4)$ is denoted as $P(A;\lambda) = \lambda^{a_1+a_2+a_3+a_4+4} - c_1\lambda^{a_1+a_2+a_3+a_4+3} + c_2\lambda^{a_1+a_2+a_3+a_4+2} - c_3\lambda^{a_1+a_2+a_3+a_4+1} + c_4\lambda^{a_1+a_2+a_3+a_4} + \cdots + c_n$. In this polynomial, c_2 represents the count of 2-minors, reflecting the total number of edges in the graph, while c_3 counts the number of triangles. As the four vertices u, v, w, x form a clique, choosing any three of these vertices will form a cycle C_3. Hence there are ${}^4C_3 = 4$ triangles in the graph. Applying this value in Lemma 1 yields the value of c_3 as 8. The coefficient c_4 can be obtained by considering those subgraphs that are either $C_4's$ (3 in number) or 2-matchings (see Table 2). For c_5 in the polynomial, we consider subgraphs that are union of a triangle and a disjoint edge. This subgraph is composed of 2 components. To each triangle of $K_4(a_1,a_2,a_3,a_4)$, there exists a partition P_i $(i \leq 4)$ whose end vertices are the graph disjoint from the triangle. Thus c_5 computed by applying Lemma 1 to these subgraphs. To compute the remaining c_i's, it is enough to consider only matchings. The coefficients c_6 and c_8 can be calculated by utilizing the matching quantities provided in Table 3 and employing Lemma 1. For c_7 in the polynomial, we should consider subgraphs as cycle C_7 or the union of a triangle and two disjoint edges. From the graph we can observe that this way of consideration is not possible thus the value of c_7 is zero. The other coefficients $c_i, i \geq 9$, require matchings of size atleast 5. This scenario does not exist in $K_4(a_1, a_2, a_3, a_4)$. Consolidating all the above individual computations of c_i's the result is obtained. □

4.4 Spectral Parameters of Balanced Multi-star Graphs $K_r(n)$

Balanced multi-star graphs $K_r(n)$ are constructed by connecting same number of pendant vertices to each apex vertex of the n copies of S_n (i.e $a_1 = a_2 = a_3 = \cdots = a_r = n$). Consider the balanced multi-star graph $K_r(n)$. Let $V(K_r(n)) = \{u_g,\ 1 \leq g \leq r\} \cup \{v_h,\ 1 \leq h \leq rn\}$ be the vertex set. Here u_g $(1 \leq g \leq r)$ are the apex vertices and v_h $(1 \leq h \leq rn)$ are the pendant vertices. Each apex vertex u_g is adjacent with n number of pendant vertices. Thus the vertex partition $\{u_g,\ 1 \leq g \leq r\}$ forms a clique while $\{v_h,\ 1 \leq h \leq rn\}$ forms an independent set. Here $K_r(n)$ has $r(n+1)$ vertices and $r\left(\frac{2n+r-1}{2}\right)$ edges. $K_6(3)$ is shown in (Fig. 4).

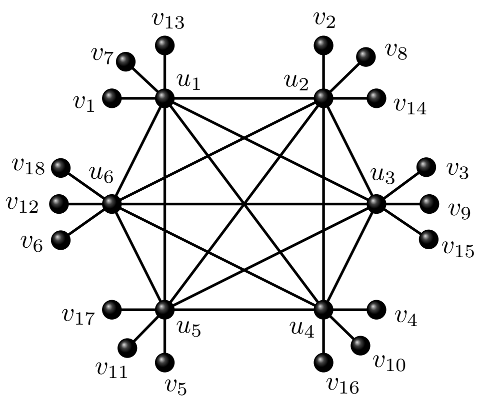

Figure 4. Balanced multi-star graph $K_6(3)$

Theorem 10. *The adjacency matrix $A = A(K_r(n))$ is given by*

$$A\left(K_r(n)\right) = \begin{bmatrix} K_r & I_1 & I_2 & I_3 & \dots & I_n \\ I_1 & 0 & 0 & 0 & \dots & 0 \\ I_2 & 0 & 0 & 0 & \dots & 0 \\ I_3 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ I_n & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

where K_r is the adjacency matrix of the complete graph of order r, I_t, $t = 1, 2, 3, \dots, n$, are the identity matrices each of order r.

Proof. To construct the adjacency matrix $A(K_r(n))$ in the requisite form, the following labelling is done to its vertices: The vertices of the clique K_r are considered first; a pendant vertex to each clique vertex labelled next. Thus the adjacency matrix of K_r is now appended with I_r. Labelling of another pendant vertex to each clique vertex will join another I_r block matrix to the existing construction. Proceeding with the same order of labelling yields the result. This process of labelling is expressed in (Fig. 4). □

Theorem 11. *The spectrum of $K_r(n)$ is given by,*

$$Sp\left(K_r(n)\right) = \left\{ \begin{array}{c} 0^{r(n-1)}, \\ \frac{(r-1)\pm\sqrt{(r-1)^2+4n}}{2}^{(1)}, \frac{-1\pm\sqrt{1+4n}}{2}^{(r-1)} \end{array} \right\}.$$

Proof. By substituting $a_1 = a_2 = a_3 = a_4 = n$ in the Theorems 1, 8, 9 we get the following characteristic polynomials

$r = 2:$
$P(A;\lambda) = \lambda^{2(n-1)}\left(\lambda^2 - \lambda - n\right)\left(\lambda^2 + \lambda - n\right).$
$r = 3:$
$P(A;\lambda) = \lambda^{3(n-1)}\left(\lambda^2 - 2\lambda - n\right)\left(\lambda^2 + \lambda - n\right)^2.$
$r = 4:$
$P(A;\lambda) = \lambda^{4(n-1)}\left(\lambda^2 - 3\lambda - n\right)\left(\lambda^2 + \lambda - n\right)^3.$
$\vdots$
$r:$
$P(A;\lambda) = \lambda^{r(n-1)}\left(\lambda^2 - (r-1)\lambda - n\right)$
$\left(\lambda^2 + \lambda - n\right)^{(r-1)}.$

On solving the three factors of $P(A;\lambda)$, the spectrum is computed . □

Theorem 12. *The graph energy of $K_r(n)$, is given by* $\mathcal{E}(K_r(n)) = \left(\sqrt{(r-1)^2 + 4n} + (r-1)\sqrt{1+4n}\right).$

Proof. The proof follows by taking the sum of absolute values of the eigenvalues computed in Theorem 11. □

4.4.1 Integral spectrum and energy of Balanced Multi-star Graphs $K_r(n)$

In the following Theorem 13 we have characterized the integral spectrum of $K_r(n)$.

Theorem 13. *A balanced multi-star $K_r(n)$, $n \geq 2$ is integral if and only if $r = \frac{n}{p} - (p-1)$ where p is a factor of n and $\frac{n}{p} > (p-1)$ and $n = s(s+1)$ for some positive integer s.*

Proof. From Theorem 11, let us consider one of the non zero eigenvalues

$$\frac{(r-1) \pm \sqrt{(r-1)^2 + 4n}}{2}. \tag{1}$$

Since, we know that p is a factor of n, we can express n as $n = kp$ for some positive integer k. By substituting $r = \frac{n}{p} - (p-1)$ and $n = kp$ into the expression 1, we get $\frac{(\frac{n}{p}-(p-1)-1)\pm\sqrt{(\frac{n}{p}-(p-1)-1)^2+4n}}{2} = \frac{(k-p)\pm\sqrt{(k-p)^2+4kp}}{2} = k, -p$. Now, let us consider the another non zero eigenvalues

$$\frac{-1 \pm \sqrt{1+4n}}{2}. \tag{2}$$

By substituting $n = s(s+1)$ into the expression 2 we get $\frac{-1\pm\sqrt{1+4s(s+1)}}{2} = s, -(s+1)$. As a result, the spectrum of the corresponding $K_r(n)$ can be given as $Sp\left(K_r(n)\right) = \left\{0^{r(n-1)}, k^{(1)}, -p^{(1)}, s^{(r-1)}, -(s+1)^{(r-1)}\right\}$. This proves that $K_r(n)$ is integral whenever $r = \frac{n}{p} - (p-1)$ and $n =$

$s(s+1)$.

Conversely, assume that $K_r(n)$ is integral. Then the eigenvalue 2 will be an integer and hence $\sqrt{1+4n}$ will be a positive integer say β. We can then express n in terms of β as $n = \frac{\beta^2-1}{4}$. To ensure that n is an integer, $\beta^2 - 1$ must be a multiple of 8, which can be written as $\beta^2 - 1 = 8\gamma$, where γ is a positive integer. Solving for β, we find $\beta = \sqrt{1+8\gamma}$. For β to be an integer, it is necessary for γ to be of the form $\frac{s(s+1)}{2}$ for a positive integer s. Substituting this expression for β back into the equation for n, we obtain $n = s(s+1)$. Now, we consider the other eigenvalue pair expressed as 1. We assume that $\sqrt{(r-1)^2+4n} = \alpha$, where α is a positive integer. We can express r in terms of α as $r = \sqrt{\alpha^2-4n}+1$ (As $r \geq 2$, we omlit the negative part of $\sqrt{\alpha^2-4n}$). As $n = s(s+1)$, it is always even. Let p be a factor of n. Then $n = kp$ for some positive integer k. Here, we can observe that r will become an integer only if $\alpha = k + p$. Substitution the α in the r yields $r = k - (p-1) = \frac{n}{p} - (p-1)$ (As $r \geq 2$, $\frac{n}{p} > (p-1)$. This completes the proof. □

Corollary 14. *If $r = \frac{n}{p} - (p-1)$ where p is a factor of n and $\frac{n}{p} > (p-1)$ and $n = s(s+1)$ for some positive integer s, then the corresponding graph energy of $K_r(n)$ is $\mathcal{E}(K_r(n)) = (k-p)(2s+1) + k + p$.*

Proof. By taking the sum of the absolute values of the eigenvalues computed in Theorem 13, the proof follows. □

Corollary 15. *If $K_r(n)$ is integral for some $r \geq 2$ then n is always even.*

Proof. The result follows as n is the product of two consecutive numbers, as stated in Theorem 13. □

5 Conclusion

The paper discusses the integral graph spectrum of balanced multi-star graphs, denoted as $K_r(n)$, where each constituent star graph possesses an equal number of pendant vertices. It establishes the characterization for what values of r and n the spectrum and energy of balanced multi-star graphs will be integers. Additionally, the paper provides generalized spectrum and energy for double-star graphs ($r = 2$), generalized characteristic polynomials for triple-star graphs ($r = 3$), and quadruple-star graphs ($r = 4$). As the characterization for the integral spectrum of balanced multi-star graphs is determined, it naturally leads to the question of what the characterization would be for the integral spectrum of unbalanced multi-star graphs. This generates interest among researchers to explore this special type of graph further. Overall, the study contributes to enhancing the understanding of graph structures and their applications, ranging from network analysis to computational biology and image processing. By delving into integral graph spectra, the paper advances knowledge in graph theory and its interdisciplinary applications.

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