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Communications to the Pseudo-Additive Probability Measure and the Induced Probability Measure Realized by $\bar{g}-Transform$

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Abstract The Theory of Pseudo-Additive Measures has been studied by analyzing and evaluating significant results. The system of pseudo-arithmetic operations (SPAO) $\{\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}},\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}}\}$ as a system generated by the generator $\bar{g}=\bar{g}_{a,r}$ is shown directly by taking results of Rybárik and Pap, but $\bar{g}-Transform$ is a further development of $g-calculus$. Using the meaning of entropy as a logarithmic measure in information theory. Through examples we present the relation between the $\bar{g}_{a,r}-entropy$ and the entropy, realized by the $\bar{g}-logarithmic\ function$, i.e. a $\bar{g}-Transformation$. The paper studies the construction of relationships between entropy and $\bar{g}-entropy$ supported by $\bar{g}-Transform$ and the connection with Shannon Entropy. For the pseudo-additive probabilistic measure $\left(\overline{\oplus}_{\bar{g}_{a,r}}-\mathrm{P}\right)-m$, using $\bar{g}-logarithmic\ \ function$ as well as in the system $\{\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}},\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}}\}$ generated by $\bar{g}$, the problem of modification of this measure by $\bar{g}-Transform$ is addressed. The modifications of the Pseudo-Additive Probability Measure $\left(\overline{\oplus}_{\bar{g}_{a,r}}-\mathrm{P}\right)-m$, $(m_{\bar{g}_{a,r}})$ and the Induced Probability Measure $(P_{\bar{g}_{a,r}})$ supported by $\bar{g}=\bar{g}_{a,r}-Transform$ are presented, showing the relationships between the two modifications of the Pseudo-Additive Probability Measure (PAPM) $(m_{\bar{g}_{a,r}})$ and the Induced Probability Measure (IPM) $(P_{\bar{g}_{a,r}})$. Further, the Bi-Pseudo-Integral for $\boldsymbol{f}_{\bar{g}_{a,r}}$ - $RMF_{\bar{g}_{a,r}}$ and the Lebesgue Integral are represented in a relationship.

Keywords Pseudo-Arithmetic Operations, Modified Function, Pseudo-Additive Measure, $\bar{g}_{a,r}-generator$, $\bar{g}_{a,r}-Transform$, $\bar{g}_{a,r}-entropy$

1. Introduction

The consistent SPAO generated by the generator g [11] has brought developments in the Theory of Pseudo-Additive Measures [2-4,7-8,30]. First, we introduce the PAO concept $\{\oplus,\odot,\ominus,\oslash\}$ on the interval $[0,\ +\infty]$ and then to the extended $\overline{\mathbb{R}}=[-\infty,+\infty]$ [2-4,7-8,11,13,15,18,30].

The binary operations $(\oplus,\odot)$ (pseudo-addition, pseudo-multiplication) by Mesiar and Rybárik [10] are respectively as PA $\oplus:[0,+\infty]^2\rightarrow[0,+\infty]$, PM $\odot:[0,+\infty]^2\rightarrow[0,+\infty]$ that [8,18,20] satisfy the system of axioms SA, ($\oplus$-A.1÷A.7) and ($\odot$-A.1÷ $A.7$). Also, nonlinear problems are treated by using PAO [2]. Using results of Aczél [1] are derived some elementary $\bar{g}=\bar{g}_{a,r}-functions$ as solutions of some functional equations. By applying the consistent SPAO [2,9] to $\bar{g}=\bar{g}_{a,r}-functions$ and their $\bar{g}-derivatives$ [2], a further development of $g-calculus$ [3,4] is obtained.

Therefore, $\bar{g}-functions$ are modifications of functions by $\bar{g}=\bar{g}_{a,r}-transform$.

We have developed a $\bar{g}-calculus$ for functions by studying the theory of Pseudo-Functional Analysis to pave the way for the composition of these functions with masses and $\bar{g}-Transformation$.

The binary operation PA, $\oplus$ on $[0,+\infty]$ that fulfils the SA (A1) ÷ (A7), is of the type $\oplus=\vee$ (max) or $\oplus=\oplus_g$, (i.e., the functions $g=g_{a,r}-generator$, $g_{a,r}(z)=a\cdot z^r, a>0$ [1,12,20] defined by the relation $z_1\oplus_{g_{a,r}} z_2=(z_1{}^r+z_2{}^r)^{1/r}$ for some values of $r\geq 1$, are the generator for PA $\oplus=\oplus_g=\oplus_{g_{a,r}}$).

Let $\bar{g}_{a,r}:[-\infty,+\infty]\longrightarrow[-\infty,+\infty]$ be a generator "continuous, monotone strictly increasing unbouded function" [11] of the PA, $\overline{\oplus}$ on $[-\infty,+\infty]$ with the specifications:

$$\bar{g}_{a,r}(0)=0_{\overline{\oplus}_{\bar{g}_{a,r}}},\ \bar{g}_{a,r}(1)=1_{\overline{\odot}_{\bar{g}_{a,r}}}, \bar{g}_{a,r}(+\infty)=+\infty,$$

and further, the convention $0\cdot(+\infty)=0$. For generator [11] $g:[0,+\infty]\longrightarrow[-\infty,+\infty]$ is presented an odd extension as:

$$\bar{g}=\bar{g}_{a,r}(z)=sgn\,z\cdot g_{a,r}(|z|),\ z\in[-\infty,\infty]).$$

$$\bar{g}-generators=\begin{cases}\bar{g}_{1,1}(z)=z, & for\ r=1, a=1\\ \bar{g}_{a,1}(z)=a\cdot z, & for\ r=1, a>0\\ \bar{g}_{1,r}(z)=z^r, & for\ r\geq 1, a=1\\ \bar{g}_{a,r}(z)=a\cdot z^r, & for\ r\geq 1, a>1\end{cases}$$

Then, generated by this generator $\bar{g}=\bar{g}_{a,r}$, the system of pseudo-arithmetical operations (SPAO) is said to be a *consistent* system [8] with the form of presentation as $\{\overline{\oplus},\overline{\odot},\overline{\ominus},\overline{\oslash}\}=\{\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}},\overline{\ominus}_{\bar{g}_{a,r}},\overline{\oslash}_{\bar{g}_{a,r}}\}$.

These extensions of SPAO and the definition of $g-calculus$ [3,4,6,7,14,18,20,22,23] lead us to the presentation of $\bar{g}_{a,r}-calculus$:

$$\{\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}},\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}}\}=$$

$$=\begin{cases}\{\overline{\oplus}_{\bar{g}_{1,1}},\overline{\odot}_{\bar{g}_{1,1}},\overline{\ominus}_{\bar{g}_{1,1}},\overline{\oslash}_{\bar{g}_{1,1}}\}\ for\ r=1, a=1\ (\overline{\oslash}_{\bar{g}_{1,1}}; z_2\neq 0)\\ \{\overline{\oplus}_{\bar{g}_{a,1}},\overline{\odot}_{\bar{g}_{a,1}},\overline{\ominus}_{\bar{g}_{a,1}},\overline{\oslash}_{\bar{g}_{a,1}}\}\ for\ r=1, a>0\ (\overline{\oslash}_{\bar{g}_{a,1}}; z_2\neq 0)\\ \{\overline{\oplus}_{\bar{g}_{1,r}},\overline{\odot}_{\bar{g}_{1,r}},\overline{\ominus}_{\bar{g}_{1,r}},\overline{\oslash}_{\bar{g}_{1,r}}\}\ for\ r\geq 1, a=1\ (\overline{\oslash}_{\bar{g}_{1,r}}; z_2\neq 0)\\ \{\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}},\overline{\ominus}_{\bar{g}_{a,r}},\overline{\oslash}_{\bar{g}_{a,r}}\}\ for\ r\geq 1, a>0\ (\overline{\oslash}_{\bar{g}_{a,r}}; z_2\neq 0)\end{cases}$$

$$\{\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}},\overline{\ominus}_{\bar{g}_{a,r}},\overline{\oslash}_{\bar{g}_{a,r}}\}\xrightarrow{\bar{g}(z)=\bar{g}_{1,1}(z)=z}\{(+),(\cdot),(-),(/)\}.$$

2. Materials and Methods

2.1. Modified Functions by $\bar{g}_{a,r}-Transform$

Definition 2.1.1. [2,9,11] *Let f be a function on $]a,b[\subseteq]-\infty,+\infty[$ and $\bar{g}_{a,r}$ be the generator of CSPAO,*

$$\{\overline{\oplus}_{\bar{g}},\overline{\odot}_{\bar{g}},\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}}\}=\{\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}},\overline{\ominus}_{\bar{g}_{a,r}},\overline{\oslash}_{\bar{g}_{a,r}}\}.$$

The modified function $f_{\bar{g}_{a,r}}$ given as $\bar{g}_{a,r}-Transform$,

$f_{\bar{g}_{a,r}}(z)=\bar{g}_{a,r}{}^{-1}\left(f\left(\bar{g}_{a,r}(z)\right)\right)$ *for every* $z\in\left(\bar{g}_{a,r}{}^{-1}(a),\bar{g}_{a,r}{}^{-1}(b)\right)$ *will be called* $\bar{g}_{a,r}-function$ *corresponding to* f.

Definition 2.1.2. [9,11] *Let f be a function on interval $]a,b[\subseteq]-\infty,+\infty[$ and $\bar{g}_{a,r}$ be the generator of CSPAO. The modified function $f_{\bar{g}_{a,r}}$ given by the transformation as $f_{\bar{g}_{a,r}}(z_1,z_2)=\bar{g}_{a,r}{}^{-1}\left(f\left(\bar{g}_{a,r}(z_1),\bar{g}_{a,r}(z_2)\right)\right)$ for every $z_1,z_2\in\left(\bar{g}_{a,r}{}^{-1}(a),\bar{g}_{a,r}{}^{-1}(b)\right)$ will be called $\bar{g}_{a,r}-function$ corresponding to the f.*

Definition 2.1.3. [2] *For the functional equation, a continuous modified function $f_{\bar{g}_{a,r}}$ which is a solution of it as in the presentation below:*

$$f_{\bar{g}_{a,r}}(z_1)\,\overline{\oplus}_{\bar{g}_{a,r}}\,f_{\bar{g}_{a,r}}(z_2)=f_{\bar{g}_{a,r}}\left(z_1\,\overline{\odot}_{\bar{g}_{a,r}}\,z_2\right)$$

and $f_{\bar{g}_{a,r}}\left(\bar{g}_{a,r}{}^{-1}(a)\right)=1$, where $a>0$, $a\neq 1$ will be called the $\bar{g}_{a,r}-logarithmic$ function and denoted by $f_{\bar{g}_{a,r}-a,log}$.

Definition 2.1.4. [2,11] *A continuous modified function $f_{\bar{g}}=f_{\bar{g}_{a,r}}$ as a solution of the functional equation in presentation below:*

$f_{\bar{g}_{a,r}}(z_1)\,\overline{\odot}_{\bar{g}_{a,r}}\,f_{\bar{g}_{a,r}}(z_2)=f_{\bar{g}_{a,r}}\left(z_1\,\overline{\oplus}_{\bar{g}_{a,r}}\,z_2\right)$, with values $r>0$, $z_1,z_2\in]-\infty,+\infty[$ will be appointed the $\bar{g}_{a,r}-power$ function, with the note $f_{\bar{g}_{a,r}-r,power}$. This function is given by:

$$f_{\bar{g}_{a,r}-r,power}(z)=\bar{g}_{a,r}{}^{-1}\left(\left(\bar{g}_{a,r}(z)\right)^r\right), r>0$$

(for $z<0$ $hold$ $\bar{g}_{a,r}(z)=-\bar{g}_{a,r}(-z)$).

2.2. The Role of $\bar{g}_{a,r}-transform$ for Modification of $\bar{g}_{a,r}-calculus$ and the Extensions of PAO $\left(\overline{\ominus}_{\bar{g}_{a,r}},\overline{\oslash}_{\bar{g}_{a,r}}\right)$

The PAO generated by the generator $\bar{g}_{a,r}$, expressed as $\bar{g}_{a,r}-$ modified function of the PAO $\{\overline{\oplus},\overline{\odot},\overline{\ominus},\overline{\oslash}\}$ by $\bar{g}_{a,r}-transform$ [6,9,11,12] according to the specific values of (a,r) are presented in the following cases.

C1. $Case\ a=1,\ r=1$:

$$\overline{\oplus}_{\bar{g}_{1,1}}(z_1,z_2)=\bar{g}_{1,1}{}^{-1}\left(\bar{g}_{1,1}(z_1)+\bar{g}_{1,1}(z_2)\right)==f_{\bar{g}_{1,1}-(+)}(z_1,z_2)$$

$$\overline{\odot}_{\bar{g}_{1,1}}(z_1,z_2)=\bar{g}_{1,1}{}^{-1}\left(\bar{g}_{1,1}(z_1)\cdot\bar{g}_{1,1}(z_2)\right)==f_{\bar{g}_{1,1}-(\cdot)}(z_1,z_2)$$

$$\overline{\ominus}_{\bar{g}_{1,1}}(z_1,z_2)=\bar{g}_{1,1}{}^{-1}\left(\bar{g}_{1,1}(z_1)-\bar{g}_{1,1}(z_2)\right)==f_{\bar{g}_{1,1}-(-)}(z_1,z_2)$$

$$\overline{\oslash}_{\bar{g}_{1,1}}(z_1,z_2)\bar{g}_{1,1}{}^{-1}\left(\bar{g}_{1,1}(z_1)/\bar{g}_{1,1}(z_2)\right)==f_{\bar{g}_{1,1}-(/)}(z_1,z_2).$$

C2. $Case\ a>0, r=1$:

$$\overline{\oplus}_{\bar{g}_{a,1}}(z_1,z_2)=\bar{g}_{a,1}{}^{-1}\left(\bar{g}_{a,1}(z_1)+\bar{g}_{a,1}(z_2)\right)==f_{\bar{g}_{a,1}-(+)}(z_1,z_2)$$

$$\overline{\odot}_{\bar{g}_{a,1}}(z_1,z_2)=\bar{g}_{a,1}{}^{-1}(\bar{g}_{a,1}(z_1)\cdot\bar{g}_{a,1}(z_2))= =f_{\bar{g}_{a,1}-(\cdot)}(z_1,z_2)$$

$$\overline{\ominus}_{\bar{g}_{a,1}}(z_1,z_2)=\bar{g}_{a,1}{}^{-1}(\bar{g}_{a,1}(z_1)-\bar{g}_{a,1}(z_2))=f_{\bar{g}_{a,1}-(-)}(z_1,z_2)$$

$$\overline{\oslash}_{\bar{g}_{a,1}}(z_1,z_2)=\bar{g}_{a,1}{}^{-1}(\bar{g}_{a,1}(z_1)/\bar{g}_{a,1}(z_2))= f_{\bar{g}_{a,1}-(/)}(z_1,z_2).$$

C3. $\boldsymbol{Case}\ a=1, r\geq 1$:

$$\overline{\oplus}_{\bar{g}_{1,r}}(z_1,z_2)=\bar{g}_{1,r}{}^{-1}(\bar{g}_{1,r}(z_1)+\bar{g}_{1,r}(z_2))= =f_{\bar{g}_{1,r}-(+)}(z_1,z_2)$$

$$\overline{\odot}_{\bar{g}_{1,r}}(z_1,z_2)=\bar{g}_{1,r}{}^{-1}(\bar{g}_{1,r}(z_1)\cdot\bar{g}_{1,r}(z_2))= =f_{\bar{g}_{1,r}-(\cdot)}(z_1,z_2)$$

$$\overline{\ominus}_{\bar{g}_{1,r}}(z_1,z_2)=\bar{g}_{1,r}{}^{-1}(\bar{g}_{1,r}(z_1)-\bar{g}_{1,r}(z_2))= =f_{\bar{g}_{1,r}-(-)}(z_1,z_2)$$

$$\overline{\oslash}_{\bar{g}_{1,r}}(z_1,z_2)=\bar{g}_{1,r}{}^{-1}(\bar{g}_{1,r}(z_1)/\bar{g}_{1,r}(z_2))= =f_{\bar{g}_{1,r}-(/)}(z_1,z_2).$$

C4. $Case\ a>0, r\geq 1$:

$$\overline{\oplus}_{\bar{g}_{a,r}}(z_1,z_2)=\bar{g}_{a,r}{}^{-1}(\bar{g}_{a,r}(z_1)+\bar{g}_{a,r}(z_2))= =f_{\bar{g}_{a,r}-(+)}(z_1,z_2)$$

$$\overline{\odot}_{\bar{g}_{a,r}}(z_1,z_2)=\bar{g}_{a,r}{}^{-1}(\bar{g}_{a,r}(z_1)\cdot\bar{g}_{a,r}(z_2))= =f_{\bar{g}_{a,r}-(\cdot)}(z_1,z_2)$$

$$\overline{\ominus}_{\bar{g}_{a,r}}(z_1,z_2)=\bar{g}_{a,r}{}^{-1}(\bar{g}_{a,r}(z_1)-\bar{g}_{a,r}(z_2))= =f_{\bar{g}_{a,r}-(-)}(z_1,z_2)$$

$$\overline{\oslash}_{\bar{g}_{a,r}}(z_1,z_2)=\bar{g}_{a,r}{}^{-1}(\bar{g}_{a,r}(z_1)/\bar{g}_{a,r}(z_2))= =f_{\bar{g}_{a,r}-(/)}(z_1,z_2).$$

2.3. The Extensions of the PAO $(\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}})$ treated by $\bar{g}_{a,r}-Transform$

The axiomatic concepts of the PAO $(\overline{\ominus}_{\bar{g}},\overline{\oslash}_{\bar{g}})=(\overline{\ominus}_{\bar{g}_{a,r}},\overline{\oslash}_{\bar{g}_{a,r}})$ [2,6,9,11] and the problems of their extensions, we are treating again supported by $\bar{g}_{a,r}-functions$, the functions f as solutions of parameterized Linear and $f_{\bar{g}_{a,r}}$ for Pseudo-Linear Functional Equation [9] (respectively, by $\bar{g}_{a,r}-transform$ we can handle the classes I and IV [9]):

$$z_1\,\overline{\ominus}_{\bar{g}_{a,r}}\,z_2= z_1\,\overline{\oplus}_{\bar{g}_{a,r}}\,(-z_2)=$$

$$=\bar{g}_{a,r}{}^{-1}(\bar{g}_{a,r}(z_1)-\bar{g}_{a,r}(z_2))=$$

$$=\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}\left(\left(\bar{g}_{a,r}(z_1)+\left(-\bar{g}_{a,r}(z_2)\right)\right)\right)\right)=$$

$$=\bar{g}_{a,r}{}^{-1}\left(\bar{g}_{a,r}(z_1)+\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}\left(-\bar{g}_{a,r}(z_2)\right)\right)\right)=$$

$$=\bar{g}_{a,r}{}^{-1}\left(\bar{g}_{a,r}(z_1)+\bar{g}_{a,r}\left(\boldsymbol{f}_{\bar{g}_{a,r}}(z_2)\right)\right)=$$

$$=z_1\,\overline{\oplus}_{\bar{g}_{a,r}}\,f_{\bar{g}_{a,r}}(z_2).$$

$$z_1\,\overline{\oslash}_{\bar{g}_{a,r}}\,\boldsymbol{z_2}=\bar{g}_{a,r}{}^{-1}\left(\frac{\bar{g}_{a,r}(z_1}{\bar{g}_{a,r}(z_2)}\right)=\bar{g}_{a,r}{}^{-1}\left(\bar{g}_{a,r}(z_1)\cdot\frac{1}{\bar{g}_{a,r}(z_2)}\right)=$$

$$=\bar{g}_{a,r}{}^{-1}\left(\bar{g}_{a,r}(z_1)\cdot\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}\left(\frac{1}{\bar{g}_{a,r}(z_2)}\right)\right)\right)=$$

$$=\bar{g}_{a,r}{}^{-1}\left(\bar{g}_{a,r}(z_1)\cdot\bar{g}_{a,r}\left(\boldsymbol{f}_{\bar{g}_{a,r}}(z_2)\right)\right)= z_1\,\overline{\odot}_{\bar{g}_{a,r}}\,f_{\bar{g}_{a,r}}(z_2).$$

2.4. Modified Pseudo-Additive Measure by $\bar{g}_{a,r}-Transform$

The meanings of the modifications of our set functions $(\overline{\oplus}_{\bar{g}_{a,r}}-\text{P})-\text{m}$ are given by categorizing these modifications in expectional cases.

We consider a set $X, X\neq\emptyset$ and $\mathcal{A}$ as $\sigma-algebra$ of sets that are included in X [3,5,6,8,15,17,20,22].

From a ***set function*** $m:\mathcal{A}\longrightarrow[0,+\infty]$ and for sequences $(A_i)_{n\in\mathbb{N}}$ of sets that are pairwise disjoint, will be named $\overline{\oplus}_{\bar{g}_{a,r}}$ - ***measure,*** $(\overline{\oplus}_{\bar{g}_{a,r}}-m)$ if holds two conditions $(\text{C}\mathbf{1}.\,m(\emptyset)=0)$ and $\left(\text{C}\mathbf{2}.\ m(\cup_{i=1}^{\infty}A_i)=\overline{\oplus}_{i=1}^{\infty}\,m(A_i)\right)$ [9,11].

Therefore, we write the form as below,

$\overline{\oplus}_{i=1}^{n}\ a_i\ =a_1\,\overline{\oplus}_{\bar{g}_{a,r}}\,a_2\,\overline{\oplus}_{\bar{g}_{a,r}}\ \ldots\overline{\oplus}_{\bar{g}_{a,r}}\,a_n$ and $\overline{\oplus}_{i=1}^{\infty}\,a_i\ =sup_n\,(\overline{\oplus}_{i=1}^{n}\ a_i)$. If the PAO, $\overline{\oplus}_{\bar{g}_{a,r}}$ is an idempotent ($\overline{\oplus}_{\bar{g}_{a,r}}-ID$), then we can omit the condition (1) and disjointness of sets [11].

Based on the definition of the $\left(\overline{\oplus}_{\bar{g}_{a,r}}-m\right)-measure$ and the $\bar{g}-function$ [2-3,5,9,10,11,16,18,19] and also on the consistent SPAO generated by $\bar{g}_{a,r}$ and $\bar{g}_{a,r}-transform$, we can modify the measure.

Definition 2.4.1. [9,11] *Let* m *be a set function* $m:\mathcal{A}\longrightarrow[0,+\infty]$ *also,* $\bar{g}_{a,r}$ *be the generator of the SPAO* $\{\overline{\oplus},\overline{\odot},\overline{\ominus},\overline{\oslash}\}$. *The function* $m_{\bar{g}_{a,r}}$ *is given by* $m_{\bar{g}_{a,r}}(A)=\bar{g}_{a,r}{}^{-1}\left(m\left(\bar{g}_{a,r}(A)\right)\right)$ *for each set* $A, A\in\left\{\bar{g}_{a,r}{}^{-1}(A_1),\ldots,\bar{g}_{a,r}{}^{-1}(A_n)\right\}$ *is called to be the* $\bar{g}_{a,r}-\left(\overline{\oplus}_{\bar{g}_{a,r}}-m\right)$ *measure function , for short-form as* $\left(m_{\bar{g}_{a,r}}\right)$*, corresponding to the set function* m*.*

By the $\bar{g}_{a,r}-calculus$, $\bar{g}_{a,r}-Transform$ and the definition of the $(\overline{\oplus}_{\bar{g}_{a,r}}-m)-measure$, we take [2,4,6,9,10]:

$$m(C)\,\overline{\oplus}_{\bar{g}_{a,r}}\,m(D)=\bar{g}_{a,r}{}^{-1}\left(m\left(\bar{g}_{a,r}(C)+\bar{g}_{a,r}(D)\right)\right)=$$

$$=\bar{g}^{\,-1}\left((+)\left(\bar{g}_{a,r}(m(C)),\bar{g}_{a,r}(m(D))\right)\right)=$$

$$=\boldsymbol{f}_{\bar{g}_{a,r}-(+)}\big(m(C),m(D)\big).$$

Moreover, by the definition of the $(\overline{\oplus}_{\bar{g}_{a,r}}-m)-measure$ and the $\bar{g}_{a,r}-calculus$ [9] we can write the relations below:

- $\boldsymbol{m}_{\bar{g}_{a,r}}(C\cup D)=\bar{g}_{a,r}{}^{-1}\left(\boldsymbol{m}\left(\bar{g}_{a,r}(C\cup D)\right)\right)=$

$= \bar{g}_{a,r}{}^{-1}\left(m\left(\bar{g}_{a,r}(C) \cup \bar{g}_{a,r}(D)\right)\right).$

- $\boldsymbol{m}\left(\bar{g}_{a,r}(C) \cup \bar{g}_{a,r}(D)\right) =$

  $= \boldsymbol{m}\left(\bar{g}_{a,r}(C)\right) \overline{\oplus}_{\bar{g}_{a,r}} \boldsymbol{m}\left(\bar{g}_{a,r}(D)\right) =$

  $= \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C)\right) \overline{\oplus}_{\bar{g}_{a,r}} \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(D)\right).$

- $\bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C \cup D)\right) =$

  $= \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C)\right) \overline{\oplus}_{\bar{g}_{a,r}} \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(D)\right).$

- $\boldsymbol{m}_{\bar{g}_{a,r}}(C \cup D) =$

  $= \bar{g}_{a,r}{}^{-1}\left(\bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C)\right) \overline{\oplus}_{\bar{g}_{a,r}} \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(D)\right)\right) =$

  $= f_{\bar{g}_{a,r}-(\overline{\oplus}\bar{g}_{a,r})}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C), \boldsymbol{m}_{\bar{g}_{a,r}}(D)\right).$

- $\boldsymbol{m}\left(\bar{g}_{a,r}(C)\right) \overline{\oplus}_{\bar{g}_{a,r}} m\left(\bar{g}_{a,r}(D)\right) = \boldsymbol{m}\left(\bar{g}_{a,r}(C \cup D)\right)$

  or $m\left(\bar{g}_{a,r}(C)\right) \overline{\oplus}_{\bar{g}_{a,r}} m\left(\bar{g}_{a,r}(D)\right) =$

  $= m\left(\bar{g}_{a,r}(C) \cup \bar{g}_{a,r}(D)\right).$

- $\boldsymbol{m}_{\bar{g}_{a,r}}(C \cup D) = f_{\bar{g}_{a,r}-m}(C \cup D) =.$

  $= f_{\bar{g}_{a,r}-(\overline{\oplus}\bar{g}_{a,r})}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C), \boldsymbol{m}_{\bar{g}_{a,r}}(D)\right)$

Proposition 2.4.2. [11] *Let $(X, \mathcal{A}, m)$ be a $\overline{\oplus}_{\bar{g}_{a,r}}-$ measure space. For a real measurable function $\boldsymbol{f}_{\bar{g}_{a,r}}$ - $RMF_{\bar{g}_{a,r}}$, the integral with respect to a $\overline{\oplus}_{\bar{g}_{a,r}}-$ measure m, for $\overline{\oplus}_{\bar{g}}=\overline{\oplus}_{\bar{g}_{a,r}} \neq \vee(max)$ (if the integral $\int_{\bar{g}^{-1}(X)}^{(\overline{\oplus}\bar{g}_{a,r}, \overline{\odot}\bar{g}_{a,r})}$ is defined) is given by the Bi-Pseudo-integral (**BP**):*

$$\left(\boldsymbol{BP}_{(\bar{g}_{a,r}-TR)} - \int_{\bar{g}_{a,r}{}^{-1}(X)}^{(\overline{\oplus}\bar{g}_{a,r}, \overline{\odot}\bar{g}_{a,r})}\right)(m_{\bar{g}_{a,r}}, \boldsymbol{f}_{\bar{g}_{a,r}} - RMF_{\bar{g}_{a,r}}) =$$

$$= \bar{g}_{a,r}{}^{-1}\left(\left(\boldsymbol{L} - \int_{\bar{g}_{a,r}{}^{-1}(X)}^{(+,\ \cdot\ )}\right)(\bar{g}_{a,r} \circ m_{\bar{g}_{a,r}}, \bar{g}_{a,r} \circ \boldsymbol{f}_{\bar{g}_{a,r}} - RMF_{\bar{g}_{a,r}})\right)$$

where the integral inside the brackets to the right side of this form:

$$\left(\boldsymbol{L} - \int_{\bar{g}_{a,r}{}^{-1}(X)}^{(+,\ \cdot\ )}\right)(\bar{g}_{a,r} \circ m_{\bar{g}_{a,r}}, \bar{g}_{a,r} \circ \boldsymbol{f}_{\bar{g}_{a,r}} - RMF_{\bar{g}_{a,r}})$$

is the Lebesgue integral also, the $\bar{g}_{a,r} \circ m_{\bar{g}_{a,r}}$ is the Lebesgue measure.

$$\left(\boldsymbol{BP}_{(\bar{g}_{a,r}-TR)} - \int_{\bar{g}_{a,r}{}^{-1}(X)}^{(\overline{\oplus}\bar{g}_{a,r}, \overline{\odot}\bar{g}_{a,r})}\right)(m_{\bar{g}_{a,r}}, \boldsymbol{f}_{\bar{g}_{a,r}}{}^{+}) \overline{\odot}_{\bar{g}_{a,r}}$$

$$\overline{\odot}_{\bar{g}_{a,r}}\left(\boldsymbol{BP}_{(\bar{g}_{a,r}-TR)} - \int_{\bar{g}_{a,r}{}^{-1}(X)}^{(\overline{\oplus}\bar{g}_{a,r}, \overline{\odot}\bar{g}_{a,r})}\right)(m_{\bar{g}_{a,r}}, \boldsymbol{f}_{\bar{g}_{a,r}}{}^{-}) =$$

$$= \left(\int_{\bar{g}_{a,r}{}^{-1}(X)}^{(\overline{\oplus}\bar{g}_{a,r}, \overline{\odot}\bar{g}_{a,r})} f_{\bar{g}_{a,r}}{}^{+} \overline{\odot}_{\bar{g}_{a,r}} dm_{\bar{g}_{a,r}}\right) \overline{\odot}_{\bar{g}_{a,r}}$$

$$\overline{\odot}_{\bar{g}_{a,r}}\left(\int_{\bar{g}_{a,r}{}^{-1}(X)}^{(\overline{\oplus}\bar{g}_{a,r}, \overline{\odot}\bar{g}_{a,r})} f_{\bar{g}_{a,r}}{}^{-} \overline{\odot}_{\bar{g}_{a,r}} dm_{\bar{g}_{a,r}}\right) =$$

$$= \left(\boldsymbol{BP}_{(\bar{g}_{a,r}-TR)} - \int_{\bar{g}_{a,r}{}^{-1}(X)}^{(\overline{\oplus}\bar{g}_{a,r}, \overline{\odot}\bar{g}_{a,r})}\right)(m_{\bar{g}_{a,r}}, \boldsymbol{f}_{\bar{g}_{a,r}} - RMF_{\bar{g}_{a,r}})$$

Therefore, the Bi-Pseudo-Integral for $\boldsymbol{f}_{\bar{g}_{a,r}}$ - $RMF_{\bar{g}_{a,r}}$ and the Lebesgue Integral relationship is presented in the form below:

$$\left(\boldsymbol{BP}_{(\bar{g}_{a,r}-TR)} - \int_{\bar{g}_{a,r}{}^{-1}(X)}^{(\overline{\oplus}\bar{g}_{a,r}, \overline{\odot}\bar{g}_{a,r})}\right)(m_{\bar{g}_{a,r}}, \boldsymbol{f}_{\bar{g}_{a,r}} - RMF_{\bar{g}_{a,r}}) =$$

$$= \bar{g}_{a,r}{}^{-1}\left(\left(\boldsymbol{L} - \int_{\bar{g}_{a,r}{}^{-1}(X)}^{(+,\ \cdot\ )}\right)(f \circ \bar{g}_{a,r}, m \circ \bar{g}_{a,r} - RMF)\right).$$

We will take into consideration the $(\overline{\oplus}_{\bar{g}_{a,r}} - \mathrm{P}) - m$ [6,16,19,24,25], the definition 2.1.1. for the $\bar{g}_{a,r} - function$ [2,13,17,19,25] and the system of the PAO generated [2,6,9,11] by the generator $\bar{g}_{a,r}$ (where the unbounded odd function $\bar{g}_{a,r}$ is continuous, monotone, strictly increasing). For more, we have modified [9,11] the measure (continuous Archimedean t-conorm) and the probability measure by $\bar{g}_{a,r} - \mathrm{transforms}$. The entropy and the study of the $\bar{g}_{a,r} - entropy$ [4] for the decomposable probability measure $\overline{\oplus}_{\bar{g}_{a,r}} - DPM$ further, are more motivated by the generators' role as $\bar{g}_{a,r} - functions$ and several relationships have been identified by $\bar{g}_{a,r} - transforms$ [2,9,11,19]. The problem of measure modification by $\bar{g}_{a,r} - transform$ is raised reasonably, and some relations between the $m_{\bar{g}_{a,r}}$ and $P_{\bar{g}_{a,r}}$ are given [9].

Definition 2.4.3. [9] *Let P be a IPM $(P = \bar{g}_{a,r} \circ m)$ and $\bar{g}_{a,r}$ be a generator of the consistent SPAO, $\{\overline{\oplus}_{\bar{g}_{a,r}}, \overline{\odot}_{\bar{g}_{a,r}}, \overline{\ominus}_{\bar{g}_{a,r}}, \overline{\oslash}_{\bar{g}_{a,r}}\}$. The function $P_{\bar{g}_{a,r}}$ given by:*

$P_{\bar{g}_{a,r}}(A) = \bar{g}_{a,r}{}^{-1}\left(P\left(\bar{g}_{a,r}(A)\right)\right)$ for each set $A \in \{\bar{g}_{a,r}{}^{-1}(A_1), \dots, \bar{g}_{a,r}{}^{-1}(A_n)\}$ is said to be modified function $\bar{g}_{a,r} - induced\ probability\ measure$, $IPM\ (P_g)$ which corresponds to the set function $P\ (P = \bar{g}_{a,r} \circ m)$ [16,21,24].

Proposition 2.4.4. [9] *Let B be a finite collections and $B_{\bar{g}_{a,r}} \subset \mathcal{A}$ with conditions of definition 2.4.1. respectively. The modified probability measure $(\overline{\oplus}_{\bar{g}_{a,r}} - P) - m$ and IPM on $\mathcal{A}$, satisfy by $\overline{\oplus}_{\bar{g}_{a,r}} - transform$ the following terms:*

1. $\boldsymbol{m}_{\bar{g}_{a,r}}(C) = \bar{g}_{a,r}{}^{-1}\left(\boldsymbol{P}_{\bar{g}_{a,r}}(C)\right)$
2. $\boldsymbol{P}_{\bar{g}_{a,r}}(C \cup D) = \boldsymbol{P}_{\bar{g}_{a,r}}(C) \overline{\oplus}_{\bar{g}_{a,r}} \boldsymbol{P}_{\bar{g}_{a,r}}(D)$
3. $\bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C \cup D)\right) =$
   $= \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C)\right) \overline{\oplus}_{\bar{g}_{a,r}} \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(D)\right).$

Application 2.4.5: For the composite function $h = \boldsymbol{t} \circ m$ we can apply [9-11] the $\bar{g}_{a,r} - transform$:

$$\left((\boldsymbol{t}\circ m)(C,D)\right)_{\bar{g}_{a,r}} = \left(\boldsymbol{t}(m(C),m(D))\right)_{\bar{g}_{a,r}} =$$
$$= \bar{g}_{a,r}{}^{-1}\left(\boldsymbol{t}\left(\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}\left(m\left(\bar{g}_{a,r}(C),\bar{g}_{a,r}(D)\right)\right)\right)\right)\right) =$$
$$= \boldsymbol{t}_{\bar{g}_{a,r}}\left(m_{\bar{g}_{a,r}}(C,D)\right) = \left(\boldsymbol{t}_{\bar{g}_{a,r}}\circ m_{\bar{g}_{a,r}}\right)(C,D) =$$
$$= \boldsymbol{t}_{\bar{g}_{a,r}}\left(m\left(\bar{g}_{a,r}(C),\bar{g}_{a,r}(D)\right)\right).$$

Application 2.4.6: For the composite function $h = \mathbf{t}\circ P$ we can apply [9-11] the $\bar{g}_{a,r} - transform$:

$$\left((\boldsymbol{t}\circ P)(C,D)\right)_{\bar{g}_{a,r}} = \left(\boldsymbol{t}(P(C),P(D))\right)_{\bar{g}_{a,r}} =$$
$$= \bar{g}_{a,r}{}^{-1}\left(\boldsymbol{t}\left(\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}\left(P\left(\bar{g}_{a,r}(C),\bar{g}_{a,r}(D)\right)\right)\right)\right)\right) =$$
$$= \boldsymbol{t}_{\bar{g}_{a,r}}\left(P_{\bar{g}_{a,r}}(C,D)\right) = \left(\boldsymbol{t}_{\bar{g}_{a,r}}\circ P_{\bar{g}_{a,r}}\right)(C,D) =$$
$$= \boldsymbol{t}_{\bar{g}_{a,r}}\left(P_{\bar{g}_{a,r}}(C),P_{\bar{g}_{a,r}}(D)\right).$$

2.5. Relations between *entropy* and $\bar{\boldsymbol{g}}_{a,r} - \boldsymbol{entropy}$ by $\bar{\boldsymbol{g}}_{a,r} - \boldsymbol{function}$

By the definition of entropy and the examples treated above, we established relations between the *entropy* and $\bar{g}_{a,r} - entropy$ by $\bar{g}_{a,r} - Transform$[4,8,13,26,27].

Definition 2.5.1. *Let* $\boldsymbol{B} = \{B_1, B_2, \dots, B_n\} \subset \mathcal{A}$ *be a* $\overline{\oplus}_{\bar{g}_{a,r}} - measurable\ partition\ of\ \boldsymbol{X}$. *Then the* $\bar{g}_{a,r} - entropy$ *[13] is determined as:*

$\boldsymbol{H}_{a,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B) = -\overline{\bigoplus}_{i=1}^{n} h_{\bar{g}_{a,r}}\left(m(B_i)\right)$, where

$$\boldsymbol{h}_{\bar{g}_{a,r}}\left(m(B)\right) = \begin{cases} 0 & if\ m(B_i) = 0 \\ m(B_i)\,\overline{\odot}_{\bar{g}_{a,r}}\, f_{\bar{g}_{a,r}-a,log}\left(m(B_i)\right) & if\ m(B_i) \neq 0 \end{cases}$$

and $f_{\bar{g}_{a,r}-a,log}\left(m(B_i)\right) = \bar{g}_{a,r}{}^{-1}\left(log_a\left(\bar{g}_{a,r}(m(B_i))\right)\right)$ *is the* $\bar{g}_{a,r} - logarithmic$ *function [9].*

Theorem 2.5.2. *Let* m *be a measure* $(\overline{\oplus}_{\bar{g}_{a,r}} - P) - DM$ *on the measurable space* $(\boldsymbol{X},\mathcal{A})$ *of NSA-type [9,13,15,16,19]. Then, on* $\mathcal{A}$ *we find* m *such an IPM that* $m = \bar{g}_{a,r}{}^{-1}\circ P$ *where the additive generator* $\bar{g}$ *is the normalized one of* $\overline{\oplus}_{\bar{g}_{a,r}} = \oplus_s$ $(\oplus_s - t - conorm)$ *[9] and*

$$\boldsymbol{H}_{a,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B) = \bar{g}_{a,r}{}^{-1}(H_{a,p}^{(+,\ \cdot)}(B))$$

for every $\overline{\oplus}_{\bar{g}_{a,r}}$*−measurable partition* $\boldsymbol{B}$*.*

This theorem provides the connection between the entropy (*H*) presented by Rybárik, Kolmogorov and Sinaj [13] and further on, we made the connection with Shannon (*SH*) entropy [9,28-29]. For the partition $\boldsymbol{B}$ on the $(\boldsymbol{X},\mathcal{A},P) - probability\ space$, the amount $\boldsymbol{H}_{a,p}^{(+,\ \cdot)}(\boldsymbol{B})$ is an entropy [9,13], so

$\boldsymbol{H}_{a,p}^{(+,\ \cdot)}(B) = -\sum_{i=1}^{n} h\left(P(B_i)\right)$ where,

$$\boldsymbol{h}\left(P(B)\right) = \begin{cases} 0 & if\ \ P(B_i) = 0 \\ P(B_i)\cdot log_a\, P(B_i) & if\ \ P(B_i) \neq 0\,. \end{cases}$$

Further, we express the relations for $\bar{g}_{a,r} - entropy\ (H)$ above [9] according to $\bar{g}_{a,r} - function$:

$\boldsymbol{H}_{a,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(A) = \bar{g}_{a,r}{}^{-1}(H_{a,p}^{(+,\ \cdot)}(A))$ or

$\bar{g}_{a,r}\left(H_{a,m}^{((\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(A)\right) = H_{a,p}^{(+,\ \cdot)}(A)$

$$\boldsymbol{H}_{a,m}^{((\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(A) =$$
$$= -\overline{\bigoplus}_{i=1}^{n} m(A_i)\,\overline{\odot}_{\bar{g}_{a,r}}\, f_{\bar{g}_{a,r}-a,log}\left(m(A_i)\right) =$$
$$= \bar{g}_{a,r}{}^{-1}(H_{a,p}^{(+,\ \cdot)}(A)).$$

$$\boldsymbol{H}_{2,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(A) =$$
$$= -\overline{\bigoplus}_{i=1}^{n} m(A_i)\,\overline{\odot}_{\bar{g}_{a,r}}\, f_{\bar{g}_{a,r}-2,log}\left(m(A_i)\right) =$$
$$= \bar{g}_{a,r}{}^{-1}(H_{2,p}^{(+,\ \cdot)}(A)) =$$
$$= SH_m^{((\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(A) = SH_p^{(+,\ \cdot)}(A).$$

$$\boldsymbol{SH}_m^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(A) = H_{2,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(A) =$$
$$= \bar{g}_{a,r}{}^{-1}\left(H_{2,p}^{(+,\ \cdot)}(A)\right) = \bar{g}_{a,r}{}^{-1}\left(SH_p^{(+,\ \cdot)}(A)\right).$$

The entropy is used as a logarithmic measure to rate the information. Therefore, the information theory transfers information in a given language or message using the entropy [30]. We present the relation between the $\bar{g}_{a,r} - entropy$ and entropy, with some examples, realized by the $\bar{g}_{a,r} - logarithmic\ function$, i.e. through a $\bar{g}_{a,r} - Transformation$.

Example 2.5.3. Change of base for the logarithmic function ($a \to b$) [9,16,26-28] as below:

$$log_b P(C) = log_b a\ \cdot log_a P(C);$$
$$log_b \bar{g}_{a,r}(m(C)) = (log_b a\,)\cdot\left(log_a \bar{g}_{a,r}(m(C)\right);$$
$$log_b \bar{g}_{a,r}\left(m(C)\right) = \left(\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}(log_b a)\right)\right)\cdot$$
$$\cdot\left(\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}\left(log_a \bar{g}_{a,r}(m(C)\right)\right)\right).$$

Applying the $\bar{g}_{a,r}{}^{-1} - function$ on both sides of the equation above, we find:

$$\bar{g}_{a,r}{}^{-1}\left(log_b \bar{g}_{a,r}(m(C)\right) =$$
$$= \bar{g}_{a,r}{}^{-1}\left(\left(\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}(log_b a)\right)\right)\cdot\right.$$
$$\left.\cdot\left(\bar{g}_{a,r}\left(\bar{g}_{a,r}{}^{-1}\left(log_a \bar{g}_{a,r}(m(C)\right)\right)\right)\right).$$
$$f_{\bar{g}_{a,r}-b,log}(m(C)) = \bar{g}_{a,r}{}^{-1}(log_b a)\,\overline{\odot}_{\bar{g}_{a,r}}\, f_{\bar{g}_{a,r}-a,log}(m(C).$$

For the relationships and crossing in different entropies mentioned above of the logarithm during the change of bases ($a \to b$), we have some rules which are conditioned by pseudo-operations $\{\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}}\}$ and $\bar{g}_{a,r} - function$:

$\boldsymbol{H}_{b,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B) =$
$= -\bar{g}_{a,r}{}^{-1}(log_b a) \overline{\odot}_{\bar{g}_{a,r}} (\overline{\oplus}_{i=1}^{n}\ m(B_i) \overline{\odot}_{\bar{g}_{a,r}} f_{\bar{g}_{a,r}-a,log}(m(B_i)) =$
$= -\bar{g}_{a,r}{}^{-1}(log_b a) \overline{\odot}_{\bar{g}_{a,r}} \bar{g}_{a,r}{}^{-1}(H_{a,p}^{(+,\ \cdot)}(B)).$
$\boldsymbol{H}_{b,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B) =$
$= -\bar{g}_{a,r}{}^{-1}(log_b a) \overline{\odot}_{\bar{g}_{a,r}} \left(\overline{\oplus}_{i=1}^{n}\ m(B_i) \overline{\odot}_{\bar{g}_{a,r}} f_{\bar{g}_{a,r}-a,log}(m(B_i))\right) =$

$$= -\bar{g}_{a,r}{}^{-1}(log_b a) \overline{\odot}_{\bar{g}_{a,r}} \left(H_{a,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B)\right).$$

Shannon entropy, as $\bar{g}_{a,r} - entropy\,(SH)$ by PAO $\{\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}}\}$ and $\bar{g}_{a,r} - function$ [9] is presented in the form:

$$\boldsymbol{SH}_{m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B) =$$

$$= -\bar{g}_{a,r}{}^{-1}(log_b 2) \overline{\odot}_{\bar{g}_{a,r}} \left(H_{2,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B)\right).$$

$$H_{b,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B) =$$

$$= -\bar{g}_{a,r}{}^{-1}(log_b a) \overline{\odot}_{\bar{g}_{a,r}} \left(H_{a,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B)\right) =$$

$$= -\bar{g}_{a,r}{}^{-1}(log_b 2) \overline{\odot}_{\bar{g}_{a,r}} \left(H_{2,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B)\right).$$

The relationship between two cases of $\bar{g}_{a,r} - entropies\,(H - SH)$ and transformed by $\bar{g}_{a,r} - function$ is presented in the form:

$$\boldsymbol{H}_{b,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B) = -\bar{g}_{a,r}{}^{-1}(log_b 2) \overline{\odot}_{\bar{g}_{a,r}} \boldsymbol{SH}_{m}^{(\oplus,\odot)}(B).$$

3. Conclusions

The modification of the Pseudo-Additive Probability Measure $(\overline{\oplus}_{\bar{g}_{a,r}} - \mathrm{P}) - m,\ (m_{\bar{g}_{a,r}})$ and the Induced Probability Measure ($P_{\bar{g}_{a,r}}$) supported by $\bar{g}_{a,r} - Transform$ indicate the relationships between the two modifications of the PAPM ($m_{\bar{g}_{a,r}}$) and the IPM ($P_{\bar{g}_{a,r}}$) realized by $\bar{g}_{a,r} - Transform.$

C. I. The PAPM $(\overline{\oplus}_{\bar{g}_{a,r}} - \mathrm{P}) - m$ and the IPM on $\mathcal{A}$ satisfy the conditions below, by $\bar{g}_{a,r} - transform$:

- $\boldsymbol{P}_{\bar{g}_{a,r}}(C) = \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C)\right)$
- $\boldsymbol{P}_{\bar{g}_{a,r}}(C \cup D) = \boldsymbol{P}_{\bar{g}_{a,r}}(C) \overline{\oplus}_{\bar{g}_{a,r}} \boldsymbol{P}_{\bar{g}_{a,r}}(D)$
- $\boldsymbol{P}_{\bar{g}_{a,r}}(C \cup D) = \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(C)\right) \overline{\oplus}_{\bar{g}_{a,r}} \bar{g}_{a,r}\left(\boldsymbol{m}_{\bar{g}_{a,r}}(D)\right).$

C. II. Stopping again into the conditions presented at first [11], i.e., $\bar{g}(z) = \bar{g}_{a,r}(z) = az^r$ for $r \geq 1,\ a > 0$ and the SA $(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}} - \mathrm{A.1} \div A.7)$ [8,11,18] is fulfilled, for $\bar{g}_{a,r} - transform$ of the PPM $(\overline{\oplus}_{\bar{g}_{a,r}} - \mathrm{P}) - m$, $(m_{\bar{g}_{a,r}})$ and the IPM $(P_{\bar{g}_{a,r}})$, we present the $(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}}) - integral$ with respect of $m_{\bar{g}_{a,r}}$, also relation with the Bi-Pseudo-Integral $(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}}) - integral$ with respect to $P_{\bar{g}_{a,r}}$ [9] as:

- $\left(\boldsymbol{BP}_{(\bar{g}_{a,r}-TR)} - \int_{\bar{g}_{a,r}{}^{-1}(z)}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}\right)(m_{\bar{g}_{a,r}},\boldsymbol{f}_{\bar{g}_{a,r}} - RMF_{\bar{g}_{a,r}}) =$

$$= \int_{\bar{g}_{a,r}{}^{-1}(z)}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})} \boldsymbol{f}_{\bar{g}_{a,r}} \overline{\odot}_{\bar{g}_{a,r}} dm_{\bar{g}_{a,r}} =$$

$$= \bar{g}_{a,r}{}^{-1}\left(\int_{\bar{g}_{a,r}{}^{-1}(z)}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})} f(\bar{g}_{a,r}) \cdot d(\boldsymbol{P}_{\bar{g}_{a,r}})\right).$$

C. III. The relationships between entropies are treated in the paper as logarithmic measures of the rate of information transfer so, the $\bar{g}_{a,r} - entropies\,(H - SH)$ transformed by $\bar{g}_{a,r} - function$ are related as below:

- $\boldsymbol{H}_{b,m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B) =$

$$= -\bar{g}_{a,r}{}^{-1}(log_b 2) \overline{\odot}_{\bar{g}_{a,r}} \boldsymbol{SH}_{m}^{(\overline{\oplus}_{\bar{g}_{a,r}},\overline{\odot}_{\bar{g}_{a,r}})}(B).$$

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