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Convergence of Spectral-Grid Method for Burgers Equation with Initial-Boundary Conditions

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Abstract In this study, initial-boundary value problem for the Burgers equation is solved using the theoretical substantiation of the spectral-grid method. Using the theory of Green's functions, an operator equation of the second kind is obtained with the corresponding initial-boundary conditions for a continuous problem. To approximately solve the differential problem, the spectral grid method is used, i.e. a grid is introduced on the integration interval, and approximate solutions of the differential problem on each of the grid elements are presented as a finite series in Chebyshev polynomials of the first kind. At the internal nodes of the grid, the requirement for the continuity of the approximate solution and its first derivative is imposed. The corresponding boundary conditions are satisfied at the boundary nodes. A discrete analogue of the operator equation of the second kind is obtained using the spectral-grid method. The convergence theorems for the spectral-grid method are proven and estimates for the method's convergence rate are obtained. To discretize the Burgers equation in time on the interval $[0,T]$, a grid with a uniform step of τ is introduced, i.e. $\varpi_\tau = \{t_m = m\tau, m = 0,1,..., K, \tau = T1/K\}$, where $T1$ - given number. Numerical calculations have been carried out at sufficiently low values of viscosity, which cannot be obtained by other numerical methods. The high accuracy and efficiency of the spectral-grid method used in solving the initial-boundary value problem for the Burgers equation is shown.

Keywords Burgers Operator, Initial and Boundary Conditions, Spectral-Grid Method, Chebyshev Polynomials of the First Kind, Green's Function, Estimate of the Rate of Convergence

1. Introduction

Numerical techniques are widely used to compute the nonlinear wave processes described by the Burgers equation. At the same time, their application to the solution of the Burgers equation faces serious difficulties. They are mainly associated with the existence of a small parameter at the highest derivative and the nonlinearity of the equation under consideration. As a result of these effects, it appears in the solution of areas of strong spatial inhomogeneity. Then the requirements imposed on the approximation properties of numerical methods increase sharply. The Burgers equation is of great importance in the mathematical modeling of the following practical problems: wave propagation in various media, soliton theory, nonlinear acoustics, nonlinear optics, plasma physics, radio physics, electronics.

In [1], the numerical solution of the Burgers equation is solved by hybrid numerical scheme based on the implicit Euler method, quasi-linearization, and homogeneous Haar wavelets. This article states that most of the numerical methods available in the scientific literature cannot reflect the physical behavior of the equations when the viscosity v goes to zero. The main goal of the article was to develop a numerical scheme to overcome the shortcomings of existing schemes. It has been established that the use of the homogeneous Haar wavelet is accurate, simple, fast, flexible, convenient and requires little computational effort.

Novel and efficient approaches to tackling the one-dimensional quasilinear Burgers equation were introduced in various studies. In [2], a method employing the nonlinear Cole-Hopf transformation to reduce the equation to a one-dimensional diffusion equation was presented. This approach involved semi-discretizing the linearized diffusion equation via the method of lines, leading to a system of ordinary differential equations in time. Solving these equations involved using the method of inverse differentiation of different orders, with an accompanying analysis of numerical errors, primarily performed at moderate kinematic viscosity values.

Additionally, a gridless technique for solving the Burgers equation in the "intrinsic" Hilbert space of the reproducing kernel was investigated in [3]. This method involved constructing derivatives' discretizations based on interpolants of Newton's basis functions, featuring a variable scale for localized sets of nodes. Its primary advantage lay in generating numerous small matrices in overlapping areas of influence rather than a large collocation matrix, resulting in a sparse matrix. The method demonstrated efficiency, accuracy, and stability, particularly in flows with high Reynolds numbers.

Further advancements included the presentation of fully implicit numerical schemes for solving both one-dimensional and two-dimensional nonstationary Burgers equations in [4]. Here, the equations were spatially discretized using a second-order finite difference method, transforming them into nonlinear systems of ordinary differential equations. Employing second-order inverse differentiation formulas enabled time advancement. Comparative analysis against exact solutions and other schemes showcased the simplicity, efficiency, and accuracy of the proposed schemes, even in scenarios involving large Reynolds numbers.

Another study in [5] described into a weak L-stable scheme for integrating the Burgers equation. This involved utilizing explicit inverse Taylor polynomial approximations and interpolating Hermite polynomial approximations to derive a circuit and formulate a vector-based formula to solve the Burger equation. Discussions revolved around the stability and convergence of this scheme.

Moreover, in [6], new finite-difference schemes for the shallow water model, formulated as a viscous Burgers-Poisson system with periodic boundary conditions, were described. These schemes, belonging to the family of three-level linearized finite difference methods, proved effective in numerical simulations for both viscous and inviscid problems.

The numerical simulation of nonlinear waves in a dissipative medium without dispersion, elucidating the challenges faced in numerically solving the Burgers equation at low viscosity values was studied in [7]. To address these issues, the spectral-grid method was proposed and studied for solving the initial-boundary value problem for the Burgers equation, demonstrating high accuracy and efficiency.

In this work, for the construction of the spectral-grid method, the Burgers equation was written in operator form with homogeneous boundary conditions and inhomogeneous initial conditions. In the present study, a continuous solution is constructed using the Green's function and approximate solutions are obtained using the spectral-grid method (SGM). This method consists of a preliminary approximation of the differential problem and subsequent exact solution of the approximate equation. In spectral-grid method, the integration interval is divided into grids and the grids may be uniform or non-uniform. The form of a linear combination of a different number of Chebyshev polynomials of the first kind is used to approximate the solutions of the problem on each of the grid elements. It is imposed that the continuity of the approximate solution and its first derivative at the inner nodes of the grid and the corresponding boundary conditions are satisfied. Both the differential problem and the approximate problem for the Burgers equations are reduced to an operator equation of the second kind. Convergence study is executed and estimates for the rate of convergence of the SGM are obtained. It follows from the estimation, the convergence of the SGM is ensured both by increasing the number of basic functions and by decreasing the grid spacing.

2. Materials and Methods

The Burgers equation is written in operator form

$$L_0 u = \frac{\partial u}{\partial t} - \frac{1}{\mu}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0, \quad -1 < x < 1, \quad t > 0 \qquad (1)$$

and is considered with homogeneous boundary conditions

$$u(-1,t) = u(1,t) = 0 \qquad (2)$$

as well as with the following initial condition

$$u(x,0) = \varphi(x) \qquad (3)$$

where μ is the viscosity.

The Green's function for the differential problem (1)-(3) has the form [8]:

$$G(\xi, x, t) = \frac{(x-\xi)^2}{2t} + \int_0^{\xi} \varphi(y)dy. \qquad (4)$$

Function is introduced as

$$f(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (5a)$$

where $u(x,t)$ is the solution to problems (1)-(3) and

$$u(x,t)=\int_{-1}^{1}G(\xi,x,t)\,f(\xi,t)d\xi \tag{5b}$$

From equation (1) we have the following formula

$$\frac{\partial^2 u}{\partial x^2}-\mu\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=0 \tag{1a}$$

Now substituting (5), (5b) into equation (1a) we have:

$$f(x,t)+\int_{-1}^{1}\left[-\mu\left(\frac{\partial G(\xi,x,t)}{\partial t}+\right.\right.$$
$$\left.\left.+G(\xi,x,t)\frac{\partial G(\xi,x,t)}{\partial x}\right)\right]f(\xi,t)d\xi=0. \tag{6}$$

Introducing the notation

$$\overline{T}(\xi,x,t)=-\mu\left(\frac{\partial G(\xi,x,t)}{\partial t}+\right.$$
$$\left.+G(\xi,x,t)\frac{\partial G(\xi,x,t)}{\partial x}\right) \tag{7}$$

The equation (6) can be written using Eqn. (7) as

$$f(x,t)+\int_{-1}^{1}\overline{T}(\xi,x,t)f(\xi,t)d\xi=0. \tag{8}$$

It follows from the properties of the Green's function that the function $\overline{T}(\xi,x,t)$ is a continuous function on

$$[-1,1]\times[-1,1]\times[-1,1]$$

except for the diagonal $x=\xi$, on which it has a finite discontinuity. If we introduce operator

$$\tilde{T}F=\int_{-1}^{1}\overline{T}(\xi,x,t)f(\xi,t)d\xi \quad \text{and} \quad F=f(x,t)$$

then equation (8) can be written in operator form:

$$F+\tilde{T}F=0$$
$$\text{Or } \left(E+\tilde{T}\right)F=0. \tag{9}$$

For numerical simulation of the differential problem given in Eqns. (1)-(3), the spectral-grid method is applied and the resulting discrete problem is brought to an operator equation of the second kind of the form given in Eqn. (9). Convergence and convergence rate estimates for the spectral-grid method for the eigenvalue problem for a nonlinear ordinary differential equation were given [9,10].

The purpose of this work is to prove the convergence and obtain the estimates for the convergence rate of the spectral-grid method for the initial-boundary value problem of evolutionary equations.

In the numerical calculation of the problem posed, the approximate solution is considered at discrete fixed points in time. Therefore, the study of the convergence of the spectral-grid method is carried out at discrete times and takes into account that the time is not indicated in the basic equations and their coefficients for simplicity of presentation, and it is also considered that the basic equations are considered on discrete time layers. Let us denote the fixed moment of time by t^*.

The integration interval is considered on [–1,1] in the spectral-grid method and the grid is written as

$$-1=x_0<x_1<\ldots<x_M=1,$$

where M is an integer. The grid may be uniform or non-uniform.

To approximate the solution of a problem on a grid using Chebyshev polynomials, you can represent the solution on each grid element $[x_{j-1}, x_j]$, $j=1,2,\ldots,M$ as a linear combination of Chebyshev polynomials of the first kind T_n:

$$u_j^{(p_j)}(x,t^*)=u_j^{(p_j)}(x)=\sum_{n=0}^{p_j}a_n^{(p_j)}T_n(\tilde{x}),$$
$$x\in\left[x_{j-1},x_j\right],\quad x=\frac{m_j}{2}+\frac{k_j}{2}\tilde{x}, \tag{10}$$

where $m_j=x_j+x_{j-1}, k_j=x_j-x_{j-1}, -1\le\tilde{x}\le1$, and k_j is the length of the jth grid element, and T_n is the Chebyshev polynomials, $a_n^{(p_j)}$ - unknown expansion coefficients. In general, $\tilde{x}$ depends on j, but here and below the index j is omitted for simplicity. p_j is the number of Chebyshev polynomials to approximate the solution of the equations (1)-(3) on the j-element of the grid $[x_{j-1}, x_j]$, where $p_j\to\infty$ for each fixed j and $p_j\ge2$, which is not be less than the order of the highest derivative of the differential equation. The maximum grid step is denoted by $\hbar=\max_{1\le j\le M}k_j=\max_{1\le j\le M}(x_j-x_{j-1})$, and the minimum number of Chebyshev polynomials approximating the solution on grid intervals is denoted by $p_-=\min_{1\le j\le M}p_j$.

Thus, the total number of Chebyshev polynomials required to approximate the solution of the differential problem (1)-(3) in all grid intervals is determined by formula

$$\overline{m}=\sum_{j=1}^{M}(p_j+1).$$

In the spectral-grid method, which is commonly used in numerical analysis for solving differential equations, including partial differential equations (PDEs), the approach involves discretizing the domain into a grid or mesh and approximating the solution at specific points within this grid. In this method, continuity of the approximate solution and its derivatives are enforced at internal nodes, while boundary conditions are applied at the boundary nodes of the grid $x_0=-1, x_M=1$, the corresponding boundary conditions

$$\begin{cases} \dfrac{\partial^s u_j^{(p_j)}(x_j)}{\partial x^s} = \dfrac{\partial^s u_{j+1}^{(p_j)}(x_j)}{\partial x^s}, \\ u_1(-1,t) = 0, \ u_M(+1,t) = 0 \end{cases} \quad (11)$$
$$s = 0,1, \quad j = 1,2,..., M-1$$

must be satisfied.

Let $U^{(\vec{p})}(x) = (u_1^{(p_1)}(x),..., u_M^{(p_M)}(x)),\ (\vec{p}) = (p_1, p_2,..., p_M)$ be the vector of the approximate solution, the coefficients $a_n^{(p_j)}$ will be determined using the requirement that residual $L_0 u_j^{(p_j)}$ be orthogonal to Chebyshev polynomials up to number (p_j–2) with weight $\rho(\tilde{x})$ on the interval [x_{j-1}, x_j], i.e.

$$\int_{x_{j-1}}^{x_j} L_0 u_j^{(p_j)}(x) T_k(\tilde{x}) \rho(\tilde{x}) dx = 0, \quad (12)$$
$$j = 1,2,..., M, \ k = 0,1,..., p_j - 2,$$

where

$$\rho(\tilde{x}) = \frac{1}{\sqrt{1-\tilde{x}^2}}.$$

It can be seen that the number of conditions given in (11) and (12) coincides with the number of unknown coefficients $a_n^{(p_j)}$ in (10) and is equal to

$$\overline{m} = \sum_{j=1}^{M} (p_j + 1).$$

The notation

$$f_j^{(p_j)}(x,t^*) = \frac{\partial^2 u_j^{(p_j)}(x,t^*)}{\partial x^2}, \ x \in [x_{j-1}, x_j] \quad (13)$$

is introduced, then

$$f_j^{(p_j)}(x,t^*) = f_j^{(p_j)}(x) = \sum_{n=0}^{p_j} a_n^{(p_j)} \left(\frac{2}{k_j}\right)^2 T_n(\tilde{x}). \quad (14)$$

It follows from the relation (11) that the function $f_j^{(p_j)}$ at the ends of the interval (x_{j-1},x_j) has a finite limit on the left and right. From the vector $U^{(\vec{p})}$, by virtue of (11), we can construct a function $u^{(p)}(x)$ defined on [–1,1] and belonging to the space on C^1[–1,1], and at the partition points x_j, the second derivative has the left and right limits.

The Green's function of the operator $\partial^2/\partial x^2$ for problem (1) is known; we denote it by $G(\xi,x,t^*)$. Let us also construct a function $f^{(p)}(x)$ defined over the entire integration interval, i.e. $x \in [-1,1]$ and satisfying the following conditions

$$f^{(p)}(x,t^*) = f_j^{(p)}(x,t^*), \ x \in [x_{j-1}, x_j]. \quad (15)$$

It is obvious that the function $f^{(p)}$ can have finite discontinuities only at internal nodes of the grid x_j and, moreover,

$$f^{(p)}(x,t^*) = f^{(p)}(x) = \frac{\partial^2 u^{(p)}}{\partial x^2} \quad (16)$$

whereas the derivative $\partial^2/\partial x^2$, due to the above reasoning, one can understand the generalized derivative by the definition of S.L. Sobolev. Since $u^{(p)}(x)$ satisfies the boundary conditions from (9), then $u^{(p)}$ is represented as

$$u^{(p)}(x) = \int_{-1}^{1} G(\xi,x,t^*) f^{(p)}(\xi) d\xi. \quad (17)$$

Let us now construct an operator equation of the form (9) for the function $f^{(p)}$. Substituting the function $u^{(p)}(x)$ from (17) into (10) and using the eqns. (15) and (13), we get

$$\begin{aligned} &\int_{x_{j-1}}^{x_j} f_j^{(p_j)}(x) T_k(\tilde{x}) \rho(\tilde{x}) dx + \\ &+ \int_{x_{j-1}}^{x_j} \left[\int_{-1}^{1} -\mu \left(\frac{\partial G}{\partial t} + G \frac{\partial G}{\partial x} \right) f_j^{(p_j)}(\xi) \right] \cdot \\ &\cdot T_k(\tilde{x}) \rho(\tilde{x}) dx = 0, \\ &j = 1,2,..., M; \ k = 0,1,..., p_j - 2. \end{aligned} \quad (18)$$

Using the introduced notation for $\tilde{T}(\xi,x,t^*)$ and splitting the integral on the interval [–1,1], into the sum of integrals over the intervals [x_{j-1}, x_j] we have

$$\begin{aligned} &\int_{x_{j-1}}^{x_j} f_j^{(p_j)}(x) T_k(\tilde{x}) \rho(\tilde{x}) dx + \\ &+ \int_{x_{j-1}}^{x_j} \left[\sum_{i=1}^{M} \tilde{T}^i f_i^{(p_i)} \right] T_k(\tilde{x}) \rho(\tilde{x}) dx = 0 \end{aligned}, \quad (19)$$

here the operators $\tilde{T}^i$ are defined by the formulas

$$\tilde{T}^i f_i^{(p_i)} = \int_{x_{j-1}}^{x_j} \tilde{T}(\xi,x,t^*) f_i^{(p_i)}(\xi) d\xi, \ x \in [x_{i-1}, x_i]. \quad (20)$$

Let us introduce the Hilbert space $L_{2,\rho}(x_{j-1}, x_j)$ with the scalar product

$$(\varphi,\psi)_{L_{2,\rho}} = \int_{x_{i-1}}^{x_i} \varphi(x) \psi(x) \rho(\tilde{x}) dx. \quad (21)$$

It is obvious that $\tilde{T}^i : L_{2,\rho}(x_{i-1}, x_i) \to C[-1,1]$, i.e. the operator $\tilde{T}^i$ maps the $L_{2,\rho}(x_{j-1}, x_j)$ Hilbert space into the space C[–1,1] of functions continuous on the interval [–1,1], similarly, $\tilde{T}^i : C(x_{i-1}, x_i) \to C[-1,1]$.

Now, consider

$$g_i(x)=\int_{x_{j-1}}^{x_j}\tilde{T}(\xi,x,t^*)f(\xi)d\xi . \qquad (22)$$

Then

$$\frac{\partial g_i}{\partial x}=\frac{\partial}{\partial x}\left(\int_{x_{i-1}}^{x}\tilde{T}(\xi,x,t^*)f(\xi)d\xi+\right.$$
$$\left.+\int_{x}^{x_i}\tilde{T}(\xi,x,t^*)f(\xi)d\xi\right)=\left(\tilde{T}_+-\tilde{T}_-\right)f(x)+$$
$$+\int_{x_{i-1}}^{x}\frac{\partial\tilde{T}(\xi,x,t^*)}{\partial x}f(\xi)d\xi+\int_{x}^{x_i}\frac{\partial\tilde{T}(\xi,x,t^*)}{\partial x}f(\xi)d\xi= \qquad (23)$$
$$=f(x)+\int_{x_{i-1}}^{x}\frac{\partial\tilde{T}(\xi,x,t^*)}{\partial x}f(\xi)d\xi+$$
$$+\int_{x}^{x_i}\frac{\partial\tilde{T}(\xi,x,t^*)}{\partial x}f(\xi)d\xi.$$

Here $\frac{\partial\tilde{T}}{\partial x}$ is limited and $\left(\tilde{T}_+-\tilde{T}_-\right)=\left(\tilde{T}(\xi,x,t^*)\Big|_{\xi=x+0}-\tilde{T}(\xi,x,t^*)\Big|_{\xi=x-0}\right)=1.$

Therefore, the inequality is true:

$$\left\|\frac{\partial g_i}{\partial x}\right\|_{L_2(-1,1)}\le\|f(x)\|_{L_2(-1,1)}+c_1\|f(x)\|_{L_2(-1,1)}+c_2\|f(x)\|_{L_2(-1,1).}$$

But $L_{2,\rho}\subset L_2$, because

$$\rho(\tilde{\xi})=\frac{1}{\sqrt{1-\tilde{\xi}^2}}.$$

Then $\|f\|_{L_2}\le\|f\|_{L_{2,\rho}}$ and thus $g_i\in W_2^1$. By the embedding theorem, g_i is continuous and even satisfies the Hölder condition with exponent 1/2. Since we proved that

$$\tilde{T}^i:L_2(x_{i-1},x_i)\to C[-1,+1],$$

Therefore

$$\tilde{T}^i:C(x_{i-1},x_i)\to C[-1,+1].$$

Let us introduce a vector as:

$$F^{(p)}=\left(f_1^{(p_1)}(x),...,f_M^{(p_M)}(x)\right)$$

whose components $f_j^{(p_j)}$ are defined on their interval and, according to (14), they are polynomials of degree (p_j–2). Therefore, they can be represented in terms of a linear combination of the first (p_j–2) Chebyshev polynomials.

In the space of such vectors, the vectors can be taken as vectors

$$\vec{T}_{i_1}=\left(T_{i_1}(\tilde{x}),0,...,0\right),\vec{T}_{i_2}=\left(0,T_{i_2}(\tilde{x}),...,0\right),...,\vec{T}_{i_M}=$$
$$=\left(0,...,0,T_{i_M}(\tilde{x})\right),$$

where $T_{i_j}(\tilde{x})$ are Chebyshev polynomials ($\tilde{x}\in[-1,1]$ and i_k=0,1,…, p_k–2, k=0,1,2,…,M).

Let us introduce the scalar product in the finite-dimensional space of $L_{2,\rho}^{(p)}$ vectors $F^{(p)}$ according to the formula

$$\left(F^{(p)},Z^{(p)}\right)_{L_{2,\rho}^{(p)}}=\sum_{j=1}^{M}\int_{x_{j-1}}^{x_j}f_j^{(p_j)}(x)z_j^{(p_j)}(x)\rho(\tilde{x})dx,\tilde{x}\in[-1,1].$$

Equation (18) through the introduced scalar product is written in the form

$$\left(F^{(p)},\vec{T}_{i_j}\right)+\left(\tilde{T}_-F^{(p)},\vec{T}_{i_j}\right)=0,$$
$$i_j=0,1,...,p_j-2;\ j=1,2,...,M \qquad (24)$$

where

$$\tilde{T}F^{(p)}=\left(\sum_{i=1}^{M}T^i f_i^{(p_i)}(x)\Big|_{x\in[x_0,x_1]},\sum_{i=1}^{M}T^i f_i^{(p_i)}(x)\Big|_{x\in[x_1,x_2]},...\right).$$

Then the Hilbert space $L_{2,\rho}^M$ of vector functions $F=(f_1(x),...,f_M(x))$ is introduced, where $f_i(x)\in L_{2,\rho}(x_{i-1},x_i)$, i.e.

$$L_{2,\rho}^M\in L_{2,\rho}(x_0,x_1)\times L_{2,\rho}(x_1,x_2)\times...\times L_{2,\rho}(x_{M-1},x_M)$$

with dot product, as in $L_{2,\rho}^{(p)}$.

Since it is always possible to construct a function $f(x)$ from the vector F according to the rule $f(x)=f_i(x)$ for $x\in[x_{i-1},x_i]$, then the space $L_{2,\rho}^M$ is isomorphic to the space $L_{2,\rho}^M(-1,1)$ of functions with the scalar product

$$(f,z)_{L_{2,\rho}^M}=\sum_{j=1}^{M}\int_{x_{j-1}}^{x_j}f(x)z(x)\rho(\tilde{x})dx.$$

We introduce projector $P_{\vec{p}}:L_{2,\rho}^M\to L_{2,\rho}^{(\vec{p})}$ according to the rule: if $F=(f_1(x),...,f_M(x))$ and $f_j(\tilde{x})=\sum_{k=0}^{\infty}c_k^{(j)}T_k(\tilde{x})$, then

$$P_{\vec{p}}F=\left(\sum_{k=0}^{p_1-2}c_k^{(1)}T_k(\tilde{x}),...,\sum_{k=0}^{p_M-2}c_k^{(M)}T_k(\tilde{x})\right).$$

The possibility of expanding the function $f_i(x)\in L_{2,\rho}(x_{i-1},x_i)$ in a Fourier series in terms of Chebyshev polynomials is substantiated [11]. This also implies a similar assertion about the expansion of vectors $F\in L_{2,\rho}^M$ in terms of vectors $T_{i_j},i_j=0,1,...,p_j-2,\ \ j=1,2,...,M$.

Using the above notation, equation (18) can be written as

$$F^{(p)} + P_{\bar{p}} T F^{(p)} = 0,\ F^{(p)} \in L^{(p)}_{2,\rho}. \quad (25)$$

The above reasoning allows us to formulate some statements about the convergence of $F^{(p)}$ to F as $(p_-)\to\infty$ (i.e., as $p_j\to\infty$ for any j, where $(p_-)=\min(p_1, p_2, \ldots, p_M)$). And due to isomorphism, the convergence of $u^{(p)}(x)$ to $u(x)$.

Theorem 1. Let F_0 be a solution to equation (9) (the existence of F_0 follows from the unique solvability of problem (1)–(3)). Then, for sufficiently large $(\vec{p})$ (i.e., large p_j for all j), equation (25) has a unique solution $F^{(p)}$ and the estimate

$$\left\| F^{(p)} - F_0 \right\|_{L^M_{2,\rho}} \le c_0 \left\| F_0 - F^0_p \right\|_{L^M_{2,\rho}}, \quad (26)$$

holds true, where F^0_p is the Fourier segment in the Chebyshev polynomials of the function $F_0 = (f^0_1, \ldots, f^0_M)$ of length "p", i.e.

$$F^0_p = (f^{(0)}_{p_1}, \ldots, f^{(0)}_{p_M}),$$

$$f^{(0)}_{p_i}(x) = \sum_{j=0}^{p_i-2} \tilde{c}^i_j T_j(\tilde{x}),\ \tilde{c}^i_j = \int_{x_{j-1}}^{x_j} f^0_i(x) T_j(\tilde{x}) \rho(\tilde{x}) d x,$$

$$\left(\left\| F_0 \right\|^0_{L^M_{2,\rho}} = \sum_{j=1}^{M} \left(\sum_{i=1}^{\infty} \left(\tilde{c}^{(j)}_i \right)^2 \right) \right),$$

$$\left\| F_0 - F^0_p \right\|_{L^M_{2,\rho}} = \sum_{j=1}^{M} \left(\sum_{i=p_j-1}^{\infty} \left(\tilde{c}^{(j)}_i \right)^2 \right);$$

c_0 is a constant depending on the norm of the inverse operator $(E+\tilde{T})^{-1}$, E is the identity operator.

Proof. The operator

$$(E+\tilde{T}) : L^M_{2,\rho} \to L^M_{2,\rho}$$

is continuously invertible, since the Green's function of problem (1) for the operator L_0 is explicitly known, and this problem is equivalent to the inversion of the operator $(E+\tilde{T})$, and $\left\| (E+\tilde{T})^{\pm 1} \right\| \le c$. Wherever not otherwise stated, the norm of the operator $\|S\|$ is understood as

$$\underset{\varphi \in L_{2,\rho}}{Sup} \left(\frac{\|S\varphi\|_{L_{2,\rho}}}{\|\varphi\|_{L_{2,\rho}}} \right).$$

Condition $\left\| (E-P_p)\tilde{T} \right\|_{L^M_{2,\rho}} \to 0$ as $(p)\to\infty$ follows from the operator $\tilde{T}$, as a vector integral operator with a piecewise continuous kernel, is completely continuous in $L^M_{2,\rho}$. The proof of the last fact is carried out similarly to the proof of the complete continuity in $L_2(-1,1)$ of an integral operator with a summable square over a variable kernel, since the weight $\rho(\tilde{x})$ has singularities of order 1/2, and the kernel is uniformly bounded.

Condition

$$\left\| F - P_p F \right\|_{L^M_{2,\rho}} \to 0$$

as $(p)\to\infty$ follows from the fact that the vectors $\vec{T}_{i_j}$ form a complete orthogonal system in $L^M_{2,\rho}$, and P_p is the projection operator on $L^{(p)}_{2,\rho}$, i.e. segment of the Fourier series in $\vec{T}_{i_j}$ from F to the $(p)^{\text{th}}$ term. The boundedness of the operator $\left\| P_p \right\|$ follows from the Bessel inequality:

$$\left\| P_p X \right\|^2_{L^M_{2,\rho}} = \sum_{i=1}^{M} \left(\sum_{i=0}^{p_j-2} \left(\tilde{c}^i_j \right)^2 \right) \le \sum_{i=1}^{M} \left(\sum_{j=0}^{\infty} \left(\tilde{c}^i_j \right)^2 \right) = \|X\|^2_{L^M_{2,\rho}},$$

where $\tilde{c}^i_j$ are the Fourier coefficients for F. Thus, $\left\| P_p \right\| \le 1$. The assertion of Theorem 1 is proved.

Theorem 2. Function $u^{(p)}(x)$ as, $(p)\to\infty$, tends to the exact solution $u_0(x)$ of problems (1)-(3) in the space norm W^2_2 and estimate

$$\left\| u^{(p)} - u_0 \right\|_{W^2_2(-1,1)} \le \tilde{c}_0 \left\| F^{(p)} - F_0 \right\|_{L^M_{2,\rho}}, \quad (27)$$

is valid, where

$$\tilde{c}_0 = c_0 \tilde{c},\ \tilde{c} = \underset{\varphi \in L_2}{Sup} \frac{\left\| \int_{-1}^{1} G(\xi, x, t^*) \varphi(\xi) d\xi \right\|_{W^2_2}}{\|\varphi\|_{L_2}},$$

$$\|\varphi\|^2_{W^2_2} = \sum_{k=0}^{M} \left\| \frac{\partial^k \varphi}{\partial x^k} \right\|^2$$

Proof. The proof of Theorem 2 follows from the convergence of $F^{(p)}$ to F_0 using the isomorphism $F \leftrightarrow f$, the equivalence of (9) and (1), and the estimate

$$\left\|u^{(p)}-u_0\right\|^2_{W_2^2(-1,1)}=$$

$$=\left\|\int_{-1}^{1}G(\xi,x,t^*)\left(f^{(p)}(\xi)-f_0(\xi)\right)d\xi\right\|^2_{W_2^2(-1,1)}\le$$

$$\le\tilde{c}^2\left\|f^{(p)}-f_0\right\|^2_{L_2(-1,1)}=$$

$$=\tilde{c}^2\sum_{j=1}^{M}\int_{x_{j-1}}^{x_j}\left(f_j^{(p_j)}(\xi)-f_0(\xi)\right)^2\rho(\tilde{\xi})\sqrt{1-\tilde{\xi}^2}\,d\xi\le$$

$$\le\tilde{c}^2\sum_{j=1}^{M}\int_{x_{j-1}}^{x_j}\left(f_j^{(p_j)}(\xi)-f_0(\xi)\right)^2\rho(\tilde{\xi})d\xi=$$

$$=\tilde{c}^2\left\|F^{(p)}-F_0\right\|^2_{L^M_{2,\rho}}\le\tilde{c}_0^2\left\|F_0-F_p^0\right\|^2_{L^M_{2,\rho}},$$

since

$$\rho(\tilde{\xi})\sqrt{1-\tilde{\xi}^2}=1 \text{ and } \sqrt{1-\tilde{\xi}^2}\le 1,$$

$$\xi=\frac{m_j}{2}+\frac{k_j}{2}\tilde{\xi},\ \tilde{\xi}\in[-1,1].$$

Theorem 3. The solution of equation (25) $F^{(p)}$ converges as $(p)\to\infty$ to the exact solution F_0 of equation (9), and the estimate (26) has the form

$$\left\|F^{(p)}-F_0\right\|^2_{L^M_{2,\rho}}\le\tilde{c}_1\left(\frac{\hbar}{p_- -2}\right)^{2(s+\alpha)}, \quad (28)$$

where

$$(p)=(p_1,...,p_M),\ p_-=\min_{1\le j\le M}p_j,\ \hbar=\max_{1\le j\le M}(x_j-x_{j-1}),$$

$$\tilde{c}_1=c_0c_1,\ c_1=\frac{\pi}{2}\left(c_s'\overline{M}\right)^2,\ c_s'=12\frac{6^s s^s}{s!}\left(\frac{s+1}{2}\right)^{\alpha}$$

the constant $\overline{M}$ is determined from the Lipschitz condition, i.e.

$$\left|f^{(s)}(x_1)-f^{(s)}(x_2)\right|\le\overline{M}\left|x_1-x_2\right|^{\alpha},$$

here the condition $(p)\to\infty$ is equivalent to the condition that $p_-\to\infty$.

Proof. Equality

$$\left\|F_p-F_0\right\|^2_{L^2_{2,\rho}}=\sum_{j=1}^{M}\int_{x_{j-1}}^{x_j}\left(f_j^{(p_j)}(\xi)-f_j^{(0)}(\xi)\right)^2\rho(\tilde{\xi})d\xi. \quad (29)$$

is true.

We use the arguments from [12] (i.e., Theorem 6). According to Toepler's theorem [12], we have

$$\int_{x_{j-1}}^{x_j}\left(f_j^{(p_j)}(\xi)-f_j^{(0)}(\xi)\right)^2\rho(\tilde{\xi})d\xi\le$$
$$\le\int_{x_{j-1}}^{x_j}\left(D_{p_j}(\xi)-f_j^{(0)}(\xi)\right)^2\rho(\tilde{\xi})d\xi, \quad (30)$$

where $D_{p_j}(\xi)$ is algebraic polynomial of degree at most (p_j–2). If we now use a more precise estimate from Jackson's theorem (i.e., estimate (150) [12]), then we obtain that such a polynomial $D_{p_j}(\xi)$ exists, and the estimate

$$\left|D_{p_j}(\xi)-f_j^{(0)}(\xi)\right|\le\frac{c_s'(x_j-x_{j-1})^{s+\alpha}}{(p_j-2)^{s+\alpha}}\overline{M},$$
$$j=1,2,...,M,\ \ p_j-2\ge s+1$$

is valid for it.

Then, continuing inequality (30), we obtain

$$\int_{x_{j-1}}^{x_j}\left(D_{p_j}(\xi)-f_j^{(0)}(\xi)\right)^2\rho(\tilde{\xi})d\xi\le$$
$$\le\int_{x_{j-1}}^{x_j}\left((c_s'\overline{M})^2\left(\frac{x_j-x_{j-1}}{(p_j-2)}\right)\right)^{2(s+\alpha)}\rho(\tilde{\xi})d\xi\le$$
$$\le(c_s'\overline{M})^2\left(\frac{\hbar}{(p_--2)}\right)^{2(s+\alpha)}\int_{x_{j-1}}^{x_j}\rho(\tilde{\xi})d\xi\le \quad (31)$$
$$\le(c_s'\overline{M})^2\left(\frac{\hbar}{(p_--2)}\right)^{2(s+\alpha)}\frac{k_j}{2}\pi=$$
$$=c_1\left(\frac{\hbar}{(p_--2)}\right)^{2(s+\alpha)}\frac{k_j}{2},$$

Here

$$c_1=\pi(c_s'\overline{M})^2,\ p_-=\min_{1\le j\le M}p_j,\ \hbar=\max_{1\le j\le M}(x_j-x_{j-1}),$$

$$\int_{x_{j-1}}^{x_j}\rho(\tilde{\xi})d\xi=\int_{x_{j-1}}^{x_j}\frac{d\xi}{\sqrt{1-\tilde{\xi}^2}}=\frac{k_j}{2}\int_{-1}^{1}\frac{d\tilde{\xi}}{\sqrt{1-\tilde{\xi}^2}}=\frac{k_j}{2}\pi,$$

since $\xi=\frac{m_j}{2}+\frac{k_j}{2}\tilde{\xi}$, and the value of the integral $\int_{-1}^{1}\frac{d\tilde{\xi}}{\sqrt{1-\tilde{\xi}^2}}=\pi$ is calculated using recursive formulas for the beta function.

Substituting estimate (32) into equality (29), we have

$$\left\|F_P-F_0\right\|^2_{L^M_{2,\rho}}\le\sum_{j=1}^{M}c_1\left(\frac{\hbar}{p_--2}\right)^{2(s+\alpha)}\frac{k_j}{2}=c_1\left(\frac{\hbar}{p_--2}\right)^{2(s+\alpha)},$$

since $\sum_{j=1}^{M} k_j = 2$.

After that, estimate (29) from Theorem 3 is a consequence of Theorem 1 and the above reasoning.

Theorem 4. The function $u^{(p)}(x)$, as $(p)\to\infty$, converges to the exact solution $u_0(x)$ of problems (1)-(3) in the space norm W_2^2 and the estimate

$$\left\|u^{(p)} - u_0\right\|^2_{W_2^2(-1,1)} \le c_2\left(\frac{\hbar}{p_- - 2}\right)^{2(s+\alpha)},$$

where $c_2 = \tilde{c}_0\tilde{c}_1$ is satisfied.

Proof. To prove the theorem, it suffices to estimate the right-hand side in (20). And it was estimated in the proof of Theorem 3.

3. Results

The spectral-grid method is applied to the numerical simulation of initial-boundary value problems for the Burgers equation. The following problem

$$\frac{\partial u}{\partial t} = \frac{1}{\mu}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}, \tag{32}$$

$$u(x,0) = -\sin\pi x, \tag{33}$$

$$u(\pm 1, t) = 0. \tag{34}$$

is considered.

The computational experiment was carried out with the following values of the characteristic parameters: $\mu=\pi\cdot 10^2$, $\tau=10^{-2}/6\pi$. The number of grid elements M=8, the total number of polynomials in these elements N=64 is fixed. The analytical solution of the problem (32)-(34) is having the following function

$$u(x,t) = 4\pi\nu\frac{\sum_{n=1}^{\infty} n a_n e^{-n^2\pi^2 t\nu}\sin n\pi x}{\left(a_0 + 2\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t\nu}\cos n\pi x\right)},$$

where $a_n = (-1)^n I_n(1/2\pi\nu)$, $I_n(z)$ is the improved Bessel function of the first kind, $\nu=1/\mu$.

Comparative studies are shown in Table 1 and Table 2. In Table 1, the results are compared using spectral-grid method with other differential method by fixing $t = 0.5,\ \mu = 1, x \in [0,1]$ and Table 2 shows the comparison for $\mu = 100,\ x \in [0,1]$. It is seen from the tables that the spectral-grid method is one of the best approximations and it is significant to find the approximate solutions.

Table 1. Calculation results ($t = 0.5,\ \mu = 1, x \in [0,1]$)

x	**Differential Method [2]** $\tau = 0.001$ $N = 100$		**Differential Method [4]** $\tau = 0.005$ $N = 100$	**Spectral-grid Method** $N = 32$ $M = 4$ $\tau = 0.005$	**Analytical Solutions**
	BDF-2	BDF-3	BDF-2		
0,1	0.002214	0.002213	0.002213	0.002211	0.002213
0,3	0.005798	0.005795	0.005796	0.005797	0.005796
0,5	0.007172	0.007168	0.007170	0.007169	0.007169
0,7	0.005806	0.005803	0.005804	0.005804	0.005804
0,9	0.002219	0.002218	0.002218	0.002217	0.002218

Table 2. Calculation results ($\mu = 100,\ x \in [0,1]$)

x	t	**Differential Method [1]** $\tau = 0.001$ $N = 100$	**Differential Method [3]** $N = 90$	**Differential Method [4]** $\tau = 0.005$ $N = 80$	**Spectral-grid Method** $N = 32$ $M = 4$ $\tau = 0.005$	**Analytical Solutions**
0.25	1	0.18820	0.18819	0.18819	0,18819	0.18819
	3	0.07511	0.07511	0.07511	0,07511	0.07511
0.5	1	0.37443	0.37443	0.37441	0,37442	0.37442
	3	0.15019	0.15018	0.15018	0,15018	0.15018
0.75	1	0.55606	0.55607	0.55607	0,55605	0.55605
	3	0.22486	0.22484	0.22484	0,22481	0.22481

Fig.1 describes the solution for the Eqns. (32)-(34) using *M*=1 - spectral method. From the figure 1, it can be seen that, at a point $x = 0$, as the value of the first derivative increases, oscillations appear in the numerical solution and at a time point $t = 1.0$ the amplitude of the oscillations increases greatly and, as a result, there is no more than one correct digit left in the numerical solution.

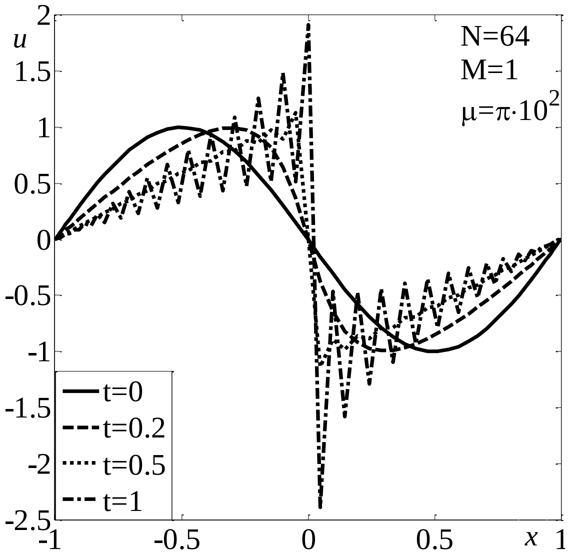

Figure 1. Spectral method

When using the spectral-grid method and the integration interval is divided into two elements: $[-1,0],[0,1]$ (shown in Fig. 2), it can be seen that at $t = 0.5$ the amplitude of oscillations of the approximate solution decreases sharply and two correct digits appear in the approximate solution. With the growth of the value of time, i.e. at $t > 0.5$, fluctuations in the approximate solution are smoothed out and the accuracy of finding the approximate solution grows and is achieved up to an accuracy of $\varepsilon \sim 10^{-4}$.

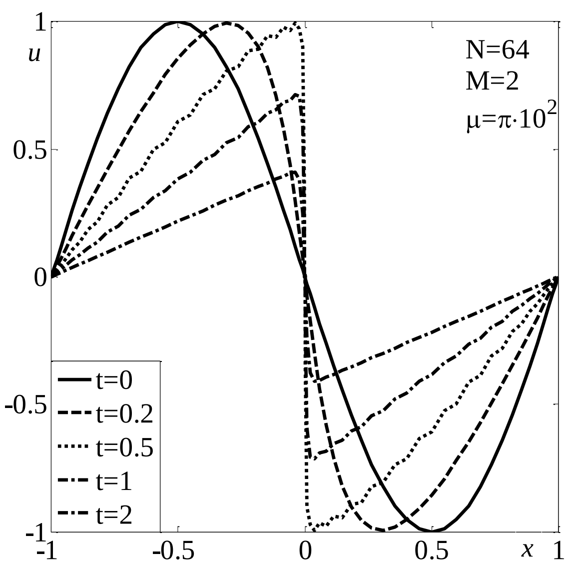

Figure 2. Spectral grid method (M=2)

Similarly, in the spectral-grid method, the number of grid elements increases near the center of segment $[-1,1]$, where the gradients of the solution are large. An uneven grid consisting of *M*=8 elements is considered (for example, on segment $[-1,0]$, the following non-uniform 4 elements are selected, similar grid elements symmetrical to these 4 elements are selected on segments [0,1]):

$$[-1,-0.47],[-0.47,-0.2],[-0.2,-0.07],$$
$$[-0.07,0]$$

in this case, the total number of Chebyshev polynomials does not increase, and the approximate solution on each of the 8 grid elements is approximated using 8 polynomials. The dynamics of changes in the approximate solution at different times *t* are shown in Fig.3.

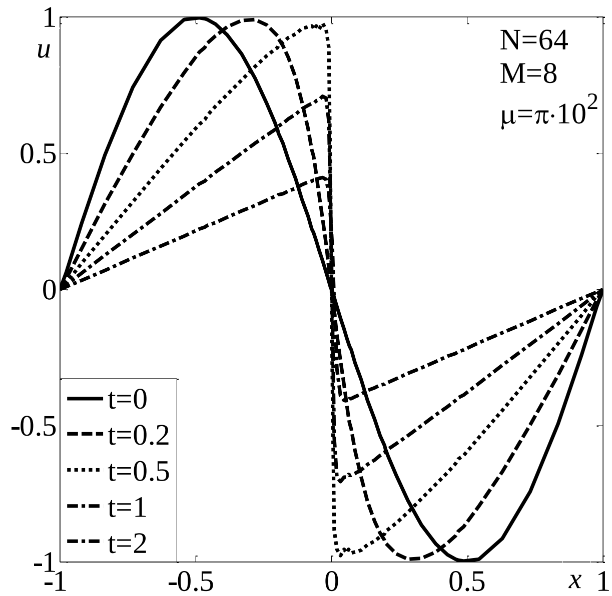

Figure 3. Spectral grid method (M=8)

It can be seen that the oscillations in the approximate solution are strongly smoothed out at different times.

Thus, the calculation results show that the division of the integration interval into elements, i.e. when using the spectral-grid method, one can find an approximate solution with a sufficiently high accuracy in areas with a high solution gradient.

Then, calculations were carried out with the same partitions into elements *M*=8, but with an increase in the polynomials used to approximate the approximate solution, i.e. with a total number of polynomials *N*=96, 12 polynomials were selected on each of the grid elements *M*=8. The obtained results are showed in Figs. 4-5.

It can be seen from the results that when a uniform mesh is used in the spectral-grid method, small fluctuations are observed in the approximate solution (Fig. 4), however, when a non-uniform mesh is selected that thickens to the center of the segment $[-1,1]$, where the gradients of the solution are large, small fluctuations appear when using a non-uniform, the grids disappear completely and the approximate solution of the Burgers equation is determined with a sufficiently high accuracy. It can be seen from the figures and tables that the fluctuations in the approximate solutions at different time *t* are smoothed out to a high degree.

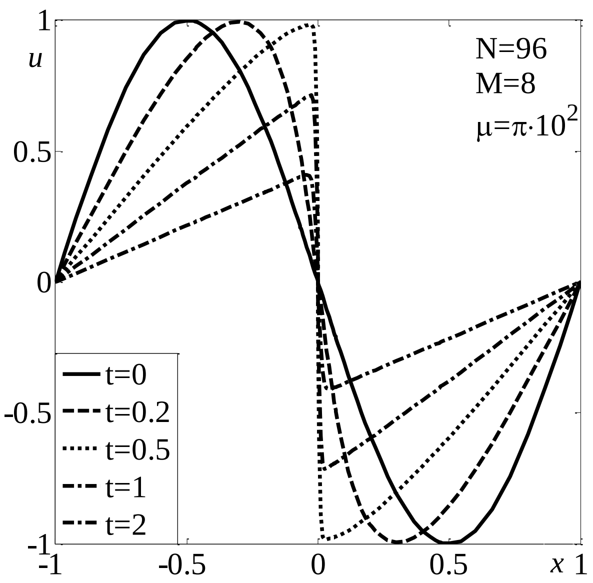

Figure 4. Spectral grid method (uniform grid)

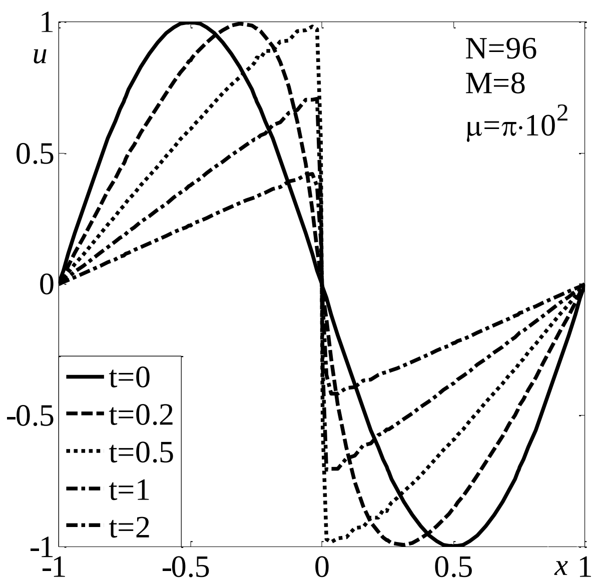

Figure 5. Spectral grid method (non-uniform grid)

4. Conclusions

In this work, the spectral-grid method is used to approximate the differential equations. To discretize the Burgers equation with the corresponding initial-boundary conditions, a grid with a uniform step was used in time, and a spectral-grid method was used in the spatial variable. Using Green's function theory, both the differential problem and the discrete problem are related to an operator equation of the second kind. Theorems on the convergence of the solution of a continuous operator equation of the second kind to a discrete operator equation of the second kind in arbitrary fixed layers in time are proved and estimates of the rate of convergence of the spectral-grid method are obtained. In particular, Theorem 1 proves the convergence of the solution of an approximate discrete operator equation of the second kind to the solution of a continuous operator equation of the second kind. Theorem 2 proves the convergence of the approximate solution obtained by the spectral-grid method to the exact solution of the differential problem.

In Theorem 3, an estimate is obtained for the rate of convergence of an approximate solution of a discrete operator equation of the second kind to a solution of a continuous operator equation of the second kind.

In Theorem 4, an estimate is obtained for the rate of convergence of the approximate solution obtained by the spectral-grid method to the exact solution of the initial-boundary value problem for the Burgers equation.

Thus, it can be seen from the calculation results that by dividing the integration interval into elements, i.e. using the spectral-grid method, the accuracy of the approximate solution can be obtained with high accuracy in areas where the gradients of the solution are large. The approximate and exact solutions are indistinguishable and they are almost significant. This is illustrated by the high accuracy of the spectral-grid method.

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