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Mixture of Ailamujia and Size Biased Ailamujia Distributions: Estimation and Application

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Abstract In this article, we introduce a new model entitled a mixture of the Ailamujia and size biased Ailamujia distributions. We present and discuss some statistical properties of this mixture of the Ailamujia and size biased Ailamujia distributions, such as moments, skewness, and kurtosis. We also provide some graphical results on the mixture of the Ailamujia and size biased Ailamujia distributions and provide some numerical results to understand the behavior of the proposed mixture and its properties. Also, we provide some reliability analysis results on the proposed mixture. The parameters of the Ailamujia and size biased Ailamujia distributions are estimated by using the maximum likelihood method. The usefulness of the proposed combination is illustrated by using a real-life dataset. We use the Ailamujia distribution and the size biased Ailamujia distribution, in addition to the mixture of the Ailamujia and size biased Ailamujia distributions to fit the real-life dataset. We use different criteria in this comparison; the results show that the proposed mixture fits the dataset better than the use of the Ailamujia distribution and the size biased Ailamujia distribution alone.

Keywords Ailamujia Distribution, Mixture Distributions, Maximum Likelihood Estimation

1 Introduction

In this article, we introduce a finite mixture of two statistical continuous distributions: the Ailamujia distribution (AD) and the size biased Ailamujia distribution (SBAD). The mixture distributions are used to propose a new model with more flexibility to model the data. The mixture distributions can be used to model data that contains subgroups. For more details on mixture distributions see [1]. The AD was proposed by Lv et al. [2] as a lifetime model and drew the attention of researchers for use in introducing new continuous univariate distributions. For example, based on the AD, Rather et al. [3] proposed the size biased Ailamujia distribution (SBAD), Jan et al. [4] proposed the weighted version of the AD, Jayakumar and Elangovan [5] defined the area biased weighted AD, Aijaz et al. [6] introduced the inverse analogue of AD, Jamal et al. [7] introduced the power version of the AD, Rather et al. [8] defined the exponentiated AD, Smadi and Ansari [9] introduced the Ailamujia inverted Weibull distribution, and Alruwaili [10] introduced a new weighted version of the AD. For the mixture distributions, Andrabi et al. [11] introduced the Qammer distribution as a lifetime distribution by mixing the Rayleigh and Ailamujia distributions.

In this article we introduce the mixture of the Ailamujia and size biased Ailamujia distributions and its properties. The article is set out as follows: Section 2 introduces the theoretical background of the mixture of AD and SBAD. Section 3 introduces some statistical properties of the proposed mixture. Sections 4 and 5 introduce some results on reliability, and present the results on the parameters estimations, respectively. The results of the data analysis and the conclusion are presented in Sections 6 and 7, respectively.

2 Mixture model

In this section, we introduce the mixture of the two components, each of which represents a different distribution. The probability density function PDF and cumulative distribution function CDF of the mixture of two components can be defined as follows:

$$f(x) = mf_1(x) + (1-m)f_2(x), \quad 0 \le m \le 1 \tag{1}$$
$$F(x) = mF_1(x) + (1-m)F_2(x), \quad 0 \le m \le 1 \tag{2}$$

where m is the mixing proportion, and $f_1(x)$ $F_1(x)$ represents the PDF and CDF of the AD, and $f_2(x)$ $F_2(x)$ represents the PDF and CDF of the SBAD.

The PDF and CDF of the AD are defined as follows:

$$f_1(x,\theta) = 4x\theta^2 \exp^{-2x\theta} \tag{3}$$
$$F_1(x,\theta) = 1-(1+2x\theta)\exp^{-2x\theta} \tag{4}$$

The PDF and CDF of the SBAD are defined as follows:

$$f_2(x,\alpha) = 4x^2\alpha^3 \exp^{-2x\alpha} \tag{5}$$
$$F_2(x,\alpha) = 1+(-1-2x\alpha(1+x\alpha))\exp^{-2x\alpha} \tag{6}$$

By substituting (3) and (5) in (1), the PDF of the mixture of AD and SBAD is defined as follows:

$$f(x,\theta,\alpha,m) = m(4x\theta^2 \exp^{-2x\theta}) + (1-m)(4x^2\alpha^3 \exp^{-2x\alpha}), \quad \alpha,\theta \ge 0, x \ge 0 \tag{7}$$

and by substituting (4) and (6) in (2), the CDF of the mixture of AD and SBAD is defined as follows:

$$F(x,\theta,\alpha,m) = m(1-(1+2x\theta)\exp^{-2x\theta}) + (1-m)(1+(-1-2x\alpha(1+x\alpha))\exp^{-2x\alpha}), \quad \alpha,\theta \ge 0, x \ge 0 \tag{8}$$

Below, Figures 1 and 2 show the PDF and CDF plots for the mixture of AD and SBAD with different values of θ and α, and fixed values of m. We can observe that from Figure 1, the plots of the PDF of the mixture show one and two local maxima.

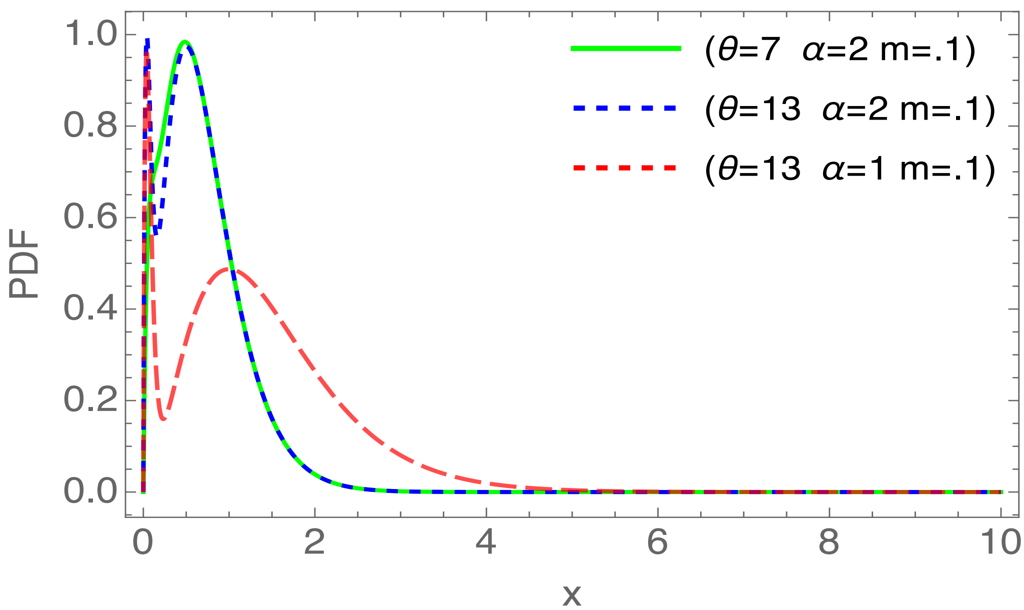

Figure 1. PDF plots of the mixture of AD and SBAD for fixed values of m, and varying values for θ and α.

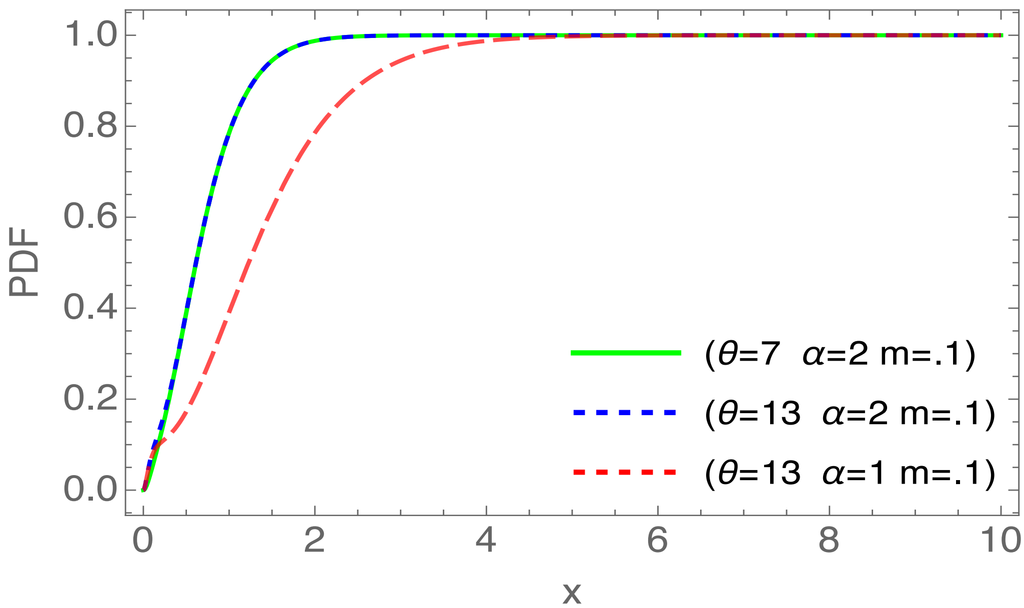

Figure 2. CDF plots of the mixture of AD and SBAD with fixed values of m, and varying values for θ and α.

3 Statistical Properties

Here, we present some statistical properties for the mixture of AD and SBAD.

Moments

Let X be a random variable following the mixture of AD and SBAB, then r^{th} moments of X is given by:

$$\mu_r^{'} = 2^{-1-r}(2m\theta^{-r} + (1-m)(2+r)\alpha^{-r})\ \Gamma[2+r] \tag{9}$$

Proof.

$$\begin{aligned} \mu_r^{'} &= E(X^r) = \int_0^\infty x^r f(x)dx & (10)\\ &= \int_0^\infty x^r \Big(m(4x\theta^2 \exp^{-2x\theta}) + (1-m)(4x^2\alpha^3 \exp^{-2x\alpha})\Big)dx & (11)\\ &= 2^{-1-r}(2m\theta^{-r} + (1-m)(2+r)\alpha^{-r})\ \Gamma[2+r] & (12) \end{aligned}$$

□

By setting $r = 1, 2, 3, 4$ in Equation (12), the first four moments of X can then be introduced as follows:

$$\begin{aligned} \mu_1^{'} = E(X) &= \frac{m}{\theta} + \frac{3(1-m)}{2\alpha} & (13)\\ \mu_2^{'} = E(X^2) &= \frac{3m}{2\theta^2} + \frac{3(1-m)}{\alpha^2} & (14)\\ \mu_3^{'} = E(X^3) &= \frac{3m}{\theta^3} + \frac{15(1-m)}{2\alpha^3} & (15)\\ \mu_4^{'} = E(X^4) &= \frac{15}{2}\Big(\frac{m}{\theta^4} + \frac{3(1-m)}{\alpha^4}\Big) & (16) \end{aligned}$$

Hence, the mean and variance of the mixture of AD and SBAD are defined as follows:

$$\begin{aligned} \mu_1 &= \frac{m}{\theta} + \frac{3(1-m)}{2\alpha} & (17)\\ var(X) = \mu_2 &= \mu_2^{'} - (\mu_1^{'})^2 & (18)\\ &= \Big(\frac{3m}{2\theta^2} + \frac{3(1-m)}{\alpha^2}\Big) - \Big(\frac{m}{\theta} + \frac{3(1-m)}{2\alpha}\Big)^2 & (19) \end{aligned}$$

Skewness and kurtosis

The skewness and kurtosis of the mixture of AD and SBAD are defined as follows, respectively:

$$SK(X) = \frac{(\frac{3m}{\theta^3}+\frac{15(1-m)}{2\alpha^3}) - 3(\frac{m}{\theta}+\frac{3(1-m)}{2\alpha})(\frac{3m}{2\theta^2}+\frac{3(1-m)}{\alpha^2}) + 2(\frac{m}{\theta}+\frac{3(1-m)}{2\alpha})^3}{\left(\left(\frac{3m}{2\theta^2}+\frac{3(1-m)}{\alpha^2}\right) - \left(\frac{m}{\theta}+\frac{3(1-m)}{2\alpha}\right)^2\right)^{3/2}} \tag{20}$$

$$Ku(X) = \frac{\frac{15}{2}\left(\frac{m}{\theta^4}+\frac{3(1-m)}{\alpha^4}\right) - 4(\frac{3m}{\theta^3}+\frac{15(1-m)}{2\alpha^3})(\frac{m}{\theta}+\frac{3(1-m)}{2\alpha}) + 6(\frac{3m}{2\theta^2}+\frac{3(1-m)}{\alpha^2})(\frac{m}{\theta}+\frac{3(1-m)}{2\alpha})^2 - 3(\frac{m}{\theta}+\frac{3(1-m)}{2\alpha})^4}{\left(\left(\frac{3m}{2\theta^2}+\frac{3(1-m)}{\alpha^2}\right) - \left(\frac{m}{\theta}+\frac{3(1-m)}{2\alpha}\right)^2\right)^2} \tag{21}$$

Below, Tables 1 and 2 show some results of the moments, variance, skewness, and kurtosis of the mixture of AD and SBAD.

Table 1. Some values of the first four moments, variance, skewness, and kurtosis of the mixture of the AD and SBAD for $\theta = 5$ and $\alpha = 1$.

m	μ_1'	μ_2'	μ_3'	μ_4'	variance	Sk	Ku
0	1.5	3	7.5	22.5	0.75	1.1547	5
0.1	1.37	2.706	6.752	20.251	0.829	1.0245	4.587
0.2	1.24	2.412	6.004	18.002	0.874	1.0339	4.418
0.3	1.11	2.118	5.257	15.753	0.885	1.1267	4.478
0.4	0.98	1.824	4.509	13.504	0.863	1.282	4.787
0.5	0.85	1.53	3.762	11.256	0.807	1.500	5.416
0.6	0.72	1.236	3.014	9.007	0.717	1.794	6.532
0 .7	0.59	0.942	2.266	6.758	0.593	2.207	8.5413
0.8	0.46	0.648	1.519	4.509	0.436	2.843	12.616
0.9	0.33	0.354	0.771	2.260	0.245	4.062	23.937
1	0.2	0.06	0.024	0.012	0.02	1.4142	6

Table 2. Some values of the first four moments, variance, skewness, and kurtosis of the mixture of the AD and SBAD for $\theta = 1$ and $\alpha = 5$.

m	μ_1'	μ_2'	μ_3'	μ_4'	variance	Sk	Ku
0	0.3	0.12	0.06	0.036	0.03	1.1547	5
0.1	0.37	0.258	0.354	0.782	0.121	4.0084	28.242
0.2	0.44	0.396	0.648	1.528	0.202	3.2468	17.963
0.3	0.51	0.534	0.942	2.275	0.273	2.7226	13.115
0.4	0.58	0.672	1.236	3.021	0.335	2.3503	10.396
0.5	0.65	0.81	1.53	3.768	0.387	2.0717	8.709
0.6	0.72	0.948	1.824	4.514	0.429	1.8567	7.605
0 .7	0.79	1.086	2.118	5.260	0.461	1.6891	6.871
0.8	0.86	1.224	2.412	6.007	0.484	1.5607	6.394
0.9	0.93	1.362	2.706	6.753	0.497	1.4686	6.115
1	1	1.5	3	7.5	0.5	1.4142	6

Table 1 shows that for $m > 0$ and $m < 1$, the largest value of skewness and kurtosis occur for $m = 0.9$ and the smallest value of skewness occur for $m = 0.1$ as $\theta = 5$ and $\alpha = 1$. Table 2 shows that for $m > 0$ and $m < 1$, the largest value of skewness and kurtosis occur for $m = 0.1$ and the smallest value of skewness and kurtosis occur for $m = 0.9$ as $\theta = 1$ and $\alpha = 5$. Also from Tables 1 and 2 we can observe that for $m = 0$, the skewness value is 1.1547 and the kurtosis value is 5, and for $m = 1$, the skewness value is 1.4142 and the kurtosis value is 6.

Mode and Median

The mode and median of the mixture of AD and SBAD can be obtained by solving the following equations:

$$Mode = 8\exp^{-2x\alpha}(1-m)x\alpha^3 - 8\exp^{-2x\alpha}(1-m)x^2\alpha^4 + 4\exp^{-2x\theta} m\theta^2 - 8\exp^{-2x\theta} mx\theta^3 = 0 \quad (22)$$

$$Median = m(1-(1+2x\theta)\exp^{-2x\theta}) + (1-m)(1+(-1-2x\alpha(1+x\alpha))\exp^{-2x\alpha}) = 0.5 \quad (23)$$

4 Reliability Analysis

The survival function of the mixture of AD and SBAD can be defined as follows:

$$Su(x) = m[1-F_1(x,\theta)] + (1-m)[1-F_2(x,\alpha)] \quad (24)$$

$$= m[1-(1-(1+2x\theta)\exp^{-2x\theta})] + (1-m)[1-(1+(-1-2x\alpha(1+x\alpha))\exp^{-2x\alpha})] \quad (25)$$

$$= m(1+2x\theta)\exp^{-2x\theta} - (1-m)(-1-2x\alpha(1+x\alpha))\exp^{-2x\alpha} \quad (26)$$

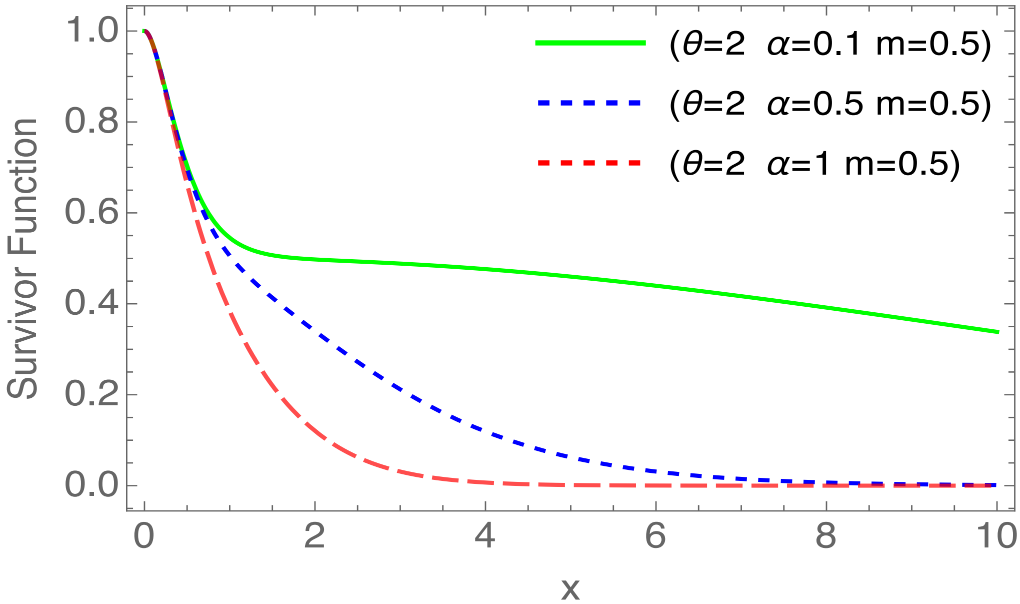

Figure 3. Survival function plots of the mixture of AD and SBAD with fixed values of m and θ, and varying values for α.

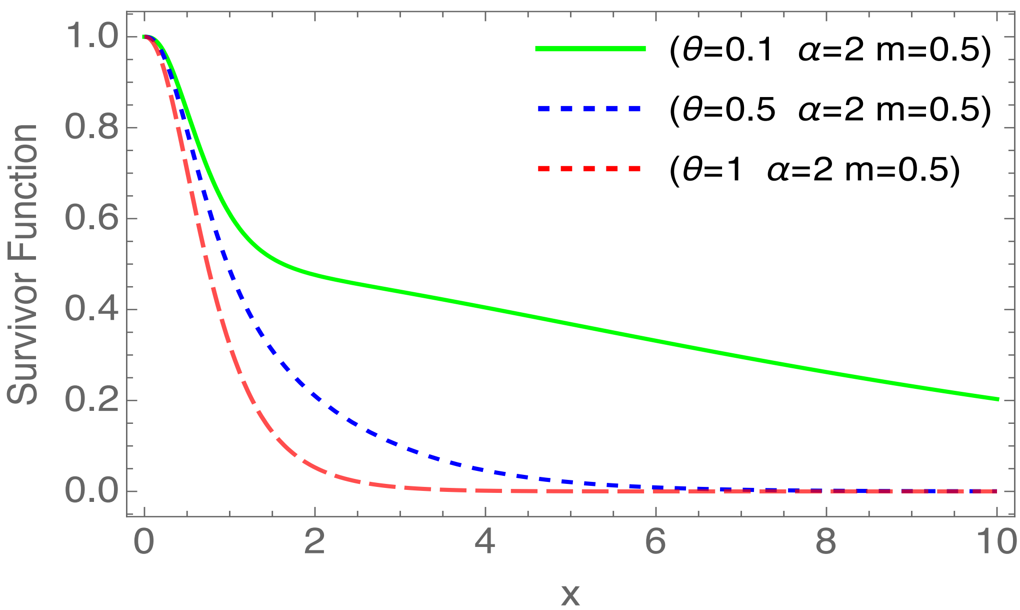

Figure 4. : Survival function plots of the mixture of AD and SBAD with fixed values of m and α, and varying values for θ.

The hazard function of the mixture of AD and SBAD can be defined as follows:

$$Hz(x) = \frac{f(x,\theta,\alpha,m)}{Su(x)} \tag{27}$$

$$= \frac{m(4x\theta^2 \exp^{-2x\theta}) + (1-m)(4x^2\alpha^3 \exp^{-2x\alpha})}{m(1+2x\theta)\exp^{-2x\theta} - (1-m)(-1-2x\alpha(1+x\alpha))\exp^{-2x\alpha}} \tag{28}$$

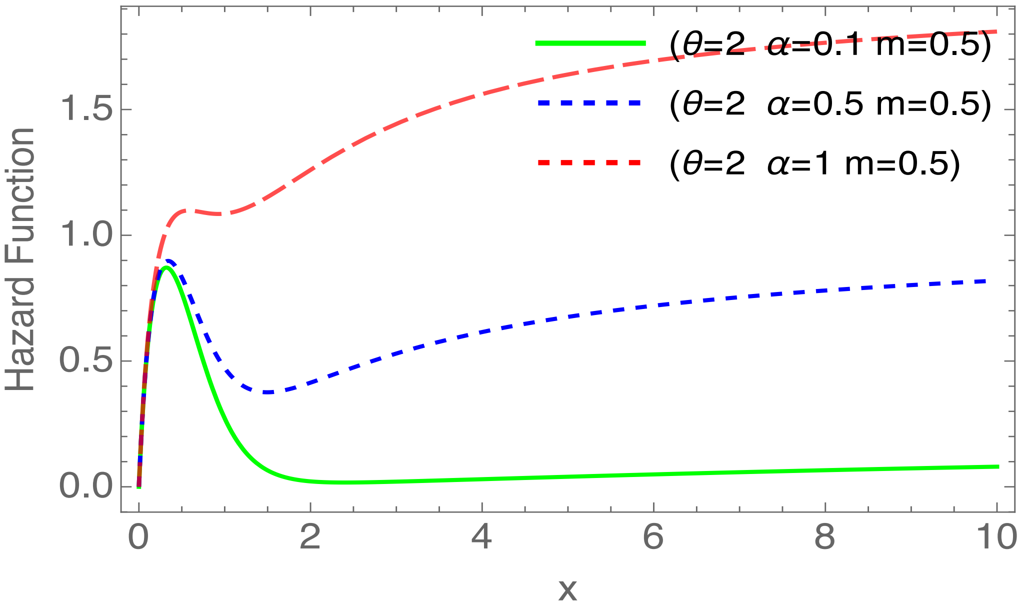

Figure 5. Hazard function plots of the mixture of AD and SBAD with fixed values of m and θ, and varying values for α.

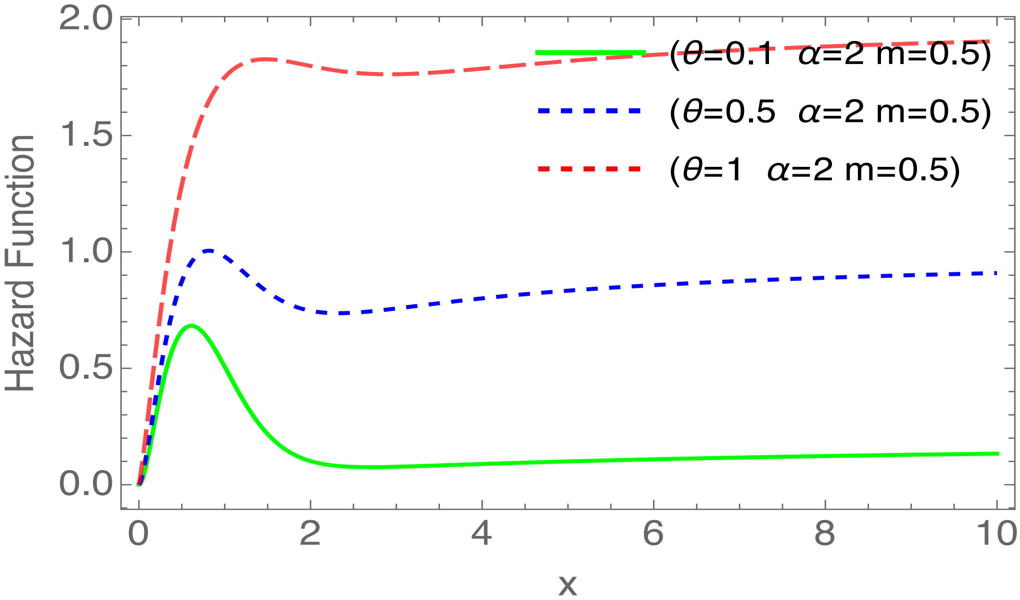

Figure 6. : Hazard function plots of the mixture of AD and SBAD with fixed values of m and α, and varying values for θ.

Figures 3 and 4 show different plots for the survival function of the mixture of AD and SBAD for fixed values of m. In both Figures 3 and 4 , the plots of the survival function show decreasing trends. Figures 5 and 6 show the different plots for the hazard function of the mixture of AD and SBAD with different behaviors.

5 Maximum Likelihood Estimation

In this section, we present the results of the parameters estimations m, θ, and α of the mixture of AD and SBAD by using the maximum likelihood approach. Let $X_1, X_2, ..., X_n$ be a random sample of size n from the mixture of AD and SBAD, then, the likelihood function is defined as:

$$L = \prod_{i=1}^{n} \left(m(4x_i\theta^2 \exp^{-2x_i\theta}) + (1-m)(4x_i^2\alpha^3 \exp^{-2x_i\alpha}) \right) \tag{29}$$

and the log-likelihood function is given by

$$\log L = \sum_{i=1}^{n} \log \left(m(4x_i\theta^2 \exp^{-2x_i\theta}) + (1-m)(4x_i^2\alpha^3 \exp^{-2x_i\alpha}) \right) \tag{30}$$

The MLEs of m, θ, and α of the mixture of AD and SBAD can be given by solving the following equations:

$$\frac{\partial \log L}{\partial m} = \sum_{i=1}^{n} \frac{\left(4x_i\theta^2 \exp^{-2x_i\theta} - 4x_i^2\alpha^3 \exp^{-2x_i\alpha} \right)}{\left(m(4x_i\theta^2 \exp^{-2x_i\theta}) + (1-m)(4x_i^2\alpha^3 \exp^{-2x_i\alpha}) \right)} = 0 \tag{31}$$

$$\frac{\partial \log L}{\partial \theta} = \sum_{i=1}^{n} \frac{-8mx_i\theta(-1+x_i\theta)\exp^{-2x_i\theta}}{\left(m(4x_i\theta^2 \exp^{-2x_i\theta}) + (1-m)(4x_i^2\alpha^3 \exp^{-2x_i\alpha}) \right)} = 0 \tag{32}$$

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{-4(1-m)x_i^2\alpha^2(2x_i\alpha-3)\exp^{-2x_i\alpha}}{\left(m(4x_i\theta^2 \exp^{-2x_i\theta}) + (1-m)(4x_i^2\alpha^3 \exp^{-2x_i\alpha}) \right)} = 0 \tag{33}$$

6 Application

In this section, we use the three models to fit a real-life dataset; namely, the proposed mixture of AD and SBAD and the regular distributions AD and SBAD. We use four different criteria as shown in Table 3 to find the best model to fit the real-life dataset that we use in this comparison. The real-life dataset was taken from [12]; it provides details on the failure times of 50 electronic components.

Table 3. Results of the parameters estimation, logL , AIC, BIC, and HQIC of the three models.

Distribution	Parameter estimate	logL	AIC	BIC	HQIC
AD	$\theta = 0.297358$	-146.691	295.383	297.295	296.111
SBAD	$\alpha = 0.446037$	-191.236	384.473	386.385	385.201
Mixture of AD and SBAD	$\theta = 3.06721$	-101.061	208.122	213.858	210.306
	$\alpha = 0.24907$				
	m= 0.46685				

Table 3 shows that there is a large difference in the values of logL, AIC, BIC, and HQIC of these three models. Besides, it is easy to observe that the lesser values of these four criteria are derived from the mixture of AD and SBAD, which indicates that the mixture of AD and SBAD outperforms AB and SBAD. Below, Figure 7 provides a visual representation of the results of the MLEs of m, θ, and α of the mixture of AD and SBAD.

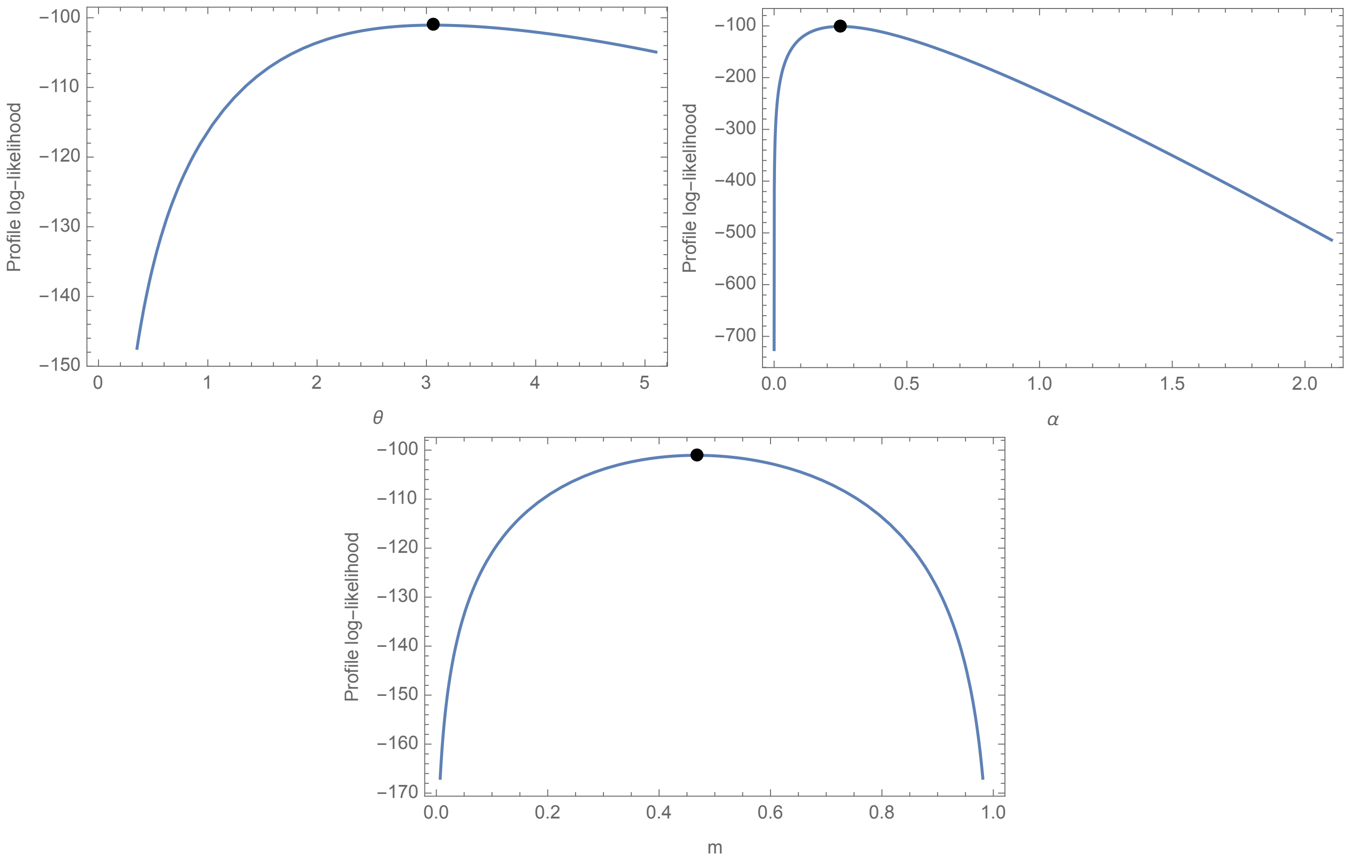

Figure 7. MLEs of m, θ, and α of the mixture of AD and SBAD

7 Conclusions

In this article, we introduce a new model: a mixture of AD and SBAD. We introduce and discuss the different statistical properties of the proposed mixture. The maximum likelihood method is used to estimate the mixture parameters. We use two statistical models in addition to the proposed mixture to fit the real-life dataset; the proposed mixture provided superior results compared to the other two models. In the future, we plan to extend the work on the proposed mixture by using Bayesian inference.

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