Mathematics and Statistics 12(2): 204-210, 2024
DOI: 10.13189/ms.2024.120210 http://www.hrpub.org

Variations of Rigidity for Abelian Groups

Inessa I. Pavlyuk[1], **Sergey V. Sudoplatov**[1,2]

[1]Department of Algebra and Mathematical Logic, Faculty of Applied Mathematics and Computer Science, Novosibirsk State Technical University, Russia
[2]Laboratory of Logical Systems, Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

Received November 9, 2023; Revised March 5, 2024; Accepted March 19, 2024

Cite This Paper in the Following Citation Styles

***(a)**: [1] Inessa I. Pavlyuk, Sergey V. Sudoplatov, "Variations of Rigidity for Abelian Groups," Mathematics and Statistics, Vol.12, No.2, pp. 204-210, 2024. DOI: 10.13189/ms.2024.120210*

***(b)**: Inessa I. Pavlyuk, Sergey V. Sudoplatov (2024). Variations of Rigidity for Abelian Groups. Mathematics and Statistics, 12(2), 204-210. DOI: 10.13189/ms.2024.120210*

Abstract A series of basic characteristics of structures and of elementary theories reflect their complexity and richness. Among these characteristics, four kinds of degrees of rigidity and the index of rigidity are considered as measures of how far the given structure is situated from rigid one, both with respect to the automorphism group and to the definable closure, for some or any subset of the universe, which has the given finite cardinality. Thus, a natural question arises on a classification of model-theoretic objects with respect to rigidity characteristics. We apply a general approach of studying the rigidity values and related classification to abelian groups and their theories. We describe possibilities of degrees and indexes of rigidity for finite abelian groups and for standard infinite abelian groups. This description is based both on general consideration of rigidity, on its application for finite structures, and on their specificity for abelian groups including Szmielew invariants, combinatorial formulas for cardinalities of orbits, links with dimensions, and on their combinations. It shows how characteristics of infinite abelian groups are related to them with respect to finite ones. Some applications for non-standard abelian groups are discussed.

Keywords Rigidity, Abelian Group, Semantic Degree of Rigidity, Syntactic Degree of Rigidity, Index of Rigidity

Mathematics Subject Classification (2020) 03C50, 03C30, 20K21

1 Introduction

The class of abelian groups is rich enough [1, 2] and admits a good elementary classification by Szmielew invariants [3, 4, 5] reducing abelian groups to standard ones which are represented by direct sums of a given collection of standard groups. It is broadly investigated both semantically, with respect to structures of abelian groups [6, 7, 8] and their syntax [9, 10].

In the present paper we continue to study families of abelian groups and their theories, starting with possibilities of closures, ranks and approximations. We apply rigidity characteristics [11] for the class of standard abelian groups describing possibilities of both semantic and syntactic degrees of rigidity, and indexes of rigidity.

The paper is organized as follows. In Section 2, we collect preliminary notions, notations and results on degrees and indexes of rigidity, and degrees of algebraicity. In Section 3, we define Szmielew invariants, standard groups, and links of Szmielew invariants. In Section 4, we describe indexes and degrees of rigidity for finite abelian groups (Corollary 4.2 and Theorem 4.11). Theorem 4.11 asserts a dichotomy for tuples of degrees of rigidity. In Section 5, degrees and indexes of rigidity for standard infinite abelian groups are found (Theorems 5.4 and 5.11). In Section 6, the considered approach is illustrated for the group of integers and its variations. Described values are based on cardinalities of orbits, Szmielew invariants including dimensions of abelian groups, on Euler function, and their combinations.

2 Preliminaries

Throughout we use standard model-theoretic and group-theoretic notions and notations [5, 16].

Let L be a countable first-order language. Throughout we consider L-structures and their complete elementary theories.

Definition. [11]. For a set A in a structure $\mathcal{M}$, $\mathcal{M}$ is called *semantically A-rigid* or *automorphically A-rigid* if any A-automorphism $f \in \mathrm{Aut}(\mathcal{M})$ is identical. The structure $\mathcal{M}$ is called *syntactically A-rigid* if $M = \mathrm{dcl}(A)$.

Obviously, if $\mathcal{M}$ is an arbitrary structure, $\mathcal{M}$ is both semantically M-rigid and syntactically M-rigid. Also, $\mathcal{M}$ is syntactically A-rigid for any $A \subseteq M$ with $M \setminus \mathrm{dcl}(\emptyset) \subseteq A$. If $\mathcal{M}$ is an arbitrary infinite linearly ordered structure, $\mathcal{M}$ is semantically A-rigid for any co-finite $A \subseteq M$.

A structure $\mathcal{M}$ is called *$\forall$-semantically / $\forall$-syntactically n-rigid* (respectively, *$\exists$-semantically / $\exists$-syntactically n-rigid*), for $n \in \omega$, if $\mathcal{M}$ is semantically / syntactically A-rigid for any (some) $A \subseteq M$ with $|A| = n$.

The least n such that $\mathcal{M}$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the *$Q$-semantical / Q-syntactical degree of rigidity*, it is denoted by $\deg^{Q\text{-sem}}_{\mathrm{rig}}(\mathcal{M})$ and $\deg^{Q\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$, respectively. Here if a set A produces the value of Q-semantical / Q-syntactical degree then we say that A *witnesses* that degree. If such n does not exist we put $\deg^{Q\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) = \infty$ and $\deg^{Q\text{-synt}}_{\mathrm{rig}}(\mathcal{M}) = \infty$, respectively.

For a set A in $\mathcal{M}$ and an expansion $\mathcal{M}_A$ of $\mathcal{M}$ by constants in A, the least n such that $\mathcal{M}_A$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the *$(Q, A)$-semantical / (Q, A)-syntactical degree of rigidity*, it is denoted by $\deg^{Q\text{-sem}}_{\mathrm{rig},A}(\mathcal{M})$ and $\deg^{Q\text{-synt}}_{\mathrm{rig},A}(\mathcal{M})$, respectively. If such n does not exist we put $\deg^{Q\text{-sem}}_{\mathrm{rig},A}(\mathcal{M}) = \infty$ and $\deg^{Q\text{-synt}}_{\mathrm{rig},A}(\mathcal{M}) = \infty$, respectively.

Any expansion $\mathcal{M}_A$ of $\mathcal{M}$ with $\deg^{\exists\text{-}s}_{\mathrm{rig}}(\mathcal{M}_A) = 0$, for $s \in \{\mathrm{sem}, \mathrm{synt}\}$, is called a *$s$-rigiditization* or simply a *rigiditization* of $\mathcal{M}$.

Clearly, any structure $\mathcal{M}$ has a s-rigiditization $\mathcal{M}_A$, and it can be realized with finite A if and only if $\deg^{\exists\text{-}s}_{\mathrm{rig}}(\mathcal{M}) \in \omega$.

Following [11] for a structure $\mathcal{M}$ we denote by $\deg_4(\mathcal{M})$ the tetrad

$$\Big(\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M}), \deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathcal{M}),$$

$$\deg^{\forall\text{-sem}}_{\mathrm{rig}}(\mathcal{M}), \deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{M}) \Big).$$

Illustrating the tuple $\deg_4(\mathcal{M})$ we take the groups $\mathbf{Z}_2$ and $\mathbf{Z}_3$. Since $\mathbf{Z}_2$ does not have non-trivial automorphisms and its elements 0 and 1 are defined by the formulae $(x + x \approx x)$ and $\neg(x + x \approx x)$ we have $\deg_4(\mathbf{Z}_2) = (0, 0, 0, 0)$. For the group $\mathbf{Z}_3$ the element 0 is again defined by the formula $(x + x \approx x)$ whereas 1 and 2 are not $\emptyset$-definable and they are connected by an automorphism. So both the sets $\{1\}$ and $\{2\}$ witness the $\exists$-semantic and $\exists$-syntactic degrees, and the sets $\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$ witness the $\forall$-semantic and $\forall$-syntactic degrees, implying $\deg_4(\mathbf{Z}_3) = (1, 1, 2, 2)$.

Remark 2.1. By the definition the degrees $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M})$ and $\deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$ represent cardinalities of smallest, that is "best", sets A guaranteeing the semantic / syntactic rigidity. Here if $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M})$ or (and) $\deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$ equal(s) 0 then the best set $\emptyset$ witnesses the correspondent degree(s) $\deg^{\forall\text{-sem}}_{\mathrm{rig}}(\mathcal{M})$ or (and) $\deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$. At the same time positive natural values $\deg^{\forall\text{-sem}}_{\mathrm{rig}}(\mathcal{M})$ and $\deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$ define cardinalities of biggest, that is "worst", sets B such that these sets witness the semantic / syntactic rigidity and for some $a \in B$ the set $B \setminus \{a\}$ does not produce the correspondent rigidity.

Fact 2.2. [11]. *Let $\mathcal{M}$ be a structure. Then:*
1. $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) \le \deg^{\forall\text{-sem}}_{\mathrm{rig}}(\mathcal{M})$.
2. $\deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathcal{M}) \le \deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$.
3. $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) \le \deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$.
4. $\deg^{\forall\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) \le \deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$.
5. $\deg^{\forall\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) = 0$ *if* $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) = 0$.
6. $\deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{M}) = 0$ *if* $\deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathcal{M}) = 0$.

Remark 2.3. [12] Since finite structures $\mathcal{M}$ are homogeneous, in such a case we have $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) = \deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$ and $\deg^{\forall\text{-sem}}_{\mathrm{rig}}(\mathcal{M}) = \deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{M})$.

Definition. [11]. For a set A in a structure $\mathcal{M}$ the *index of rigidity* of $\mathcal{M}$ over A, denoted by $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}/A)$, is the supremum of cardinalities for the set of solutions of algebraic types $\mathrm{tp}(a/A)$ for $a \in M$. We put $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}) = \mathrm{ind}_{\mathrm{rig}}(\mathcal{M}/\emptyset)$. Here we assume that $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}) = 0$ if $\mathcal{M}$ does not have algebraic types $\mathrm{tp}(a)$ for $a \in M$.

Remark 2.4. [12] Since for any theory T of a finite structure $\mathcal{M}$ all types are algebraic, $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}) \in \omega \setminus \{0\}$.

Definition. [13, 14, 15] 1. A tuple $\bar{b}$ is *defined* by a formula $\varphi(\bar{x}, \bar{a})$ of T with parameters $\bar{a}$, if $\varphi(\bar{x}, \bar{a})$ has the unique solution $\bar{b}$.

A tuple $\bar{b}$ is *defined* by a type p if $\bar{b}$ is the unique tuple which realizes p. It is *definable* over a set A if $\mathrm{tp}(\bar{b}/A)$ defines it.

2. For a set A of a theory T the union of sets of solutions of formulae $\varphi(x, \bar{a})$, $\bar{a} \in A$, such that $\models \exists^{=n} x\, \varphi(x, \bar{a})$ for some $n \in \omega$ (respectively $\models \exists^{=1} x\, \varphi(x, \bar{a})$) is said to be an *algebraic* (*definable* or *definitional*) *closure* of A. The algebraic closure of A is denoted by $\mathrm{acl}(A)$ and its definable (definitional) closure, by $\mathrm{dcl}(A)$.

In such a case we say that the formulae $\varphi(x, \bar{a})$ *witness* that algebraic / definable (definitional) closure, and these formulae are called *algebraic / defining*.

Any element $b \in \mathrm{acl}(A)$ (respectively, $b \in \mathrm{dcl}(A)$) is called *algebraic* (*definable* or *definitional*) over A. If the set A is fixed or empty, we just say that b is *algebraic* (*definable*, or definitional).

3. If $\mathrm{dcl}(A) = \mathrm{acl}(A)$, $\mathrm{cl}_1(A)$ denotes their common value.

4. If $A = \mathrm{acl}(A)$ (respectively, $A = \mathrm{dcl}(A)$) then A is called *algebraically* (*definably*) closed.

5. A type p is *algebraic* (*defining*) if it is realized by finitely many tuples (unique one) only, i.e., it contains an algebraic (defining) formula φ. This formula φ can be chosen with the minimal number of solutions, and in such a case φ isolates p. The number of these solutions is called the *degree* $\deg(p)$ of p.

6. The complete algebraic types $p(x) \in S(A)$ are exactly ones of the form $\mathrm{tp}(a/A)$, where a is algebraic over A. The *degree* of a over A, $\deg(a/A)$ is the degree of $\mathrm{tp}(a/A)$.

Remark 2.5. [13]. The pairs $\langle M, \mathrm{acl}\rangle$ and $\langle M, \mathrm{dcl}\rangle$ satisfy the following properties:

(i) the reflexivity: it is witnessed by the formula $x \approx y$;

(ii) the transitivity: if the formulae $\varphi_1(x_1, \overline{a}), \ldots, \varphi_n(x_n, \overline{a})$ witnessed that $b_1, \ldots, b_n \in \mathrm{acl}(A)$ (respectively, $b_1, \ldots, b_n \in \mathrm{dcl}(A)$) and the formula $\psi(x, b_1, \ldots, b_n)$ witnesses that $c \in \mathrm{acl}(\{b_1, \ldots, b_n\})$ (respectively, $c \in \mathrm{dcl}(\{b_1, \ldots, b_n\})$) then the formula

$$\exists x_1, \ldots, x_n \left(\psi(x, x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n} \varphi_i(x_i, \overline{a}) \right)$$

witnesses that $c \in \mathrm{acl}(\mathrm{acl}(A))$ (respectively, $c \in \mathrm{dcl}(\mathrm{dcl}(A))$);

(iii) the finite character: if a formula $\varphi(x, \overline{a})$ witnesses that $a \in \mathrm{acl}(A)$ (respectively, $a \in \mathrm{dcl}(A)$) then $a \in \mathrm{acl}(A_0)$ for the finite $A_0 \subseteq A$ consisting of coordinates in $\overline{a}$.

Definition. [18] 1. For $n \in \omega \setminus \{0\}$ and a set A an element b is called *n-algebraic* over A, if $a \in \mathrm{acl}(A)$ and it is witnessed by a formula $\varphi(x, \overline{a})$, for $\overline{a} \in A$, with at most n solutions.

2. The set of all n-algebraic elements over A is denoted by $\mathrm{acl}_n(A)$.

3. If $A = \mathrm{acl}_n(A)$ then A is called *n-algebraically* closed.

4. A type p is *n-algebraic* if it is realized by at most n tuples only, i.e., $\deg(p) \leq n$.

5. The complete n-algebraic types $p(x) \in S(A)$ are exactly ones of the form $\mathrm{tp}(a/A)$, where a is n-algebraic over A, i.e., with $\deg(a/A) \leq n$. Here $\deg(a/A) = k \leq n$ defines the *n-degree* $\deg_n(a/A)$ of $\mathrm{tp}(a/A)$.

6. If $\mathrm{acl}(A) = \mathrm{acl}_n(A)$ then minimal such n is called the *degree of algebraization* over the set A and it is denoted by $\deg_{\mathrm{acl}}(A)$. If that n does not exist then we put $\deg_{\mathrm{acl}}(A) = \infty$. The supremum of values $\deg_{\mathrm{acl}}(A)$ with respect to all sets A of a given theory T is denoted by $\deg_{\mathrm{acl}}(T)$ and called the *degree of algebraization* of the theory T.

7. Following [17] theories T with $\deg_{\mathrm{acl}}(T) = 1$, i.e., with defined $\mathrm{cl}_1(A)$ for any set A of T, are called *quasi-Urbanik*, and the models $\mathcal{M}$ of T are *quasi-Urbanik*, too.

Proposition 2.6. *For any theory T and $n \in \omega$ the following conditions are equivalent:*

(1) $\deg_{\mathrm{acl}}(T) = n$*;*

(2) *for any model $\mathcal{M} \models T$ and $A \subseteq M$,* $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}/A) \leq n$ *and there is a (finite) set $B \subseteq M$ with* $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}/B) = n$.

Proof. (1) $\Rightarrow$ (2). Since $\deg_{\mathrm{acl}}(T) = n$ any algebraic A-definable set is reduced to subsets of cardinalities at most n and there is a set B such that some B-definable set has exactly n elements which can not be divided into smaller ones over B. Thus $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}/A) \leq n$ and $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}/B) = n$.

(2) $\Rightarrow$ (1). Since any algebraic type p with k realizations $a_1, \ldots, a_k$ is isolated by a formula φ with the set $\{a_1, \ldots, a_k\}$ of its solutions, the condition (2) implies that the operator acl is expressed by these formulae φ with $k \leq n$ and that n is minimally possible. Hence $\deg_{\mathrm{acl}}(T) = n$. $\square$

3 Abelian groups, their elementary theories and Szmielew invariants

Let $\mathcal{A}$ be an abelian group in the language $\Sigma = \langle +^{(2)}, -^{(1)}, 0^{(0)} \rangle$. Then $k\mathcal{A}$ denotes its subgroup $\{ka \mid a \in A\}$ and $\mathcal{A}[k]$ denotes the subgroup $\{a \in A \mid ka = 0\}$. Let P be the set of all prime numbers. If $p \in P$ and $pA = \{0\}$ then $\dim\mathcal{A}$ denotes the dimension of the group $\mathcal{A}$, considered as a vector space over a field with p elements. The following numbers, for arbitrary $p \in P$ and $n \in \omega \setminus \{0\}$ are called the *Szmielew invariants* for the group A [5, 3]:

$$\alpha_{p,n}(\mathcal{A}) = \min\{\dim((p^n\mathcal{A})[p]/(p^{n+1}\mathcal{A})[p]), \omega\},$$

$$\beta_p(\mathcal{A}) = \min\{\inf\{\dim((p^n\mathcal{A})[p] \mid n \in \omega\}, \omega\},$$

$$\gamma_p(\mathcal{A}) = \min\{\inf\{\dim((\mathcal{A}/\mathcal{A}[p^n])/p(\mathcal{A}/\mathcal{A}[p^n])) \mid n \in \omega\}, \omega\},$$

$$\varepsilon(\mathcal{A}) \in \{0, 1\},$$

and $\varepsilon(\mathcal{A}) = 0 \Leftrightarrow (n\mathcal{A} = \{0\}$ for some $n \in \omega, n \neq 0)$.

It is known [5, Theorem 8.4.10] that two abelian groups are elementary equivalent if and only if they have same Szmielew invariants. In addition, the following proposition holds.

Proposition 3.1. [5, Proposition 8.4.12] *Let for any p and n the cardinals $\alpha_{p,n}$, β_p, $\gamma_p \leq \omega$, and $\varepsilon \in \{0, 1\}$ be given. Then there is an abelian group $\mathcal{A}$ such that the Szmielew invariants $\alpha_{p,n}(\mathcal{A})$, $\beta_p(\mathcal{A})$, $\gamma_p(\mathcal{A})$, and $\varepsilon(\mathcal{A})$ are equal to $\alpha_{p,n}$, β_p, γ_p, and ε, respectively, if and only if the following conditions hold:*

(1) *if for prime p the set $\{n \mid \alpha_{p,n} \neq 0\}$ is infinite then $\beta_p = \gamma_p = \omega$;*

(2) *if $\varepsilon = 0$ then for any prime p, $\beta_p = \gamma_p = 0$ and the set $\{\langle p, n \rangle \mid \alpha_{p,n} \neq 0\}$ is finite.*

We denote by $\mathbf{Q}$ the additive group of rational numbers, $\mathbf{Z}_{p^n}$ — the cyclic group of the order p^n, $\mathbf{Z}_{p^\infty}$ — the quasi-cyclic group of all complex roots of 1 of degrees p^n for all $n \geq 1$, R_p — the group of irreducible fractions with denominators which are mutually prime with p. The groups $\mathbf{Q}$, $\mathbf{Z}_{p^n}$, R_p, $\mathbf{Z}_{p^\infty}$ are called *basic*. Below the notations of these groups will be identified with their universes.

Since abelian groups with same Szmielew invariants have same theories, any abelian group $\mathcal{A}$ is elementary equivalent to a group

$$\oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p\mathbf{Z}_{p^\infty}^{(\beta_p)} \oplus \oplus_p R_p^{(\gamma_p)} \oplus \mathbf{Q}^{(\varepsilon)}, \qquad (1)$$

where $\mathcal{B}^{(k)}$ denotes the direct sum of k subgroups isomorphic to a group $\mathcal{B}$. Thus, any theory of an abelian group has a model represented by a direct sum of based groups. The groups of form (1) are called *standard*.

Recall that any complete theory of an abelian group is based by the set of positive primitive formulas [5, Lemma 8.4.5], reduced to the set of the following formulas:

$$\exists y(m_1x_1 + \ldots + m_nx_n \approx p^k y), \qquad (2)$$

$$m_1x_1 + \ldots + m_nx_n \approx 0, \qquad (3)$$

where $m_i \in \mathbf{Z}$, $k \in \omega$, p is a prime number [4], [5, Lemma 8.4.7]. Formulas (2) and (3) witness that Szmielew invariants define theories of abelian groups modulo Proposition 3.1.

In view of Proposition 3.1 and equations (2) and (3) we have the following:

Remark 3.2. Theories of abelian groups are forced by sentences implied by formulas of form (2) and (3) and describing dimensions with respect to $\alpha_{p,n}$, β_p, γ_p, ε as well as bounds for orders p^k of elements and possibilities for divisions of elements by p^k. Moreover, various values of Szmielew invariants are separated by some sentences modulo Proposition 3.1.

4 Degrees and indexes of rigidity for finite abelian groups

Definition. [18] Let $\mathcal{M}$ be a structure, $A \subseteq M$. Recall that an A-*automorphism* of $\mathcal{M}$ is an automorphisms $f \in \mathrm{Aut}(\mathcal{M})$ fixing A pointwise. The set of all A-automorphisms for $\mathcal{M}$ is denoted by $\mathrm{Aut}(\mathcal{M}/A)$.

For an element $a \in M$, the *orbit* $O(a/A)$ with respect to the automorphism group $\mathrm{Aut}(\mathcal{M})$ is the set of all elements $b \in M$ connected with a by A-automorphisms $f \in \mathrm{Aut}(\mathcal{M}/A)$: $f(a) = b$ and $f(a') = a'$ for any $a' \in A$.

We write $O(a)$ instead of $O(a/\emptyset)$.

We denote by $o(\mathcal{M})$ the maximal cardinality of orbits $O(a)$, i.e., the value of $\deg_{\mathrm{acl}}(\emptyset)$.

Let $T = \mathrm{Th}(\mathcal{M})$ for a finite structure $\mathcal{M}$. Since all models of T are pairwise isomorphic, the value $o(\mathcal{M})$ does not depend on the choice of model $\mathcal{M} \models T$ and it is denoted by $o(T)$.

Proposition 4.1. [18] *If T is a theory of a finite structure then* $\deg_{\mathrm{acl}}(T) = \deg_{\mathrm{acl}}(\emptyset) = o(T)$.

Propositions 2.6 and 4.1 immediately imply:

Corollary 4.2. *For any theory T of a finite structure $\mathcal{M}$ and a set $A \subseteq M$,* $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}) = o(T)$ *and* $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}/A) \leq o(T)$, *with* $\mathrm{ind}_{\mathrm{rig}}(\mathcal{M}/M) = 1$.

Recall that any finite abelian group $\mathcal{S}$ is represented as a direct sum $\oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$ [16, Theorem 8.1.2]. Recall also [19] that *Euler function* $\varphi(n)$ is defined as follows: $\varphi(n) = |\{m \in \mathbf{Z}_n \mid (m,n) = 1\}|$.

Theorem 4.3. [18] *For any finite abelian group* $\mathcal{A} = \oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$,

$$\deg_{\mathrm{acl}}(\mathrm{Th}(\mathcal{A})) = \prod_{p,n}(p^{n\alpha_{p,n}} - (p^n - \varphi(p^n))^{\alpha_{p,n}}).$$

Corollary 4.4. [18] *A finite abelian group $\mathcal{A}$ is quasi-Urbanik if $\mathcal{A}$ is either a singleton or isomorphic to* $\mathbf{Z}_2$.

Clearly, one-element and two-element groups $\mathcal{G}$ are rigid, with $\deg_4(\mathcal{G}) = (0,0,0,0)$. Besides, abelian groups with at least three elements have non-trivial automorphisms. Thus, Corollary 4.4 has the following extension:

Corollary 4.5. *For any finite abelian group $\mathcal{A}$ the following conditions are equivalent:*

(1) $\mathcal{A}$ is quasi-Urbanik;

(2) $\deg_4(\mathcal{A}) = (0,0,0,0)$;

(3) $|\mathcal{A}| \leq 2$.

In view of Corollary 4.2 and Theorem 4.3 we obtain:

Corollary 4.6. *For any finite abelian group* $\mathcal{A} = \oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$,

$$\mathrm{ind}_{\mathrm{rig}}(\mathcal{A}) = \prod_{p,n}(p^{n\alpha_{p,n}} - (p^n - \varphi(p^n))^{\alpha_{p,n}}).$$

The following Fact uses Fact 2.3 and actually it is noticed in [11].

Fact 4.7. *For any finitely generated algebra $\mathcal{M}$ with n generating elements,* $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{A}) \leq \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{A}) \leq n$.

Following [16], for a finite abelian group $\mathcal{A}$ we denote by $\mathrm{rk}(\mathcal{A})$ its *rank*, i.e. the minimal number of its generating elements. Clearly, if $\mathcal{A} = \oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$ then $\mathrm{rk}(\mathcal{A}) = \sum\limits_{p,n}\alpha_{p,n}$.

Proposition 4.8. *For any finite abelian group* $\mathcal{A} = \oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$, $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{A}) = \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{A}) = \delta$, *with* $\delta = \mathrm{rk}(\mathcal{A}) - 1$ *if* $\alpha_{2,1} = 1$, *and with* $\delta = \mathrm{rk}(\mathcal{A})$ *if* $\alpha_{2,1} \neq 1$.

Proof. We have $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{A}) = \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{A})$ in view of Remark 2.4.

The case $\delta = 0$, with $\mathrm{rk}(\mathcal{A}) = 1$, is implied by Corollary 4.5. If $|\mathcal{A}| > 2$ and $\alpha_{2,1} = 1$ then $\mathcal{A}$ is neither semantically rigid nor syntactically rigid with $\mathrm{dcl}(\emptyset) = \mathbf{Z}_2$. It means that the generator for $\mathbf{Z}_2$ is not used for $\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathcal{A}) = \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathcal{A})$ whereas generators for other $\mathbf{Z}_{p^n}$ in $\mathcal{A}$ are taken. Since these generators are independent we obtain $\delta = \mathrm{rk}(\mathcal{A}) - 1$. If $\alpha_{2,1} \neq 1$ and then a basis of generators witnesses the value δ implying $\delta = \mathrm{rk}(\mathcal{A})$. □

Lemma 4.9. *For any natural $n \geq 1$, prime p with $(p,n) \neq (2,1)$ and natural $\alpha_{p,n} > 0$,*

$$\deg_{\mathrm{rig}}^{\forall\text{-sem}}\left(\mathbf{Z}_{p^n}^{(\alpha_{p,n})}\right) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}\left(\mathbf{Z}_{p^n}^{(\alpha_{p,n})}\right) =$$

$$= p^{n(\alpha_{p,n}-1)}p^{n-1} + 1. \tag{4}$$

Proof. We have

$$\deg_{\mathrm{rig}}^{\forall\text{-sem}}\left(\mathbf{Z}_{p^n}^{(\alpha_{p,n})}\right) = \deg_{\mathrm{rig}}^{\forall\text{-synt}}\left(\mathbf{Z}_{p^n}^{(\alpha_{p,n})}\right)$$

in view of Remark 2.4. We denote this value by δ.

Taking the group $\mathbf{Z}_{p^n}$ we observe that it has a maximal proper subgroup which is isomorphic to $\mathbf{Z}_{p^{n-1}}$. Elements of a coset $a + \mathbf{Z}_{p^{n-1}}$, for $a \in \mathbf{Z}_{p^n} \setminus \mathbf{Z}_{p^{n-1}}$, are connected by $\mathbf{Z}_{p^{n-1}}$-automorphisms such that f fixing the element a the only possibility for f is to be identical. Thus $\delta = |\mathbf{Z}_{p^{n-1}}| + 1$ that corresponds to the worth choice of a subset in $\mathbf{Z}_{p^n}$ producing the rigidity following Remark 2.1.

In general case, for $\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$, the worth choice means that we take all elements of the subgroup generated by $\mathbf{Z}_{p^n}^{(\alpha_{p,n}-1)}$, for first $\alpha_{p,n} - 1$ copies $\mathbf{Z}_{p^n}$, and by $\mathbf{Z}_{p^{n-1}}$ in the last copy of $\mathbf{Z}_{p^n}$. The subgroup $\mathbf{Z}_{p^n}^{(\alpha_{p,n}-1)} \oplus \mathbf{Z}_{p^{n-1}}$ has $p^{n(\alpha_{p,n}-1)}p^{n-1}$ elements, and adding an arbitrary element in $\mathbf{Z}_{p^n}^{(\alpha_{p,n})} \setminus (\mathbf{Z}_{p^n}^{(\alpha_{p,n}-1)} \oplus \mathbf{Z}_{p^n})$ we obtain the required equality (4) for $\forall$-degrees of rigidity. □

Proposition 4.10. *For any finite abelian group* $\mathcal{A} = \oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})}$ *with some* $\alpha_{p,n} > 0$ *and* $\mathcal{A} \not\simeq \mathbf{Z}_2$, $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{A}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) = \zeta$, *where*

$$\zeta = q^{n_0(\alpha_{q,n_0}-1)}q^{n_0-1}\times$$

$$\times \prod_{\alpha_{p,n}>0, p\neq q, \text{ or } p=q \text{ and } n_0>n} p^{n\alpha_{p,n}}, \qquad (5)$$

if $\alpha_{2,1} = 1$, *and*

$$\zeta = q^{n_0(\alpha_{q,n_0}-1)}q^{n_0-1}\times$$

$$\times \prod_{\alpha_{p,n}>0, p\neq q, \text{ or } p=q \text{ and } n_0>n} p^{n\alpha_{p,n}} + 1, \qquad (6)$$

if $\alpha_{2,1} \neq 1$, *q is the maximal prime number with* $\alpha_{q,n} > 0$ *and* n_0 *is maximal one among n with* $\alpha_{q,n} > 0$.

Proof. Similarly Lemma 4.9 we have $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{A}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A})$. Now we find a maximal proper subgroup $\mathcal{A}_0$ of $\mathcal{A}$. It corresponds to the choice of maximal prime number q with $\alpha_{q,n} > 0$ and maximal n_0 among n with $\alpha_{q,n} > 0$ and produces

$$\mathcal{A}_0 = \mathbf{Z}_{q^{n_0}}^{(\alpha_{q,n_0}-1)} \oplus \mathbf{Z}_{q^{n_0-1}}\oplus$$

$$\oplus \oplus_{\alpha_{pn}>0, p\neq q, \text{ or } p=q \text{ and } n_0>n} \mathbf{Z}_{p^n}^{(\alpha_{p,n})}.$$

Lemma 4.9 is responsible for $q^{n(\alpha_{q,n_0}-1)}q^{n_0-1}$ elements of the subgroup $\mathbf{Z}_{q^{n_0}}^{(\alpha_{q,n_0}-1)} \oplus \mathbf{Z}_{q^{n_0-1}}$. Now

$$|\mathcal{A}| = q^{n(\alpha_{q,n_0}-1)}q^{n_0-1}\times$$

$$\times \prod_{\alpha_{pn}>0, p\neq q, \text{ or } p=q \text{ and } n_0>n} p^{n\alpha_{p,n}}.$$

Now we fix all elements in $\mathcal{A}_0$. If $\alpha_{2,1} = 1$ then, since $\mathrm{dcl}(\emptyset) = \mathbf{Z}_2$, the set A_0 witnesses the value $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{A}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A})$ by Remark 2.1 and we obtain the equality (5).

If $\alpha_{2,1} \neq 1$ then fixing all elements in $\mathcal{A}_0$ and an element in $A \setminus A_0$ we obtain a set B witnessing the value $\deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{A}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A})$, by Remark 2.1, and satisfying

$$|B| = q^{n(\alpha_{q,n_0}-1)}q^{n_0-1}\times$$

$$\times \prod_{\alpha_{pn}>0, p\neq q, \text{ or } p=q \text{ and } n_0>n} p^{n\alpha_{p,n}} + 1.$$

Thus we have the equality (6). □

For a finite abelian group $\mathcal{A}$ we denote by $\delta(\mathcal{A})$ the value δ in Proposition 4.8, and by $\zeta(\mathcal{A})$ the value for $\forall$-degrees in Proposition 4.10. Summarizing these propositions we conclude:

Theorem 4.11. *For any finite abelian group* $\mathcal{A}$ *either*

$$\deg_4(\mathcal{A}) = (0,0,0,0)$$

if $|A| \leq 2$, *or*

$$\deg_4(\mathcal{A}) = (\delta(\mathcal{A}), \delta(\mathcal{A}), \zeta(\mathcal{A}), \zeta(\mathcal{A})),$$

otherwise.

Remark 4.12. Following Propositions 4.8 and 4.10, the only possibility for a finite abelian group $\mathcal{A}$ with $|A| > 2$ to have $\delta(\mathcal{A}) = \zeta(\mathcal{A})$ is not to have nontrivial proper subgroups, i.e. to have the form $\mathbf{Z}_p$ for some prime $p > 2$. In such a case $\deg_4(\mathcal{A}) = (1,1,1,1)$. Thus, in view of Corollary 4.5, for a finite abelian group $\mathcal{A}$, $\deg_4(\mathcal{A})$ is identical if either $\mathcal{A}$ is a singleton or $\mathcal{A} = \mathbf{Z}_p$ for some prime p.

5 Degrees and indexes of rigidity for standard infinite abelian groups

In this section we spread results of Section 4 for standard infinite abelian groups.

At first we consider some general assertions for degrees of rigidity for infinite abelian groups.

The arguments for Proposition 4.8 imply the following:

Proposition 5.1. *If* $\mathcal{A}$ *is an abelian group of finite rank* $r = \mathrm{rk}(\mathcal{A}) > 0$ *then* $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{A}) = r - 1$ *if* $\alpha_{2,1}(\mathcal{A}) = 1$, *and* $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{A}) = r$ *if* $\alpha_{2,1}(\mathcal{A}) \neq 1$.

Proposition 5.2. *If* $\mathcal{A}$ *is an abelian group of infinite rank* $\mathrm{rk}(\mathcal{A})$ *then*

$$\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{A}) = \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) = \infty.$$

Proof. In view of Fact 2.2 it suffices to show that $\deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{A}) = \infty$. It is implied by $\mathrm{rk}(\mathcal{A}) \geq \omega$ since definable closures of finite subsets of A do not cover bases of A which are infinite and independent over their finite parts. □

Proposition 5.3. *If* $\mathcal{A}$ *is an infinite abelian group of finite rank* $r = \mathrm{rk}(\mathcal{A})$ *and* $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ *then one of the following conditions holds:*

1) $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) = 2$, *if* $r = 1$, *with some singleton* $\mathcal{A}_i$;

2) $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) = 3$, *if* $r = 2$ *and some* $\mathcal{A}_i$ *contains a copy of* $\mathbf{Z}_2$ *with* $\alpha_{2,1}(\mathcal{A}) = 1$ *such that either* $|\mathcal{A}_{3-i}| = 1$ *or* $|\mathcal{A}_i| = 2$;

3) $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) = \infty$, *if* $r = 2$ *and* $\alpha_{2,1}(\mathcal{A}) \neq 1$, *or* $r \geq 3$.

Proof. We have $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) \geq 1$ since singletons and $\mathbf{Z}_2$ are unique abelian groups consisting of $\emptyset$-definable elements. Thus the smallest sets witnessing the value $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A})$ consist of at least two elements, i.e. $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) \geq 2$.

Now we consider the following cases.

Case 1: $\mathrm{rk}(\mathcal{A}) = 1$. It implies that each nonzero element a generates all elements of $\mathcal{A}$ via its linear combinations producing some component $\mathcal{A}_j$ and the correspondent singleton $\mathcal{A}_{3-j}$. Thus the set $\{0, a\}$ witnesses the value $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) = 2$.

Case 2: $\mathrm{rk}(\mathcal{A}) = 2$. It implies that two nonzero independent elements a_1, a_2 generate all elements of $\mathcal{A}$ via their linear combinations.

If some a_i belongs to $\mathrm{dcl}(\emptyset)$ then $\{0, a_i\}$ is the universe of a copy of $\mathbf{Z}_2$ and $\mathcal{A}$ is generated by a_{3-i} modulo this copy. Now either some $\mathcal{A}_j$ is that copy of $\mathbf{Z}_2$ of $\mathcal{A}_j$ is a singleton, and the set $\{0, a_1, a_2\}$ witnesses the value $\deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{A}) = 3$.

If a_1, a_2 do not belong to $\mathrm{dcl}(\emptyset)$, i.e. $\mathrm{dcl}(\emptyset) = \{0\}$ then each a_i does not belong to the subgroup of $\mathcal{A}$ generated by a_{3-i} by linear combinations and its definable closure. Since some of this subgroup is infinite we obtain finite sets of unbounded cardinalities whose definable closures do not cover $\mathcal{A}$ implying $\deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{A}) = \infty$.

Case 3: $\mathrm{rk}(\mathcal{A}) \geq 3$. In view of the arguments for the previous case we find an element a generating a subgroup of $\mathcal{A}$ witnessing $\deg^{\forall\text{-synt}}_{\mathrm{rig}}(\mathcal{A}) = \infty$. □

Since standard abelian groups are homogeneous, and Q-semantic and Q-syntactic degrees of rigidity coincide for $Q \in \{\exists, \forall\}$, Propositions 5.1, 5.2, 5.3 imply the following:

Theorem 5.4. *For any standard infinite abelian group $\mathcal{A}$ one of the following conditions holds:*

1) $\deg_4(\mathcal{A}) = (1, 1, 2, 2)$, *if* $\mathrm{rk}(\mathcal{A}) = 1$*;*

2) $\deg_4(\mathcal{A}) = (1, 1, 3, 3)$, *if* $\mathrm{rk}(\mathcal{A}) = 2$ *and* $\alpha_{2,1}(\mathcal{A}) = 1$*;*

3) $\deg_4(\mathcal{A}) = (r - 1, r - 1, \infty, \infty)$, *if* $\mathrm{rk}(\mathcal{A}) = r > 2$ *is finite and* $\alpha_{2,1}(\mathcal{A}) = 1$*;*

4) $\deg_4(\mathcal{A}) = (r, r, \infty, \infty)$, *if* $\mathrm{rk}(\mathcal{A}) = r > 2$ *is finite and* $\alpha_{2,1}(\mathcal{A}) \neq 1$*;*

5) $\deg_4(\mathcal{A}) = (\infty, \infty, \infty, \infty)$, *if* $\mathrm{rk}(\mathcal{A})$ *is infinite.*

Below in this section we describe indexes for standard infinite abelian groups $\mathcal{A}$.

Lemma 5.5. *If $\mathcal{A}$ is torsion-free then* $\mathrm{ind}_{rig}(\mathcal{A}) = 1$.

Proof. Since $\mathcal{A}$ is torsion-free, i.e. all $\alpha_{p,n}$ and β_p equal to 0, the only possibility for finite orbits in $\mathcal{A}$ is the singleton $\{0\}$ implying $\mathrm{ind}_{rig}(\mathcal{A}) = 1$. □

Lemma 5.6. *If some $\beta_p(\mathcal{A}) > 0$ is finite then* $\mathrm{ind}_{rig}(\mathcal{A}) = \omega$.

Proof. Quasi-finite groups generate unbounded cardinalities of finite orbits. Since $\beta_p(\mathcal{A} \in \omega$, $\mathcal{A}$ contains finitely many subgroups $\mathbf{Z}_{p^\infty}$ producing these unbounded cardinalities and implying $\mathrm{ind}_{rig}(\mathcal{A}) = \omega$. □

Lemma 5.7. *If all positive $\beta_p(\mathcal{A})$ are infinite and $\mathcal{A}$ has the trivial, one-element cyclic part then* $\mathrm{ind}_{rig}(\mathcal{A}) = 1$.

Proof. Since positive $\beta_p(\mathcal{A})$ are infinite and $\mathcal{A}$ does not contain subgroups $\mathbf{Z}_{p^n}$ or contains infinitely many ones, $\mathcal{A}$ has unique finite orbit and it is equal to $\{0\}$. Thus $\mathrm{ind}_{rig}(\mathcal{A}) = 1$. □

Lemma 5.8. *If all positive $\beta_p(\mathcal{A})$ are infinite and $\mathcal{A}$ has finite cyclic part $\oplus_{p,n}\mathbf{Z}^{(\alpha_{p,n})}_{p^n}$ then*

$$\mathrm{ind}_{\mathrm{rig}}(\mathcal{A}) = \prod_{p,n}(p^{n\alpha_{p,n}} - (p^n - \varphi(p^n))^{\alpha_{p,n}}).$$

Proof follows applying Lemma 5.7 and Corollary 4.6 which give the largest cardinality of finite orbits. □

Notice that infinite values $\alpha_{p,n}$ do not influence algebraic types. Thus strengthening Lemma 5.8 we obtain:

Lemma 5.9. *If all positive $\beta_p(\mathcal{A})$ are infinite and $\mathcal{A}$ finitely many positive natural $\alpha_{p,n}$ then*

$$\mathrm{ind}_{\mathrm{rig}}(\mathcal{A}) = \prod_{\alpha_{p,n}\in\omega}(p^{n\alpha_{p,n}} - (p^n - \varphi(p^n))^{\alpha_{p,n}}). \quad (7)$$

Corollary 5.10. $\mathrm{ind}_{\mathrm{rig}}(\mathcal{A})$ *is finite if $\mathcal{A}$ has finitely many positive natural $\alpha_{p,n}$ and does not have positive natural β_p.*

Proof. If $\mathcal{A}$ has finitely many positive natural $\alpha_{p,n}$ and does not have positive natural β_p then $\mathrm{ind}_{\mathrm{rig}}(\mathcal{A}) \in \omega$ by Lemma 5.9. Conversely, if $\mathcal{A}$ has infinitely many positive natural $\alpha_{p,n}$ or has positive $\beta_p \in \omega$ then in each case $\mathcal{A}$ has unbounded finite cardinalities of finite orbits as for Lemma 5.6 implying $\mathrm{ind}_{\mathrm{rig}}(\mathcal{A}) = \omega$. □

Summarizing Assertions 5.5–5.10 we get the following:

Theorem 5.11. *For any standard infinite abelian group $\mathcal{A}$ either $\mathrm{ind}_{\mathrm{rig}}(\mathcal{A})$ is finite and satisfies the formula* (7)*, if all positive $\beta_p(\mathcal{A})$ are infinite and $\mathcal{A}$ has finitely many positive natural $\alpha_{p,n}$, or $\mathrm{ind}_{\mathrm{rig}}(\mathcal{A}) = \omega$, otherwise.*

6 Illustrations

By the definition the group $\mathbf{Z}$ is non-standard with $\gamma_p = 1$ for any prime p [20]. This group has a poor automorphism group $\mathrm{Aut}(\mathbf{Z})$ which is isomorphic to $\mathbf{Z}_2$ [16]. At the same time $\mathrm{dcl}(\emptyset) = \{0\}$ and $\mathrm{rk}(\mathbf{Z}) = 1$. Thus by Propositions 5.1. 5.3 and the homogeneity of $\mathbf{Z}$, $\deg_4(\mathbf{Z}) = (1, 1, 2, 2)$. In particular, $\deg_4(\mathbf{Z}) = \deg_4(\mathbf{Z}_3)$.

Now for the group $\mathbf{Z}^{(\alpha)}$, $\alpha > 1$, we have $\mathrm{rk}(\mathbf{Z}^{(\alpha)}) = \alpha$. Again applying Propositions 5.1. 5.3 and the homogeneity of the group, we obtain $\deg_4(\mathbf{Z}^{(\alpha)}) = (\alpha, \alpha, \infty, \infty)$, if α is finite, and $\deg_4(\mathbf{Z}^{(\alpha)}) = (\infty, \infty, \infty, \infty)$, if α is infinite.

7 Discussion

The presented approach allows to classify abelian groups and their theories with respect to rigidity characteristics, semantic and syntactic degrees as well as indexes of rigidity. It shows how rich can be automorphism groups and definable closures of given abelian groups. Obtained results give a possibility to classify similar structures having various dimensions and orbits.

8 Conclusions

We described possibilities for degrees and indexes of rigidity both for finite and infinite abelian groups in terms of Szmielew invariants and the Euler function. It is clarified that these characteristics depend on cardinalities of finite orbits. It is illustrated for the group of integers and its variations. It would be natural to describe degrees and indexes of rigidity for other non-standard abelian groups as well as for ordered abelian groups.

Acknowledgements

This research was carried out in the framework of the State Contract of Sobolev Institute of Mathematics, Project No. FWNF-2022-0012, and partially supported by Science

Committee of Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19677451).

REFERENCES

[1] Fuchs L., “Infinite Abelian groups. Volume I,” Academic Press, New York, London, 1970.

[2] Fuchs L., “Infinite Abelian groups. Volume II,” Academic Press, New York, London, 1973.

[3] Szmielew W., “Elementary properties of Abelian groups,” Fundamenta Mathematicae, vol. 41, pp. 203–271, 1955. URL: https://eudml.org/doc/213354

[4] Eklof P. C., Fischer E. R., “The elementary theory of Abelian groups,” Annals of Mathematical Logic, vol. 4, pp. 115–171, 1972. DOI: 10.1016/0003-4843(72)90013-7

[5] Ershov Yu. L., Palyutin E. A., “Mathematical logic,” Fizmatlit, Moscow, 2011. [in Russian]

[6] Chekhlov A. R., Danchev P. V., “On the socles of strongly inert subgroups of abelian p-groups,” Siberian Mathematical Journal, vol. 64, no. 2, pp. 459468, 2023. DOI: 10.1134/S0037446623020179

[7] Chekhlov A. R., Danchev P. V., “On abelian groups having isomorphic proper characteristic subgroups”, Journal of Commutative Algebra, vol. 15, no. 4, pp. 481–496, 2023. DOI: 10.1216/jca.2023.15.481

[8] Pushkova T. A., Sebel’din A. M. “On the question of the definability of certain classes of completely decomposable aelian torsion-free groups by their homomorphism groups,” Mathematical Notes, vol. 113, no. 5, pp. 700-703, 2023. DOI: 10.1134/S0001434623050097

[9] Gurevich Y., Schmitt P. H., “The theory of ordered abelian groups does not have the independence property,” Transactions of the American Mathematical Society, vol. 284, pp. 171–182, 1984. DOI: 10.2307/1999281

[10] Berger S., Block, A. C., Löwe B., “The modal logic of abelian groups,” Algebra Universalis, vol. 84, no. 25, 12 p., 2023. DOI: 10.1007/s00012-023-00821-9

[11] Sudoplatov S. V., “Variations of rigidity,” Bulletin of Irkutsk State University. Series Mathematics, vol. 47, pp. 119-136, 2024. DOI: 10.26516/1997-7670.2024.47.119

[12] Sudoplatov S. V., “Conditional characteristics of rigidity”, in Algebra and model theory 14. Collection of papers, NSTU, Novosibirsk, 2023, pp. 143–150. URL: https://erlagol.ru/tom-14-2023/?lang=en

[13] Shelah S., “Classification theory and the number of non-isomorphic models,” North-Holland, Amsterdam, 1990.

[14] Hodges W., “Model theory,” Cambridge University Press, Cambridge, 1993.

[15] Tent K., Ziegler M., “A Course in Model Theory (Lecture Notes in Logic. No. 40),” Cambridge University Press, Cambridge, 2012.

[16] Kargapolov M. I., Merzljakov J. I., “Fundamentals of the Theory of Groups,” Springer, New York, 2011.

[17] Zil’ber B. I., “Hereditarily transitive groups and quasi-Urbanik structures,” American Mathematical Society Translations: Series 2, vol. 195, pp. 165–186, 1999. DOI: 10.1090/trans2/195

[18] Pavlyuk In. I., Sudoplatov S. V., “On algebraic and definable closures for finite structures”, in Algebra and model theory 14. Collection of papers, NSTU, Novosibirsk, 2023, pp. 87–94. URL: https://erlagol.ru/tom-14-2023/?lang=en

[19] Vinogradov I. M., “Elements of Number Theory,” Dover Publications, Inc., Mineola, New York, 1954.

[20] Popkov R. A., “Distribution of countable models for the theory of the group of integers,” Siberian Mathematical Journal, vol. 56, no. 1, pp. 185–191, 2015. DOI: 10.1134/S0037446615010152