Mathematics and Statistics 12(1): 41-47, 2024 http://www.hrpub.org
DOI: 10.13189/ms.2024.120106

Complex Neutrosophic Fuzzy Set

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Received April 9, 2023; Revised November 28, 2023; Accepted December 14, 2023

Cite This Paper in the following Citation Styles

***(a)**: [1] V.KALAIYARASAN, K.MUTHUNAGAI, "Complex Neutrosophic Fuzzy Set," Mathematics and Statistics, Vol.12, No.1, pp. 41-47, 2024. DOI: 10.13189/ms.2024.120106*

***(b)**: V.KALAIYARASAN, K.MUTHUNAGAI (2024). Complex Neutrosophic Fuzzy Set. Mathematics and Statistics, 12(1), 41-47. DOI: 10.13189/ms.2024.120106*

Abstract Complex number system is an extension of the real number system which came into existence during the attempts to find a solution for cubic equations. A set characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one is called a Fuzzy set. A new development of Fuzzy system is a Complex Fuzzy system in which the membership function is complex- valued and the range of which is represented by the unit disk. The fuzzy similarity measure helps us to find the closeness among the Fuzzy sets. Due to the wide range of applications to various fields, Fuzzy Multi Criteria Decision Making (FMCDM) has gained its importance in Fuzzy set theory. A combination of Complex Fuzzy set, Fuzzy similarity measure and Fuzzy Multi Criteria Decision Making has resulted in this research contribution. In this article, we have introduced and investigated Complex neutrosophic fuzzy set, which involves complex- valued neutrosophic component. We have discussed two real life examples, one on selecting the best variety of a seed that gives the maximum yield and profit in a short period of time and another on choosing the best company to invest. Similarity measure between Complex neutrosophic fuzzy sets has been used to take a decision.

Keywords Complex Fuzzy Set, Complex Intuitionistic Fuzzy Set, Complex Neutrosophic Set, Complex Neutrosophic Fuzzy Set

Mathematical Subject Classification (2010): 30G35, 90C70

1 Introduction

It was Cardano [1], who introduced complex number $a + \sqrt{-b}$ into algebra. Later Euler $(1707 - 1783)$ introduced the notation $i = \sqrt{-1}$ and visualized complex numbers using rectangular coordinates. In 1796, Carl Friedrich Gauss gave a geometric representation for complex numbers and he introduced the term complex number. Caspar Wessel presented an article "On the Analytic Representation of Direction: An Attempt" in 1797, in which, he obtained a suitable representation for complex numbers.

A graph is a visual representation of data. The notion of graph theory was initially presented by Euler in 1736. In the annals of mathematics, the answer obtained by Euler to the well-known Konigsberg bridge issue has been recognised as the first theorem of Graph theory. Thereafter Polya, Denes Konig (1936), and Frank Harary (1969) developed Graph theory and its Fundamental Results.

Classical set theory has been extended to Fuzzy set theory whose elements have membership degree. Fuzzy means things that are vague or not clear. For details one can refer to [2],[3],[4]. It is an approach to computing, based on the degrees of truth. Membership function was not sufficient enough to describe the complexity of the object characteristics, and accordingly, an additional component called a non-membership function came into existence. By adding this new component Krassimir T. Atanassov, [5] proposed the extended form of fuzzy set, called "Intuitionistic fuzzy set"in 1986. The idea of Intuitionistic fuzzy set is more meaningful as well as intensive due to the presence of degrees of truth and falsity . The Fuzzy set and Intuitionistic fuzzy set could not handle indeterminate and inconsistent information. The idea of Neutrosophic set is a direct extension of the ideas of the conventional set, fuzzy set and Intuitionistic fuzzy set. The Neutrosophic set is identified using a truth membership function (T), an indeterminate membership function (I) and a false membership function (F) independently. The values of T, I and F are within the non-standard unit interval $]^{-}0, 1^{+}[$. In 1998, Smarandache [6] introduced the novel concept of Neutrosophic set.

Fuzzy membership grade is used to express inaccurate and inadequate or unsure information. Neutrosophic numbers are used to express indefinite and inconsistent information. But Fuzzy set or Neutrosophic set could not express both simultaneously. That is where the necessity of fuzzy Neutrosophic

numbers arose and in 2020, Sujit Das et al. [7] introduced Neutrosophic Fuzzy set. Neutrosophic fuzzy set is a combination of fuzzy set and neutrosophic set.
Fuzzy sets or intuitionistic fuzzy sets were unable to manage the periodicity of inaccurate, inconsistent, and unsure information. These theories have been useful to a variety of fields of science, but both sets have one key flaw: the inability to model two-dimensional events. In 2002, Ramot et al. [8] introduced a sophisticated fuzzy set namely, Complex Fuzzy Set (CFS) to overcome this challenge. A unit disc in a complex plane represents the range of complex membership grades in a CFS. Complex fuzzy sets and Intuitionistic fuzzy sets were not able to depict the partial ignorance of the data and its variations at a given phase of time during their execution. Besides, in our day-to-day life, uncertainty and vagueness of the data occur concurrently with changes to the phase (periodicity) of the data. In 2012, Alkouri et al. [9] introduced complex intuitionistic fuzzy sets to handle this complicated situation. It is an extension of the complex fuzzy set with a complex-valued non-membership grade. Complex intuitionistic fuzzy sets simultaneously deal with uncertainty and periodicity.
To overcome the difficulty of handling indeterminate and inconsistent information of the periodic nature, M.Ali et al.[10] and F. Smarandache et al. [11] proposed the concept of a complex Neutrosophic set in 2015. The concept of a complex Neutrosophic set is defined by complex valued membership functions representing the truth, indeterminacy and falsity. Aiyared Iampan et al. [12] have discussed about Anti-Intuitionistic Fuzzy Soft b-Ideals in BCK/BCI-Algebras and Nik Muhammad Farhan Hakim Nik Badrul Alam et al. [13] have discussed about Forecasting Model Based on Intuitionistic Fuzzy Sets with Fuzzy Time Series.

2 Preliminaries

This section consists of the definitions and the statments of few theorems, which we require for our work.
Fuzzy set[2]: A fuzzy set A in U(Universal set) is characterized by a membership function $\mu_A(v)$ which associates with each point in U, a real number in the interval $[0,1]$ with the values of $\mu_A(v)$ representing the grade of membership of v in A and is denoted by $A=\{v,\mu_A(v):v\in U\}$.
Intuitionistic Fuzzy set[5]: The membership function $\mu_A(v)$ was not adequate enough to explain the complexity of objects. So there was a necessity to introduce a non-membership function $\nu_A(v)$. The fuzzy set inclusive of the non-membership function with $0\leq\mu_A(v)+\nu_A(v)\leq 1, \quad \forall\, v\in U$ becomes Intuitionistic.
Neutrosophic Fuzzy set[7]: To express uncertainty, indeterminate and inconsistent information, an additional component called indeterminacy with $\{^{-}0 \leq \sup\mu_{T_A}(v)+\sup\mu_{I_A}(v)+\sup\mu_{F_A}(v)\leq 3^{+}, \quad \forall\, v\in U\}$ was proposed. Thus the Intutionistic Fuzzy set was made Neutrosophic by adding the component, indeterminacy.
Complex Fuzzy set[8]: Complex Fuzzy set is an extension of the fuzzy set, the range of which extends from a closed interval [0,1] to a disc of radius one in the complex plane. Here the membership function is complex valued given by, $\mu_A(v) = r_A(v)e^{i\omega_A(v)}$ where $v\in U,\ r_A(v)\in[0,1],\ \omega_A(v)\in[0,2\pi]$.
Complex Intuitionistic Fuzzy set[9]: Addition of a complex-valued non-membership grade $\nu_A(v)$, to a complex fuzzy set makes it a complex intuitionistic fuzzy set. Here $|\mu_A(v)+\nu_A(v)|\leq 1$, where $\mu_A(v)=r_A(v).e^{i\alpha_A(v)}$, and $\nu_A(v)=s_A(v).e^{i\beta_A(v)}, r_A(v)$ and $s_A(v)\in[0,1],\ \alpha_A(v)$ and $\beta_A(v)\in[0,2\pi],\forall\, v\in U$.
Complex Neutrosophic set[10]: An additional component of a complex-valued indeterminacy membership grade $I_A(v)$, to a complex intuitionistic set makes it a complex neutrosophic set. Here $T_A(v)=r_A(v).e^{i\mu_A(v)}$, $I_A(v)=s_A(v).e^{i\nu_A(v)}$, $F_A(v)=t_A(v).e^{i\omega_A(v)}$.where $r_A(v),s_A(v),t_A(v)\in[0,1]$ such that $^{-}0\leq r_A(v)+s_A(v)+t_A(v)\leq 3^{+}$ and $|T_A(v)+I_A(v)+F_A(v)|\leq 3$.
Operators on Fuzzy sets: A fuzzy operator blends two fuzzy sets to give a new fuzzy set. Equality, containment, complement, intersection and union are the most commonly used fuzzy operators. We now recall the definitions of the basic operators on Complex Fuzzy System which we need for our study.

Definition[8]: Let A_1 and A_2 be two Complex Fuzzy sets on U. Let their membership functions be
$\mu_A(v)=r_{A_1}(v)e^{i\omega_{A_1}(v)}$ and $\nu_A(v)=r_{A_2}(v)e^{i\omega_{A_2}(v)}$. Then

1. $A_1\cup A_2 = r_{A_1\cup A_2}(v).e^{i\omega_{A_1\cup A_2}(v)}$
   $=\max(r_{A_1}(v),r_{A_2}(v)).e^{i\max(\omega_{A_1}(v),\omega_{A_2}(v))},\forall\, v\in U.$
2. $A_1\cap A_2 = r_{A_1\cap A_2}(v).e^{i\omega_{A_1\cap A_2}(v)}$
   $=\min(r_{A_1}(v),r_{A_2}(v)).e^{i\min(\omega_{A_1}(v),\,\omega_{A_2}(v))},\forall\, v\in U.$
3. $A_1^c=\{v,1-r_{A_1}(v).e^{i\left(2\pi-\omega_{A_1}(v)\right)},\forall\, v\in U\}$, where A_1^c is the complement of A_1.
4. $A_1\circ A_2=\{v,(r_{A_1}(v).r_{A_2}(v)).e^{i2\pi\left(\omega_{\mu_1}(v).\omega_{\mu_2}(v)\right)},\forall\, v\in U\}$. Here $A_1\circ A_2$ denotes the composition of A_1 and A_2, where $\mu_1=\mu_{A_1/2\pi},\mu_2=\mu_{A_2/2\pi}$.

Definition[9]: Let A_1 and A_2 be two Complex Intuitionistic Fuzzy sets. Then

1. $A_1\cup A_2=\{v,(\mu_{A_1\cup A_2}(v)e^{i\alpha_{A_1\cup A_2}(v)},\nu_{A_1\cup A_2}(v)$ $e^{i\beta_{A_1\cup A_2}(v)}):\forall\, v\in U\}$ where
   $\mu_{A_1\cup A_2}(v)=\max(r_{A_1}(v),r_{A_2}(v))e^{i\max\left(\alpha_{A_1}(v),\alpha_{A_2}(v)\right)}$,
   $\nu_{A_1\cup A_2}(v)=\min(s_{A_1}(v),s_{A_2}(v))e^{i\min\left(\beta_{A_1}(v),\beta_{A_2}(v)\right)}$.
2. $A_1\cap A_2=\{(v,\mu_{A_1\cap A_2}(v)e^{i\alpha_{A_1\cap A_2}(v)},\nu_{A_2}(v)$ $e^{i\beta_{A_1\cap A_2}(v)}):\forall v\in U\}$ where
   $\mu_{A_1\cap A_2}(v)=\min(r_{A_1}(v),r_{A_2}(v))e^{i\min\left(\alpha_{A_1}(v),\alpha_{A_2}(v)\right)}$,
   $\nu_{A_1\cup A_2}(v)=\max(s_{A_1}(v),s_{A_2}(v))e^{i\max\left(\beta_{A_1}(v),\beta_{A_2}(v)\right)}$.
3. $A_1^c=\{(v,\nu_{A_1}(v)e^{i\left(2\pi-\beta_{A_1}(v)\right)},\mu_{A_1}(v)$ $e^{i\left(2\pi-\alpha_{A_1}(v)\right)}:v\in U\}$ where $\mu_{A_1^c}(v)=1-r_{A_1}(v)=$

$s_{A_1}(v) = \nu_{A_1}(v), \nu_{A_1^c}(v) = 1 - s_{A_1}(v) = r_{A_1}(v) = \mu_{A_1}(v)$.

Definition[10]: Let A_1 and A_2 be two Complex Neutrosophic sets. Then

1. $A_1 \cup A_2 = \{(v, T_{A_1\cup A_2}(v), I_{A_1\cup A_2}(v), F_{A_1\cup A_2}(v)) : v \in U\}$ where
$T_{A_1\cup A_2}(v) = \max(r_{A_1}(v), r_{A_2}(v)).e^{i\max\left(\mu_{A_1}(v),\mu_{A_2}(v)\right)}$,
$I_{A_1\cup A_2}(v) = \min(s_{A_1}(v), s_{A_2}(v)).e^{i\min\left(\nu_{A_1}(v),\nu_{A_2}(v)\right)}$,
$F_{A_1\cup A_2}(v) = \min(t_{A_1}(v), t_{A_2}(v)).e^{i\min\left(\omega_{A_1}(v),\omega_{A_2}(v)\right)}$.
2. $A_1 \cap A_2 = \{(v, T_{A_1\cap A_2}(v), I_{A_1\cap A_2}(v), F_{A_1\cap A_2}(v)) : v \in U\}$ where
$T_{A_1\cap A_2}(v) = \min(r_{A_1}(v), r_{A_2}(v)).e^{i\min\left(\mu_{A_1}(v),\mu_{A_2}(v)\right)}$,
$I_{A_1\cap A_2}(v) = \max(s_{A_1}(v), s_{A_2}(v)).e^{i\max\left(\nu_{A_1}(v),\nu_{A_2}(v)\right)}$,
$F_{A_1\cap A_2}(v) = \max(t_{A_1}(v), t_{B_2}(v)).e^{i\max\left(\omega_{A_1}(v),\omega_{A_2}(v)\right)}$.
3. $A_1 \circ A_2 = \{(v, T_{A_1\circ A_2}(v), I_{A_1\circ A_2}(v), F_{A_1\circ A_2}(v)) : v \in U\}$ where
$T_{A_1\circ A_2}(v) = r_{A_1\circ A_2}.e^{i\mu_{A_1\circ A_2}(v)}$
$= [r_{A_1}(v).r_{A_2}(v)].e^{\left(\mu_{A_1}(v).\mu_{A_2}(v)\right)}$,
$I_{A_1\circ A_2}(v) = s_{A_1\circ A_2}.e^{i\nu_{A_1\circ A_2}(v)}$
$= [s_{A_1}(v).s_{A_2}(v)].e^{i\left(\nu_{A_1}(v).\nu_{A_2}(v)\right)}$,
$F_{A_1\circ A_2}(v) = t_{A_1\circ A_2}e^{i\omega_{A_1\circ A_2}(v)}$
$= [t_{A_1}(v).t_{A_2}(v)].e^{i\left(\omega_{A_1}(v).\omega_{A_2}(v)\right)}$.
4. $A_1^c = \{(v, T_{\tilde{A_1}}(v), I_{\tilde{A_1}}(v), F_{\tilde{A_1}}(v)) : v \in U\}$ where
$T_{\tilde{A_1}}(v) = t_{A_1}(v).e^{i\left(2\pi-\omega_{A_1}(v)\right)}$,
$I_{\tilde{A_1}}(v) = 1 - s_{A_1}(v).e^{i\left(2\pi-\nu_{A_1}(v)\right)}$,
$F_{\tilde{A_1}}(v) = r_{A_1}(v).e^{i\left(2\pi-\mu_{A_1}(v)\right)}$ in which the amplitude terms are $r_{\tilde{A_1}}(v)) = t_{A_1}(v)$, $s_{\tilde{A_1}}(v)) = (1 - s_{A_1}(v))$, and $t_{\tilde{A_1}}(v) = r_{A_1}(v)$ and the phase terms are $\mu_{\tilde{A_1}}(v) = \mu_{A_1}(v), (2\pi - \mu_{A_1}(v))$ or $\mu_{A_1}(v) + \pi, \nu_{\tilde{A_1}}(v) = \nu_{A_1}(v), (2\pi - \nu_{A_1}(v))$ or $\nu_{A_1}(v) + \pi, \omega_{\tilde{A_1}}(v) = \omega_{A_1}(v), (2\pi - \omega_{A_1}(v))$ or $\omega_{A_1}(v) + \pi$.

3 Complex neutrosophic fuzzy set

3.1 Definition

A complex neutrosophic fuzzy set (CNFS) A defined on a Universe of discourse U is the one which is characterized by a complex-valued fuzzy membership, truth, indeterminacy, and falsity functions which are respectively denoted using $\mu_A(v), T_A(v,\mu), I_A(v,\mu)$ and $F_A(v,\mu), \forall\ v \in U$ and their modulus of the sum lying within the unit circle in the complex plane. They are defined by $\mu_A(v) = q_A(u).e^{i\omega_{\mu_A}(v)}$, $T_A(v) = r_A(v).e^{i\omega_{T_A}(v)}$, $I_A(u) = s_A(v).e^{i\omega_{I_A}(v),}$ and $F_A(v) = t_A(v).e^{i\omega_{F_A}(v)}$, where the complex membership values of the amplitude terms $\mu_A(v), r_A(v), s_A(v)$ and $t_A(v)$ and the phase terms $\omega_{\mu_A}(v), \omega_{T_A}(v), \omega_{I_A}(v), \omega_{F_A}(v)$are real valued and $q_A(v), r_A(v), s_A(v), t_A(v) \in [0,1]$ such that $^{-}0 \leq q_A(v) + r_A(v) + s_A(v) + t_A(v) \leq 4^{+}$. The complex neutrosophic fuzzy set A is represented using

$$A = \{(v, \mu_A(v) = z_\mu, T_A(v,\mu) = z_T, I_A(v,\mu) = z_I, F_A(v,\mu) = z_F) : v \in U\}$$

where

$$\mu_A(v) : U \to \{z_\mu : z_\mu \in \mathbb{C}, |z_\mu| \leq 1\},$$
$$T_A(v,\mu) : U \to \{z_T : z_T \in \mathbb{C}, |z_T| \leq 1\},$$
$$I_A(v,\mu) : U \to \{z_I : z_I \in \mathbb{C}, |z_I| \leq 1\},$$
$$F_A(v,\mu) : U \to \{z_F : z_F \in \mathbb{C}, |z_F| \leq 1\}.$$

and

$$|\mu_A(v) + T_A(v,\mu) + I_A(v,\mu) + F_A(v,\mu)| \leq 4$$

3.2 Numerical example of a complex neutrosophic fuzzy set

Let $Y = \{y_1, y_2, y_3, y_4\}$ be universe of discourse. Then A is a complex neutrosopic fuzzy set in Y given by

$$A = \frac{0.1.e^{i.0.2}, 0.2.e^{i.0.1}, 0.6.e^{i.0.8}, 0.7.e^{i.0.5}}{y_1} + \frac{0.6.e^{i.0.8}, 0.3.e^{i.3\pi/4}, 0.5.e^{i.0.3}, 0.7.e^{i.0.2}}{y_2} + \frac{0.7.e^{i.0}, 0.2.e^{i.0.9}, 0.1.e^{i.2\pi/3}, 0.2.e^{i.0.6}}{y_3} + \frac{0.9.e^{i.0.1}, 0.4.e^{i.\pi}, 0.7.e^{i.0.8}, 0.8.e^{i.0.3}}{y_4}$$

3.3 Set theoretic operations on a complex neutrosophic fuzzy set

3.3.1 Union and Intersection of complex neutrosophic fuzzy sets

Let the two complex neutrosophic fuzzy sets A_1 and A_2 be represented by

$$A_1 = \{(v), \mu_{A_1}(v) = z_{\mu_{A_1}}, T_{A_1}(v,\mu) = z_{T_{A_1}}, I_{A_1}(v,\mu) = z_{I_{A_1}}, F_{A_1}(u,\mu) = z_{F_{A_1}} : v \in U\} \text{ and}$$
$$A_2 = \{(v, \mu_{A_2}(v) = z_{\mu_{A_2}}, T_{A_2}(v,\mu) = z_{T_{A_2}}, I_{A_2}(v,\mu) = z_{I_{A_2}}, F_{A_2}(v,\mu) = z_{F_{A_2}} : v \in U\}.$$

Their Union and Intersection are given by $A_1 \cup A_2 = \{(v, \mu_{A_1\cup A_2}(v), T_{A_1\cup A_2}(v), I_{A_1\cup A_2}(v), F_{A_1\cup A_2}(v)) : v \in U\}$ and $A_1 \cap A_2 = \{(v, \mu_{A_1\cap A_2}(v), T_{A_1\cap A_2}(v), I_{A_1\cap A_2}(v), F_{A_1\cap A_2}(v)) : v \in U\}$ The fuzzy membership, truth, indeterminancy and falsity functions of union and intersection $\mu_{A_1\cup A_2}(v), T_{A_1\cup A_2}(v), I_{A_1\cup A_2}(v)$ and $F_{A_1\cup A_2}(v))$ are defined by

$$\mu_{A_1\cup A_2}(v) = [q_{A_1}(v) \vee q_{A_2}(v)].e^{i\omega_{\mu_{A_1\cup A_2}}(v)},$$
$$T_{A_1\cup A_2}(v) = [r_{A_1}(v) \vee r_{A_2}(v)].e^{i\omega_{T_{A_1\cup A_2}}(v)},$$
$$I_{A_1\cup A_2}(v) = [s_{A_1}(v) \wedge s_{A_2}(v)].e^{i\omega_{I_{A_1\cup A_2}}(v)},$$
$$F_{A_1\cup A_2}(v) = [t_{A_1}(v) \wedge t_{A_1}(v)].e^{i\omega_{F_{A_1\cup A_2}}(v)}.$$

and

$$\mu_{A_1\cap A_2 B}(v) = [q_{A_1}(v) \wedge q_{A_2}(v)].e^{i\omega_{\mu_{A_1\cap A_2}}(v)},$$
$$T_{A_1\cap A_2}(v) = [r_{A_1}(v) \wedge r_{A_2}(v)].e^{i\omega_{T_{A_1\cap A_2}}(v)},$$

$$I_{A_1 \cap A_2}(v) = [s_{A_1}(v) \vee s_{A_2}(v)].e^{i\omega_{I_{A_1 \cap A_2}}(v)},$$
$$F_{A_1 \cap A_2}(v) = [t_{A_1}(v) \vee t_{A_2}(v)].e^{i\omega_{F_{A_1 \cap A_2}}(v)}.$$

where $\vee$ and $\wedge$ signify the maximum and minimum operators respectively. To calculate phase term by following the properties (3.3.3)

3.3.2 Complement of complex neutrosophic fuzzy set

Let $A = \{(v, \mu_A(v), T(v,\mu), I(v,\mu), F(v,\mu)) : v \in U\}$ be a complex neutrosophic fuzzy set in U. The complement of the complex neutrosophic fuzzy set A is denoted by $c(A)$ and is defined by

$$c(A) = \{v, 1 - \mu_A(v), F_A(v,\mu), 1 - I_A(v,\mu), T_A(v,\mu)\}$$

3.3.3 Properties

Let A_1 and A_2 be two complex neutrosophic fuzzy sets in U with complex- valued fuzzy membership, truth, indeterminacy and falsity functions are $\{\mu_{A_1}(v), T_{A_1}(v,\mu), I_{A_1}(v,\mu), F_{A_1}(v,\mu)\}$, $\{\mu_{A_2}(v), T_{A_2}(v,\mu), I_{A_2}(v,\mu), F_{A_2}(v,\mu)\}$respectively. The union and intersection of the complex neutrosophic fuzzy sets A_1 and A_2 denoted by $A_1 \cup A_2$ and $A_1 \cap A_2$ are associated with the function

$$\varphi : \{(a, a_T, a_I, a_F) : a, a_T, a_I, a_F \in \mathbb{C}, |a + a_T + a_I + a_F| \leq 4, |a|, |a_T|, |a_I|, |a_F| \leq 1\} \times \{(b, b_T, b_I, b_F) : b, b_T, b_I, b_F \in \mathbb{C}, |b + b_T + b_I + b_F| \leq 4, |b|, |b_T|, |b_I|, |b_F| \leq 1\} \rightarrow \{(c, c_T, c_I, c_F) : c, c_T, c_I, c_F \in \mathbb{C}, |c + c_T + c_I + c_F| \leq 4, |c|, |c_T|, |c_I|, |c_F| \leq 1\},$$

where $a, b, c, a', b', c', a''b'', c'', a''', b''', c'''$ are the complex fuzzy membership and complex truth, indeterminacy and falsity memberships of $A_1, A_2, A_1 \cup A_2$ and $A_1 \cap A_2$ respectively. Hence by assigning complex values to φ, we have $\forall \ v \in U$

$$\varphi(\mu_{A_1}(v), \mu_{A_2}(v)) = \mu_{A_1 \cup A_2}(v) = z,$$
$$\varphi(T_{A_1}(v), T_{A_2}(v)) = T_{A_1 \cup A_2}(v) = z_T,$$
$$\varphi(I_{A_1}(v), I_{A_2}(v)) = I_{A_1 \cup A_2}(v) = z_I,$$
$$\varphi(F_{A_1}(v), F_{A_2}(v)) = F_{A_1 \cup A_2}(v) = z_F,$$
$$\varphi(\mu_{A_1}(v), \mu_{A_2}(v)) = \mu_{A_1 \cap A_2}(v) = z^*,$$
$$\varphi(T_{A_1}(v), T_{A_2}(v)) = T_{A_1 \cap A_2}(v) = z^*_T,$$
and
$$\varphi(I_{A_1}(v), I_{A_2}(v)) = I_{A_1 \cap A_2}(v) = z^*_I,$$
$$\varphi(F_{A_1}(v), F_{A_2}(v)) = F_{A_1 \cap A_2}(v) = z^*_F$$

Here the function φ must be following the axiomatic condition of Ref. [10].

3.3.4 Composition of complex neutrosophic fuzzy set

Let us consider two complex neutrosophic fuzzy sets A_1 and A_2 on U with $\mu_{A_1}(v) = q_{A_1}(v).e^{i\omega_{\mu_{A_1}}(v)}$, $T_{A_1}(v) = r_{A_1}(v).e^{i\omega_{T_{A_1}}(v)}$, $I_{A_1}(v) = s_{A_1}(v).e^{i\omega_{I_{A_1}}(v)}$, $F_{A_1}(v) = t_{A_1}(v).e^{i\omega_{F_{A_1}}(v)}$, and $\mu_{A_2}(v) = q_{A_2}(v).e^{i\omega_{\mu_{A_2}}(v)}$, $T_{A_2}(v) = r_{A_2}(v).e^{i\omega_{T_{A_2}}(v)}$, $I_{A_2}(v) = s_{A_2}(v).e^{i\omega_{I_{A_2}}(v)}$, $F_{A_2}(v) = t_{A_2}(v).e^{i\omega_{F_{A_2}}(v)}$ as their complex-valued fuzzy membership, truth membership,indeterminacy membership and falsehood membership functions. The complex-valued fuzzy membership, truth membership, indeterminacy membership and falsehood membership functions of the product of the complex neutrosophic fuzzy sets A_1 and A_2 denoted by $A_1 \circ A_2$ are,

$$\mu_{A_1 \circ A_2}(v) = q_{A_1 \circ A_2}(v).e^{i\omega_{\mu_{A_1 \circ A_2}}(v)} = \left(q_{A_1}(u).q_{A_2}(v)\right).e^{i2\pi\left(\omega_{\mu_1}(v).\omega_{\mu_2}(v)\right)}$$
$$T_{A_1 \circ A_2}(v) = r_{A_1 \circ A_2}(v).e^{i\omega_{T_{A_1 \circ A_2}}(v)} = \left(r_{A_1}(v).r_{A_2}(v)\right).e^{i2\pi\left(\omega_{T_1}(v).\omega_{T_2}(v)\right)}$$
$$I_{A_1 \circ A_2}(v) = s_{A_1 \circ A_2}(v).e^{i\omega_{I_{A_1 \circ A_2}}(v)} = \left(s_{A_1}(v).s_{A_2}(v)\right).e^{i2\pi\left(\omega_{I_1}(v).\omega_{I_2}(v)\right)}$$
$$F_{A_1 \circ A_2}(v) = t_{A_1 \circ A_2}(v).e^{i\omega_{F_{A_1 \circ A_2}}(v)} = \left(t_{A_1}(v).t_{A_2}(v)\right).e^{i2\pi\left(\omega_{F_1}(v).\omega_{F_2}(v)\right)}$$

where $\mu_1 = \mu_{A_1/2\pi}$, $\mu_2 = \mu_{A_2/2\pi}$, $T_1 = T_{A_1/2\pi}$, $T_2 = T_{A_2/2\pi}$, $I_1 = I_{A_1/2\pi}$, $I_2 = I_{A_2/2\pi}$, $F_1 = F_{A_1/2\pi}$, $F_1 = F_{A_2/2\pi}$.

3.4 Proposition

1. Let A_1, A_2, A_3 be complex neutrosophic fuzzy sets on U. Then
   (1) $(A_1 \cup A_2) \cap A_3 = (A_1 \cap A_2) \cup (A_1 \cap A_3)$
   (2) $(A_1 \cap A_2) \cup A_3 = (A_1 \cup A_2) \cap (A_1 \cap A_3)$
   (3) $(A_1 \cup A_2) \cap A_1 = A_1$
   (4) $(A_1 \cap A_2) \cup A_1 = A_1$
   (5)$(A_1 \cup A_1)^c = (A_1)^c \cap (A_2)^c$,
   (6)$(A_1 \cap A_2)^c = (A_1)^c \cup (A_2)^c$
   (8) $(A_1 \cup A_2) \cup A_3 = (A_1) \cup ((A_2 \cup A_3)$,
   (9)$(A_1 \cap A_2) \cap A_3 = (A_1) \cap ((A_2 \cap A_3)$

4 Similarity Measure on CNFS

In this section, the Hamming distance, Euclidean distance and the similarity measures between the CNFS have been discussed. Let the two CNFS $A_1 = \{v_i, \mu_{A_1}(v_i), T_{A_1}(v_i), I_{A_1}(v_i), F_{A_1}(v_i)\}$,and $A_2 = \{v_i, \mu_{A_2}(v_i), T_{A_2}(v_i), I_{A_2}(v_i), F_{A_2}(v_i)\}(i = 1, 2, 3.....n)$ be defined over the universe $U = \{v_1, v_2, v_3, ...v_n\}$. Following are the definitions of some of the distance measures between A_1 and A_2.

4.0.1 Definition

The Hamming Distance between two CNFS A_1 and A_2 is defined by

$$d_1(A_1, A_2) = \frac{1}{4} \sum_{v_i \in U} \Big[|\mu_{A_1}(v_i) - \mu_{A_2}(v_i)| + |T_{A_1}(v_i) - T_{A_2}(v_i)| + |I_{A_1}(v_i) - I_{A_2}(v_i)| + |F_{A_1}(v_i) - F_{A_2}(v_i)| \Big]$$

4.0.2 Definition

The Normalised Hamming Distance between A_1 and A_2 is given by

$$d_2(A_1, A_2) = \frac{1}{4n} \sum_{v_i \in U} \Big[|\mu_{A_1}(v_i) - \mu_{A_2}(v_i)| + |T_{A_1}(v_i) - T_{A_2}(v_i)| + |I_{A_1}(v_i) - I_{A_2}(v_i)| + |F_{A_1}(v_i) - F_{A_2}(v_i)| \Big]$$

4.0.3 Definition

The Euclidean Distance measure between A_1 and A_2 is defined to be

$$d_3(A_1, A_2) = \left[\frac{1}{4} \sum_{v_i \in U} \Big[|\mu_{A_1}(v_i) - \mu_{A_1}(v_i)|^2 + |T_{A_1}(v_i) - T_{A_2}(v_i)|^2 + |I_{A_1}(v_i) - I_{A_2}(v_i)|^2 + |F_{A_1}(v_i) - F_{A_2}(v_i)|^2 \Big] \right]^{1/2}$$

4.0.4 Definition

The Normalised Euclidean Distance between A_1 and A_2 is

$$d_4(A_1, A_2) = \left[\frac{1}{4n} \sum_{v_i \in U} \Big[|\mu_{A_1}(v_i) - \mu_{A_2}(v_i)|^2 + |T_{A_1}(v_i) - T_{A_2}(v_i)|^2 + |I_{A_1}(v_i) - I_{A_2}(v_i)|^2 + |F_{A_1}(v_i) - F_{A_2}(v_i)|^2 \Big] \right]^{1/2}$$

4.1 Properties

The distances d_j $(j = 1, 2, 3, 4)$ defined above between CNFS A_1 and A_2 satisfy the properties mentioned below (D2-D4):
(D_1) $d_j(A_1, A_2) \geq 0$;
(D_2) $d_j(A_1, A_2) = 0 \; iff \; A_1 = A_2$
(D_3) $d_j(A_1, A_2) = d_j(A_2, A_1)$;
(D_4) If $A_1 \subseteq A_2 \subseteq A_3$, A_3 is a CNFS in U then $d_j(A_1, A_3) \geq d_j(A_1, A_2)$ and $d_j(A_1, A_3) \geq d_j(A_2, A_3)$.

4.1.1 Proof

Easily we can verify properties $(D_1 - D_3)$, as similarity measures are simply generated from the distance measures. For the proof of (D_4) one can check [7] .Similarity measures $S_1(A_1, A_2)$ and $S_2(A_1, A_2)$ between CNFS A_1 and A_2 are given by

$$S_1(A_1, A_2) = 1 - \frac{1}{4n} \sum_{v_i \in U} \Big[|\mu_{A_1}(v_i) - \mu_{A_2}(v_i)| + |T_{A_1}(v_i) - T_{A_2}(v_i)| + |I_{A_1}(v_i) - I_{A_2}(v_i)| + |F_{A_1}(v_i) - F_{A_2}(v_i)| \Big]$$

$$S_2(A_1, A_2) = 1 - \left[\frac{1}{4n} \sum_{v_i \in U} \Big[|\mu_{A_1}(v_i) - \mu_{A_2}(v_i)|^2 + |T_{A_1}(v_i) - T_{A_2}(v_i)|^2 + |I_{A_1}(v_i) - I_{A_2}(v_i)|^2 + |F_{A_1}(v_i) - F_{A_2}(v_i)|^2 \Big] \right]^{1/2}$$

The similarity measure $S_k(A_1, A_2), (k = 1, 2)$ has the following properties

1. $0 \leq S_k(A_1, A_2) \leq 1$
2. $S_k(A_1, A_2) = 1 \iff A_1 = A_2$
3. $S_k(A_1, A_2) = S_k(A_2, A_1)$
4. If $A_1 \subseteq A_2 \subseteq A_3, A_3$ is CNFS in U, then $S_k(A_1, A_3) \leq S_k(A_1, A_2)$ and $S_k(A_1, A_3) \leq S_k(A_2, A_3)$.

4.1.2 Application of Complex Neutrosophic Fuzzy Set to MCDM

1. **Example**
Let us consider a stockholder who is willing to invest in a corporation. The stockholder has initially selected five corporations based on their changes of profits and losses periodically. The investor wants to select a company depending on certain criteria namely risk (C_1), Availability of green materials (C_2), Availability of labours (C_3), Market demand (C_4), Production quality (C_5) of the five industries. Automobile industry (I_1) has good profit in three to four months, Food manufacturing industry (I_2) has good profit in two to five months, Electronic manufacturing industry (I_3) has good profit in six to seven months, Oil industry (I_4) has good profit in six to eight months, and Pharmaceutical industry (I_5) has good profit in five to nine months. The company evaluation and criteria data were obtained from [7].
Table (1) contains the similarity measures between the five industries and five criteria. The ideal alternative (company) is computed from Table (1) as

a*= (0.0224, 0.0288, 0.0032, 0.0032) (0.0192, 0.0096, 0.0115, 0.0064) (0.0499, 0.0594, 0.0000, 0.0000) (0.0487, 0.0461 ,0.0448, 0.0192) (0.0512, 0.0448, 0.0128, 0.0192).
The similarity measures among the ideal alternative and the individual alternatives are obtained as

$\boldsymbol{S}_1(a*, I_1) = 0.9862, \boldsymbol{S}_1(a*, I_2) = 0.9880, \boldsymbol{S}_1(a*, I_3) = 0.9869, \boldsymbol{S}_1(a*, I_4) = 0.9867,$
$\boldsymbol{S}_1(a*, I_5) = 0.9890$

$\boldsymbol{S}_2(a*, I_1) = 0.9829, \boldsymbol{S}_2(a*, I_2) = 0.9847, \boldsymbol{S}_2(a*, I_3) = 0.9812, \boldsymbol{S}_2(a*, I_4) = 0.9810,$
$\boldsymbol{S}_2(a*, I_5) = 0.9848$

Hence in both the similarity measures, **Oil Company** is selected for the investment purpose within the periods and Figure 1 depicts the results obtained in example 1.

Table 1. Evaluation of Companies in accordance with the criteria

	C_1	C_2	C_3	C_4	C_5
I_1	(0.03, 0.022, 0.013, 0.010)	(0.010, 0.009, 0.013, 0.022)	(0.048, 0.030, 0.006, 0.012)	(0.038, 0.032, 0.058, 0.026)	(0.013, 0.013, 0.019, 0.051)
I_2	(0.019, 0.016, 0.026, 0.006)	(0.008, 0.005, 0.012, 0.025)	(0.048, 0.053, 0.000, 0.012)	(0.045, 0.038, 0.061, 0.029)	(0.019, 0.019, 0.032, 0.048)
I_3	(0.010, 0.029, 0.003, 0.003)	(0.019, 0.010, 0.019, 0.029)	(0.039, 0.015, 0.024, 0.030)	(0.049, 0.046 ,0.054 ,0.019)	(0.022, 0.019, 0.029, 0.050)
I_4	(0.016, 0.026 ,0.019, 0.010)	(0.003, 0.003, 0.026, 0.006)	(0.050, 0.059, 0.030, 0.000)	(0.042, 0.036, 0.051, 0.022)	(0.006, 0.006, 0.051, 0.038)
I_5	(0.022, 0.021 ,0.006 ,0.026)	(0.006, 0.006, 0.014, 0.021)	(0.030, 0.042, 0.024, 0.036)	(0.032, 0.026, 0.045, 0.038)	(0.051, 0.045, 0.013, 0.019)

Table 2. Evaluation of Seeds in accordance with the criteria

	C_1	C_2	C_3	C_4
s_1	(0.007 0.007 0.008 0.020)	(0.018 0.018 0.012 0.007)	(0.014 0.013 0.007 0.010)	(0.017 0.015 0.012 0.021)
s_2	(0.010 0.010 0.011 0.018)	(0.019 0.020 0.018 0.004)	(0.002 0.002 0.016 0.021)	(0.013 0.013 0.019 0.019)
s_3	(0.007 0.008 0.017 0.014)	(0.024 0.023 0.007 0.012)	(0.016 0.015 0.015 0.008)	(0.024 0.026 0.017 0.009)
s_4	(0.008 0.007 0.014 0.017)	(0.019 0.018 0.018 0.010)	(0.015 0.016 0.010 0.007)	(0.019 0.017 0.022 0.015)
s_5	(0.010 0.008 0.019 0.022)	(0.021 0.020 0.012 0.012)	(0.001 0.001 0.008 0.020)	(0.010 0.012 0.015 0.021)
s_6	(0.008 0.008 0.014 0.017)	(0.022 0.023 0.018 0.004)	(0.020 0.020 0.012 0.003)	(0.029 0.029 0.024 0.007)

Table 3. Evaluation of Seeds in accordance with the criteria

C_5	C_6	C_7	C_8	C_9
(0.026 0.031 0.016 0.005)	(0.006 0.006 0.005 0.003)	(0.001 0.001 0.001 0.027)	(0.024 0.023 0.012 0.001)	(0.029 0.027 0.013 0.009)
(0.027 0.027 0.013 0.009)	(0.005 0.004 0.004 0.005)	(0.001 0.000 0.001 0.023)	(0.025 0.026 0.007 0.002)	(0.030 0.030 0.016 0.006)
(0.029 0.033 0.009 0.004)	(0.009 0.008 0.010 0.005)	(0.001 0.001 0.001 0.021)	(0.022 0.021 0.013 0.010)	(0.027 0.027 0.024 0.010)
(0.011 0.009 0.024 0.027)	(0.016 0.016 0.007 0.004)	(0.008 0.007 0.002 0.017)	(0.025 0.026 0.001 0.002)	(0.029 0.029 0.020 0.008)
(0.016 0.016 0.026 0.011)	(0.014 0.015 0.007 0.007)	(0.000 0.000 0.002 0.020)	(0.021 0.022 0.022 0.006)	(0.031 0.030 0.016 0.006)
(0.022 0.024 0.020 0.013)	(0.014 0.015 0.005 0.008)	(0.002 0.001 0.001 0.012)	(0.023 0.023 0.012 0.004)	(0.033 0.034 0.013 0.003)

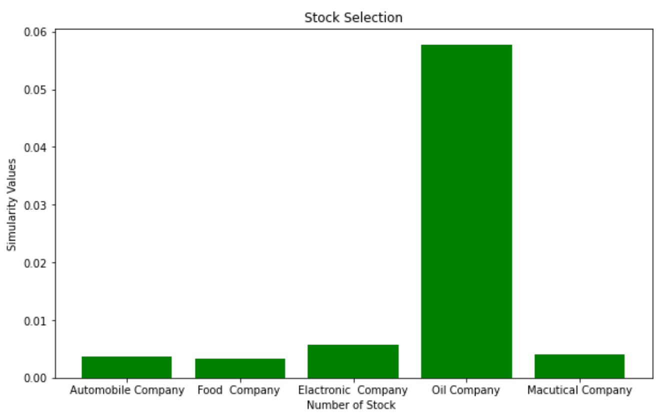

Figure 1. Best Stock Selection.

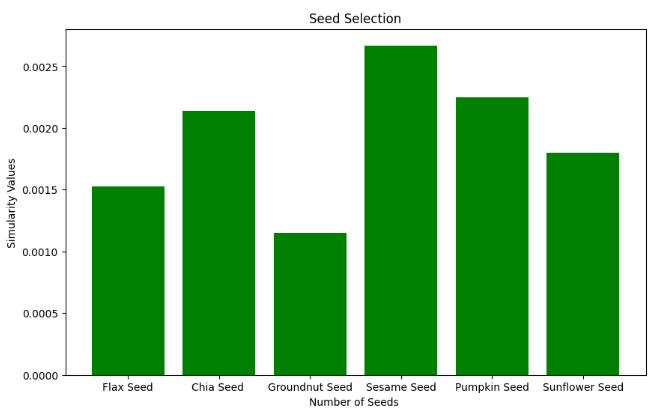

Figure 2. Best Seed Selection.

2. **Example** Let us consider a farmer willing to select a seed for farming in some period. He has initially selected six seeds based on their profit and loss as they change periodically due to the prevailing environmental conditions. He has chosen nine criteria like Sun light (C_1), Soil fertile (C_2), Humidity (C_3), Fertilizer (C_4), Pesticides (C_5), Pollination (C_6), Atmosphere (C_7), Genotype (C_8), Weed control (C_9). Let the six seeds be Flax Seed (s_1),Chia seed (s_2), Ground seed (s_3), Sesame seed (s_4), Pumpkin seed (s_5), Sunflower seed (s_6).
Tables (2) and (3) contain the similarity measures between the six seeds and nine criteria.The data used for computing similarity measures between the six seeds and nine criteria have been taken from https://www.agrifarming.in, https://greenplanet.eolss.net, https://www.sesco.net and some of articles [14],[15].

The ideal alternative (seeds) is obtained from the two tables as
a*= (0.0102 0.0099 0.0083 0.0138)(0.0237 0.0234 0.0069 0.0041(0.0295 0.0291 0.0120 0.0069)(0.0292 0.0329 0.0091 0.0037) (0.0159 0.0155 0.0044 0.0032) (0.0083 0.0069 0.0008 0.0124) (0.0254 0.0259 0.0014 0.0014) (0.0329 0.0340 0.0128 0.0026).
The similarity measures among the ideal alternative and the individual alternatives are obtained as

$\boldsymbol{S}_1(a*, s_1) = 0.9948$, $\boldsymbol{S}_1(a*, s_2) = 0.9938$, $\boldsymbol{S}_1(a*, s_3) = 0.9955$, $\boldsymbol{S}_1(a*, s_4) = 0.9942$, $\boldsymbol{S}_1(a*, s_5) = 0.9922$, $\boldsymbol{S}_1(a*, s_6) = 0.9968$.

$\boldsymbol{S}_2(a*, s_1) = 0.9933$, $\boldsymbol{S}_2(a*, s_2) = 0.9916$, $\boldsymbol{S}_2(a*, s_3) = 0.9943$, $\boldsymbol{S}_2(a*, s_4) = 0.9915$, $\boldsymbol{S}_2(a*, s_5) = 0.9899$,

$\boldsymbol{S}_2(a*, s_6) = 0.9950.$

Sesame seed is chosen for crop farming based on both similarity measures. It is the best crop that yields the maximum profit and Figure 2 represents the results of example 2.

5 Conclusions

After introducing and studying the properties of Complex Neutrosophic Fuzzy sets, we have applied the same into two real life Decision making problems and given a conclusion. We have also visualized the same using bar graphs.

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