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Homomorphism of $\vec{\mathcal{J}}\rho$ Neutrosophic Fuzzy Subgroup over a Finite Group

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Abstract Neutrosophic fuzzy sets are an extension of fuzzy sets. Fuzzy sets can only handle vague information, and it cannot deal with incomplete and inconsistent information. But neutrosophic fuzzy sets and their combinations are one technique for handling incomplete and inconsistent information. Neutrosophic fuzzy set theory provides the groundwork for a whole group of new mathematical theories and summarizes both the traditional and fuzzy counterparts. Following this, the area of neutrosophic fuzzy sets is being developed intensively, with the goal of strengthening the foundations of the theory, creating new applications, and enhancing its practicality in a range of real-life scenarios. Further, neutrosophic fuzzy sets are characterized by three components. One is truth ($\succ$), the second is indeterminacy ($\bowtie$), and the third is falsity ($\lhd$). In this paper, we have examined the idea of homomorphism of implication-based ($\vec{\mathcal{J}}\rho$) neutrosophic fuzzy subgroups over a finite group. Then, $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroups over a finite group and $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy normal subgroups over a finite group were defined. Finally, we have demonstrated some basic properties of homomorphism of $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroups over a finite group in this study.

Keywords $\vec{\mathcal{J}}\rho$ Neutrosophic Fuzzy Subgroup, $\vec{\mathcal{J}}\rho$ Neutrosophic Fuzzy Normal Subgroup, Homomorphism of $\vec{\mathcal{J}}\rho$ Neutrosophic Fuzzy Subgroup

1. Introduction

B. Basumatary [1] studied the neutrosophic bitopological group in 2020. Cetkin V [2] created a neutrosophic subgroup homomorphic image and pre image in 2015. In 2022, Dhanasekar et al. [3] established the fuzzy integers using the weighted arithmetic mean approach. J. Martina Jency [4] focused on the concept of subgroupoid in a fuzzy neutrosophic set. Rosenfeld [5] defined the concept of fuzzy subgroupoid and fuzzy subgroups in 1971. Selvarathi. M [6] defined the concept of a $\vec{\mathcal{J}}\rho$ - fuzzy normal subgroup of a group in 2015, as well as some basic properties. In 2017, Selvarathi. M [7] developed the idea of a $\vec{\mathcal{J}}\rho$ intuitionistic fuzzy subgroup of a finite group. Florentin Smarandache [8] was the first to present Neutrosophic sets, which handled with belongingness of truth values, indeterminate values and false values in 1999. Mingsheng Ying [9] developed a new approach for fuzzy topology using fuzzy logic in 1991. Yogashanthi et al. [10] described the intuitionistic fuzzy variant of the critical path technique in 2022. The $\vec{\mathcal{J}}\rho$ - fuzzy subgroup of group was established by X. H. Yuan et al. [11] in 2003. In this study, we have defined the idea of homomorphism of $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup over a finite group and also established some basic properties.

2. Preliminaries

Definition 2.1. [5]

Let $\Im$ be a group then $\mu: \Im \rightarrow [0,1]$ is called as fuzzy subgroup if

(i) $\mu(\omega\sigma) \geq \min(\mu(\omega), \mu(\sigma))$ for all $\omega, \sigma \in \Im$
(ii) $\mu(\omega^{-1}) \geq \mu(\omega)$ for all $\omega \in \Im$.

Let W be the universe of discourse and $(\mathfrak{I}, *)$ be a finite group. The truth value of a fuzzy proposition α is symbolized by [α] in fuzzy logic. The fuzzy logical and the corresponding set-theoretical notations used in this paper are

$$(\omega \in A) = A(\omega)$$

$$(\alpha \wedge \beta) = \min\{[\alpha],[\beta]\}$$

$$(\alpha \rightarrow \beta) = 1- \alpha + \alpha\beta$$

$$(\forall \omega\ \alpha(\omega)) = \inf\omega \in W\ [\alpha(\omega)]$$

$$(\exists \omega\ \alpha(\omega)) = \sup\omega \in W\ [\alpha(\omega)]$$

$\vec{\mathfrak{J}}\alpha$ if and only if [α] =1 for all valuations.

Implications used here are that of Reichenbach implication operator.

Definition 2.2. [6]

If a fuzzy subset A of a group $\mathfrak{I}$ satisfies

(i) $\vec{\mathfrak{J}}(\omega \in A) \wedge (\sigma \in A) \rightarrow (\omega\sigma \in A)$ for any $\omega, \sigma \in \mathfrak{I}$
(ii) $\vec{\mathfrak{J}}(\omega \in A) \rightarrow (\omega^{-1} \in A)$ for any $\omega \in \mathfrak{I}$

then A is called a fuzzifying subgroup of $\mathfrak{I}$. The concept of ρ-tautology was introduced by Ying [9], i.e., $\vec{\mathfrak{J}}\rho(\alpha)$ if and only if $(\alpha) \geq \rho$ for all valuations.

Definition 2.3. [6]

Let A be a fuzzy subset of a group $\mathfrak{I}$ and $\rho \in (0, 1]$ is a fixed number. If for any $\omega, \sigma \in \mathfrak{I}$

(i) $\vec{\mathfrak{J}}\rho(\omega \in A) \wedge (\sigma \in A) \rightarrow (\omega\sigma \in A)$
and
(ii) $\vec{\mathfrak{J}}\rho(\omega \in A) \rightarrow (\omega^{-1} \in A)$

then A is called a $\vec{\mathfrak{J}}\rho$ - fuzzy subgroup of $\mathfrak{I}$.

Definition 2.4. [8]

A neutrosophic fuzzy set $\mathcal{N}$ on the universe of discourse W is characterized by the degree of membership $\mathcal{N}_{\succ}$, the degree of indeterminacy $\mathcal{N}_{\bowtie}$ and the degree of non-membership $\mathcal{N}_{\lhd}$ where the function, $\succ, \bowtie, \lhd$: W → [0, 1]. It can be written as

$$\mathcal{N} = \{(\omega, \mathcal{N}_{\succ}(\omega), \mathcal{N}_{\bowtie}(\omega), \mathcal{N}_{\lhd}(\omega)), \omega \in W\}$$

and

$$\mathcal{N}_{\succ}(\omega), \mathcal{N}_{\bowtie}(\omega), \mathcal{N}_{\lhd}(\omega) \in [0, 1]$$

such that

$$0 \leq \mathcal{N}_{\succ}(\omega) + \mathcal{N}_{\bowtie}(\omega) + \mathcal{N}_{\lhd}(\omega) \leq 3$$

3. $\vec{\mathfrak{J}}\rho$ Neutrosophic Fuzzy Subgroup Over a Finite Group

Definition 3.1.

Let $(\mathfrak{I}, *)$ be a finite group. A neutrosophic fuzzy set $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\lhd})$ of a finite group $\mathfrak{I}$ is called a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy subgroup over a finite group $\mathfrak{I}$, if it satisfies for any $\omega, \sigma \in \mathfrak{I}$

(i) $\vec{\mathfrak{J}}\rho((\omega \in \mathcal{N}) \wedge (\sigma \in \mathcal{N})) \rightarrow (\omega\sigma \in \mathcal{N})$

i.e.

$$\vec{\mathfrak{J}}\rho((\omega \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ})) \rightarrow (\omega\sigma \in \mathcal{N}_{\succ})$$

$$\vec{\mathfrak{J}}\rho((\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})) \rightarrow (\omega\sigma \in \mathcal{N}_{\bowtie})$$

$$\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{\lhd}) \rightarrow ((\omega \in \mathcal{N}_{\lhd}) \vee (\sigma \in \mathcal{N}_{\lhd}))$$

(ii) $\vec{\mathfrak{J}}\rho(\omega \in \mathcal{N}) \rightarrow (\omega^{-1} \in \mathcal{N})$

i.e.

$$\vec{\mathfrak{J}}\rho(\omega \in \mathcal{N}_{\succ}) \rightarrow (\omega^{-1} \in \mathcal{N}_{\succ})$$

$$\vec{\mathfrak{J}}\rho(\omega \in \mathcal{N}_{\bowtie}) \rightarrow (\omega^{-1} \in \mathcal{N}_{\bowtie})$$

$$\vec{\mathfrak{J}}\rho(\omega^{-1} \in \mathcal{N}_{\lhd}) \rightarrow (\omega \in \mathcal{N}_{\lhd})$$

Where $(\omega \in \mathcal{N}_{\succ})$ denotes the truth membership value, $(\omega \in \mathcal{N}_{\bowtie})$ denotes the indeterminacy membership value and $(\omega \in \mathcal{N}_{\lhd})$ denotes the falsity membership value such that

$$0 \leq (\omega \in \mathcal{N}_{\succ}) + (\omega \in \mathcal{N}_{\bowtie}) + (\omega \in \mathcal{N}_{\lhd}) \leq 3$$

Example 3.2.

$\mathfrak{I} = \{0,1,2,3,4,5,6,7\}$ is a finite group of order 8 with respect to the addition modulo 8. Cayley's closure table 1 of the group $\mathfrak{I}$ is given below

Table 1. Closure table

$+_8$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

Consider the neutrosophic fuzzy set $\mathcal{N}: \mathfrak{I} \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ is defined by

$(0 \in \mathcal{N}) = (0.325, 0.300, 0.110)$
$(1 \in \mathcal{N}) = (0.315, 0.265, 0.350)$
$(2 \in \mathcal{N}) = (0.310, 0.260, 0.365)$
$(3 \in \mathcal{N}) = (0.305, 0.225, 0.360)$
$(4 \in \mathcal{N}) = (0.295, 0.235, 0.355)$
$(5 \in \mathcal{N}) = (0.280, 0.230, 0.340)$
$(6 \in \mathcal{N}) = (0.275, 0.215, 0.335)$
$(7 \in \mathcal{N}) = (0.250, 0.210, 0.400)$

Given below are the tables 2, 3 and 4 of values for ∧ and ∨ of truth membership, indeterminacy membership and falsity membership.

Table 2. Truth table

$\wedge$	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[0]	0.325	0.315	0.310	0.305	0.295	0.280	0.275	0.250
[1]	0.315	0.315	0.310	0.305	0.295	0.280	0.275	0.250
[2]	0.310	0.310	0.310	0.305	0.295	0.280	0.275	0.250
[3]	0.305	0.305	0.305	0.305	0.295	0.280	0.275	0.250
[4]	0.295	0.295	0.295	0.295	0.295	0.280	0.275	0.250
[5]	0.280	0.280	0.280	0.280	0.280	0.280	0.275	0.250
[6]	0.275	0.275	0.275	0.275	0.275	0.275	0.275	0.250
[7]	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250

Table 3. Intermediate table

$\wedge$	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[0]	0.300	0.265	0.260	0.225	0.235	0.230	0.215	0.210
[1]	0.265	0.265	0.260	0.225	0.235	0.230	0.215	0.210
[2]	0.260	0.360	0.260	0.225	0.235	0.230	0.215	0.210
[3]	0.225	0.225	0.225	0.225	0.225	0.225	0.215	0.210
[4]	0.235	0.235	0.235	0.235	0.235	0.230	0.215	0.210
[5]	0.230	0.230	0.230	0.225	0.230	0.230	0.215	0.210
[6]	0.215	0.215	0.215	0.215	0.215	0.215	0.215	0.210
[7]	0.210	0.210	0.210	0.210	0.210	0.210	0.210	0.210

Table 4. Falsity table

$\vee$	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[0]	0.110	0.350	0.365	0.360	0.355	0.340	0.335	0.400
[1]	0.350	0.350	0.365	0360	0.355	0.350	0.350	0.400
[2]	0.365	0.365	0.365	0.365	0.365	0.365	0.365	0.400
[3]	0.360	0.360	0.365	0.360	0.360	0.360	0.360	0.400
[4]	0.355	0.355	0.365	0.360	0.355	0.355	0.355	0.400
[5]	0.340	0.350	0.365	0.360	0.355	0.340	0.340	0.400
[6]	0.335	0.350	0.365	0.360	0.355	0.340	0.335	0.400
[7]	0.400	0.400	0.400	0.400	0.400	0.400	0.400	0.400

With $\rho = 0.4$ and the implication operator is that of Reichenbach, then $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\lhd})$ is a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy subgroup over a finite group.

Theorem 3.3.

Let $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\lhd})$ be a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy set. Then $\mathcal{N}$ is a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy subgroup over a finite group $\mathfrak{I}$ if and only if for all $\omega, \sigma \in \mathfrak{I}$

$$\vec{\mathfrak{J}}\rho((\omega \in \mathcal{N}) \wedge (\sigma \in \mathcal{N})) \to (\omega\sigma^{-1} \in \mathcal{N})$$

i.e.

$$\vec{\mathfrak{J}}\rho((\omega \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ})) \to (\omega\sigma^{-1} \in \mathcal{N}_{\succ})$$

$$\vec{\mathfrak{J}}\rho((\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})) \to (\omega\sigma^{-1} \in \mathcal{N}_{\bowtie})$$

$$\vec{\mathfrak{J}}\rho(\omega\sigma^{-1} \in \mathcal{N}_{\lhd}) \to ((\omega \in \mathcal{N}_{\lhd}) \vee (\sigma \in \mathcal{N}_{\lhd}))$$

Proof:

Let $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\lhd})$ be a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy subgroup over a finite group $\mathfrak{I}$.

Let $\omega, \sigma \in \mathfrak{I}$

$$\vec{\mathfrak{J}}\rho((\omega \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ})) \to (\omega \in \mathcal{N}_{\succ}) \wedge (\sigma^{-1} \in \mathcal{N}_{\succ})$$
$$\to (\omega\sigma^{-1} \in \mathcal{N}_{\succ})$$

$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})) \rightarrow (\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma^{-1} \in \mathcal{N}_{\bowtie})$$
$$\rightarrow (\omega\sigma^{-1} \in \mathcal{N}_{\bowtie})$$
$$\vec{\mathcal{J}}\rho(\omega\sigma^{-1} \in \mathcal{N}_{\triangleleft}) \rightarrow (\omega \in \mathcal{N}_{\triangleleft}) \vee (\sigma^{-1} \in \mathcal{N}_{\triangleleft})$$
$$\rightarrow (\omega \in \mathcal{N}_{\triangleleft}) \vee (\sigma \in \mathcal{N}_{\triangleleft})$$

Conversely,
Let for each $\omega, \sigma \in \mathfrak{I}$ such that
Assume that

$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ})) \rightarrow (\omega\sigma^{-1} \in \mathcal{N}_{\succ}) \qquad (1)$$
$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})) \rightarrow (\omega\sigma^{-1} \in \mathcal{N}_{\bowtie}) \qquad (2)$$
$$\vec{\mathcal{J}}\rho(\omega\sigma^{-1} \in \mathcal{N}_{\triangleleft}) \rightarrow ((\omega \in \mathcal{N}_{\triangleleft}) \vee (\sigma \in \mathcal{N}_{\triangleleft}) \qquad (3)$$

To prove: $\mathcal{N}$ is a $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup over a finite group $\mathfrak{I}$.

Let $\sigma = \omega$ in (1)

$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ})) \rightarrow (\omega \in \mathcal{N}_{\succ}) \wedge (\omega \in \mathcal{N}_{\succ})$$
$$\rightarrow (\omega\omega^{-1} \in \mathcal{N}_{\succ})$$
$$\rightarrow (e \in \mathcal{N}_{\succ})$$

Therefore

$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ})) \rightarrow (e \in \mathcal{N}_{\succ})$$
$$\vec{\mathcal{J}}\rho(\sigma \in \mathcal{N}_{\succ}) \rightarrow ((e \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ}))$$
$$\rightarrow (e\sigma^{-1} \in \mathcal{N}_{\succ})$$
$$\rightarrow (\sigma^{-1} \in \mathcal{N}_{\succ})$$

Therefore

$$\vec{\mathcal{J}}\rho(\sigma \in \mathcal{N}_{\succ}) \rightarrow (\sigma^{-1} \in \mathcal{N}_{\succ})$$
$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ})) \rightarrow (\omega \in \mathcal{N}_{\succ}) \wedge (\sigma^{-1} \in \mathcal{N}_{\succ})$$
$$\rightarrow (\omega(\sigma^{-1})^{-1} \in \mathcal{N}_{\succ})$$
$$\rightarrow (\omega\sigma \in \mathcal{N}_{\succ})$$

Therefore

$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\succ}) \wedge (\sigma \in \mathcal{N}_{\succ})) \rightarrow (\omega\sigma \in \mathcal{N}_{\succ})$$

Let $\sigma = \omega$ in (2)

$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})) \rightarrow (\omega \in \mathcal{N}_{\bowtie}) \wedge (\omega \in \mathcal{N}_{\bowtie})$$
$$\rightarrow (\omega\omega^{-1} \in \mathcal{N}_{\bowtie})$$
$$\rightarrow (e \in \mathcal{N}_{\bowtie})$$

Therefore

$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})) \rightarrow (e \in \mathcal{N}_{\bowtie})$$
$$\vec{\mathcal{J}}\rho(\sigma \in \mathcal{N}_{\bowtie}) \rightarrow ((e \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})$$
$$\rightarrow (e\sigma^{-1} \in \mathcal{N}_{\bowtie})$$
$$\rightarrow (\sigma^{-1} \in \mathcal{N}_{\bowtie})$$

Therefore

$$\vec{\mathcal{J}}\rho(\sigma \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma^{-1} \in \mathcal{N}_{\bowtie})$$
$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})) \rightarrow (\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma^{-1} \in \mathcal{N}_{\bowtie})$$
$$\rightarrow (\omega(\sigma^{-1})^{-1} \in \mathcal{N}_{\bowtie})$$
$$\rightarrow (\omega\sigma \in \mathcal{N}_{\bowtie})$$

Therefore

$$\vec{\mathcal{J}}\rho((\omega \in \mathcal{N}_{\bowtie}) \wedge (\sigma \in \mathcal{N}_{\bowtie})) \rightarrow (\omega\sigma \in \mathcal{N}_{\bowtie})$$

Let $\sigma = \omega$ in (3)

$$\vec{\mathcal{J}}\rho(\omega\omega^{-1} \in \mathcal{N}_{\triangleleft}) \rightarrow ((\omega \in \mathcal{N}_{\triangleleft}) \vee (\omega \in \mathcal{N}_{\triangleleft}))$$
$$\vec{\mathcal{J}}\rho(e \in \mathcal{N}_{\triangleleft}) \rightarrow (\omega \in \mathcal{N}_{\triangleleft})$$
$$\vec{\mathcal{J}}\rho(\omega^{-1} \in \mathcal{N}_{\triangleleft}) \rightarrow (e\omega^{-1} \in \mathcal{N}_{\triangleleft})$$
$$\rightarrow (e \in \mathcal{N}_{\triangleleft}) \vee (\omega \in \mathcal{N}_{\triangleleft})$$
$$\rightarrow (\omega \in \mathcal{N}_{\triangleleft})$$

Therefore

$$\vec{\mathcal{J}}\rho(\omega^{-1} \in \mathcal{N}_{\triangleleft}) \rightarrow (\omega \in \mathcal{N}_{\triangleleft})$$
$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\triangleleft}) \rightarrow (\omega\,(\sigma^{-1})^{-1} \in \mathcal{N}_{\triangleleft})$$
$$\rightarrow ((\omega \in \mathcal{N}_{\triangleleft}) \vee (\sigma^{-1} \in \mathcal{N}_{\triangleleft}))$$
$$\rightarrow ((\omega \in \mathcal{N}_{\triangleleft}) \vee (\sigma \in \mathcal{N}_{\triangleleft}))$$

Therefore

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\triangleleft}) \rightarrow ((\omega \in \mathcal{N}_{\triangleleft}) \vee (\sigma \in \mathcal{N}_{\triangleleft}))$$

Thus, $\mathcal{N}$ is a $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup over a finite group $\mathfrak{I}$.

Definition 3.4.

A $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\triangleleft})$ of $\mathfrak{I}$ is called a $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy normal subgroup over a finite group $\mathfrak{I}$ if

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}) \rightarrow (\sigma\omega \in \mathcal{N}) \text{ for all } \omega, \sigma \in \mathfrak{I}$$

i.e.

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\succ}) \rightarrow (\sigma\omega \in \mathcal{N}_{\succ})$$
$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma\omega \in \mathcal{N}_{\bowtie})$$
$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\triangleleft}) \rightarrow (\sigma\omega \in \mathcal{N}_{\triangleleft})$$

Theorem 3.5.

Let $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\triangleleft})$ of $\mathfrak{I}$ be a $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy normal subgroup over a finite group $\mathfrak{I}$. Then the following conditions are equivalent.

(i) $\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}) \rightarrow (\sigma\omega \in \mathcal{N})$ for all $\omega, \sigma \in \mathfrak{I}$
(ii) $\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}) \rightarrow (\sigma \in \mathcal{N})$ for all $\omega, \sigma \in \mathfrak{I}$

Let $\omega, \sigma \in \mathfrak{I}$
To prove: (i) ⇒ (ii)
Assume that: $\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}) \rightarrow (\sigma\omega \in \mathcal{N})$
i.e.

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\succ}) \rightarrow (\sigma\omega \in \mathcal{N}_{\succ})$$
$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma\omega \in \mathcal{N}_{\bowtie})$$
$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\triangleleft}) \rightarrow (\sigma\omega \in \mathcal{N}_{\triangleleft})$$

To prove: $\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}) \rightarrow (\sigma \in \mathcal{N})$ for all $\omega, \sigma \in \mathfrak{I}$

i.e.

$$\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}_{\succ}) \rightarrow (\sigma \in \mathcal{N}_{\succ}) \text{ for all } \omega, \sigma \in \Im$$

$$\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma \in \mathcal{N}_{\bowtie}) \text{ for all } \omega, \sigma \in \Im$$

$$\vec{\mathcal{J}}\rho(\sigma \in \mathcal{N}_{\lhd}) \rightarrow (\omega\sigma\omega^{-1} \in \mathcal{N}_{\lhd}) \text{ for all } \omega, \sigma \in \Im$$

(i) $\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}_{\succ}) \rightarrow (\omega\sigma(\omega^{-1}) \in \mathcal{N}_{\succ})$

$\rightarrow (\omega^{-1}\omega\sigma \in \mathcal{N}_{\succ})$ by using (i)

$\rightarrow (\sigma \in \mathcal{N}_{\succ})$

Therefore

$$\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}_{\succ}) \rightarrow (\sigma \in \mathcal{N}_{\succ})$$

(ii) $\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}_{\bowtie}) \rightarrow (\omega\sigma(\omega^{-1}) \in \mathcal{N}_{\bowtie})$

$\rightarrow (\omega^{-1}\omega\sigma \in \mathcal{N}_{\bowtie})$ by using (i)

$\rightarrow (\sigma \in \mathcal{N}_{\bowtie})$

Therefore

$$\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma \in \mathcal{N}_{\bowtie})$$

(iii) $\vec{\mathcal{J}}\rho(\sigma \in \mathcal{N}_{\lhd}) \rightarrow (\omega^{-1}\omega\sigma \in \mathcal{N}_{\lhd})$ by using (i)

$\rightarrow (\omega\sigma\omega^{-1} \in \mathcal{N}_{\lhd})$

Therefore

$$\vec{\mathcal{J}}\rho(\sigma \in \mathcal{N}_{\lhd}) \rightarrow (\omega\sigma\omega^{-1} \in \mathcal{N}_{\lhd})$$

Let $\omega, \sigma \in \Im$
To prove: (ii) ⇒ (i)
Assume that: $(\omega\sigma\omega^{-1} \in \mathcal{N}) \rightarrow (\sigma \in \mathcal{N})$ for all $\omega, \sigma \in \Im$
i.e.

$$\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}_{\succ}) \rightarrow (\sigma \in \mathcal{N}_{\succ}) \text{ for all } \omega, \sigma \in \Im$$

$$\vec{\mathcal{J}}\rho(\omega\sigma\omega^{-1} \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma \in \mathcal{N}_{\bowtie}) \text{ for all } \omega, \sigma \in \Im$$

$$\vec{\mathcal{J}}\rho(\sigma \in \mathcal{N}_{\lhd}) \rightarrow (\omega\sigma\omega^{-1} \in \mathcal{N}_{\lhd}) \text{ for all } \omega, \sigma \in \Im$$

To prove: $\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}) \rightarrow (\sigma\omega \in \mathcal{N})$
i.e.

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\succ}) \rightarrow (\sigma\omega \in \mathcal{N}_{\succ})$$

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma\omega \in \mathcal{N}_{\bowtie})$$

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\lhd}) \rightarrow (\sigma\omega \in \mathcal{N}_{\lhd})$$

(i) $\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\succ}) \rightarrow ((\omega\sigma)(\omega\omega^{-1}) \in \mathcal{N}_{\succ})$

$\rightarrow (\omega(\sigma\omega)\omega^{-1} \in \mathcal{N}_{\succ})$

$\rightarrow (\sigma\omega \in \mathcal{N}_{\succ})$ by using (ii)

Therefore

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\succ}) \rightarrow (\sigma\omega \in \mathcal{N}_{\succ})$$

(ii) $\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\bowtie}) \rightarrow ((\omega\sigma)(\omega\omega^{-1}) \in \mathcal{N}_{\bowtie})$

$\rightarrow (\omega(\sigma\omega)\omega^{-1} \in \mathcal{N}_{\bowtie})$

$\rightarrow (\sigma\omega \in \mathcal{N}_{\bowtie})$ by using (ii)

Therefore

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma\omega \in \mathcal{N}_{\bowtie})$$

(iii) $\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\lhd}) \rightarrow (\omega^{-1}(\omega\sigma)(\omega^{-1})^{-1} \in \mathcal{N}_{\lhd})$

$\rightarrow (\omega^{-1}(\omega\sigma)\omega \in \mathcal{N}_{\lhd})$

$\rightarrow (\omega^{-1}\omega(\sigma\omega) \in \mathcal{N}_{\lhd})$

$\rightarrow (\sigma\omega \in \mathcal{N}_{\lhd})$ by using (ii)

Therefore

$$\vec{\mathcal{J}}\rho(\omega\sigma \in \mathcal{N}_{\lhd}) \rightarrow (\sigma\omega \in \mathcal{N}_{\lhd})$$

Homomorphism of $\vec{\mathcal{J}}\rho$ Neutrosophic Fuzzy Subgroup over a Finite Group

Definition 3.6.

Let f: $\Im \rightarrow \Im$ be a function where $\Im$ is a finite group. Let $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\lhd})$ and $\mathcal{B} = (\mathcal{B}_{\succ}, \mathcal{B}_{\bowtie}, \mathcal{B}_{\lhd})$ be two $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroups over the finite group of $\Im$. Then the $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup $\mathcal{B}$ of $f(\Im)$ is defined by
$\vec{\mathcal{J}}\rho(\exists\omega\ \{(\omega \in \mathcal{N})\};\ \omega \in f^{1}(\sigma)) \rightarrow (\sigma \in \mathcal{B})$ for all $\omega, \sigma \in \Im$
i.e.

$$\vec{\mathcal{J}}\rho(\exists\omega\ \{(\omega \in \mathcal{N}_{\succ})\};\ \omega \in f^{1}(\sigma)) \rightarrow (\sigma \in \mathcal{B}_{\succ})$$

$$\vec{\mathcal{J}}\rho(\exists\omega\ \{(\omega \in \mathcal{N}_{\bowtie})\};\ \omega \in f^{-1}(\sigma)) \rightarrow (\sigma \in \mathcal{B}_{\bowtie})$$

$$\vec{\mathcal{J}}\rho(\forall\omega\ \{(\omega \in \mathcal{N}_{\lhd})\};\ \omega \in f^{-1}(\sigma)) \rightarrow (\sigma \in \mathcal{B}_{\lhd})$$

Similarly, if $\mathcal{B}$ is a $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup over a finite group of $f(\Im)$, then the $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup $\mathcal{N} = f \circ \mathcal{B}$ in $\Im$ is defined as

$$\vec{\mathcal{J}}\rho(f(\omega) \in \mathcal{B}) \rightarrow (\omega \in \mathcal{N})$$

i.e.

$$\vec{\mathcal{J}}\rho(f(\omega) \in \mathcal{B}_{\succ}) \rightarrow (\omega \in \mathcal{N}_{\succ})$$

$$\vec{\mathcal{J}}\rho(f(\omega) \in \mathcal{B}_{\bowtie}) \rightarrow (\omega \in \mathcal{N}_{\bowtie})$$

$$\vec{\mathcal{J}}\rho(f(\omega) \in \mathcal{B}_{\lhd}) \rightarrow (\omega \in \mathcal{N}_{\lhd})$$

for all $\omega \in \Im$, and is called the preimage of $\mathcal{B}$ under f.

Theorem 3.7.

Let $\Im_1$ and $\Im_2$ be finite groups and f: $\Im_1 \rightarrow \Im_2$ be a group homomorphism. If $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\lhd})$ is a $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup over a finite group of $\Im_1$. Then the $\mathcal{B}$, image of $\mathcal{N}$ is a $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup over a finite group of $\Im_2$.

Proof:

Let $\omega_1, \omega_2 \in \Im_1$ and $\sigma_1, \sigma_2 \in \Im_2$ such that $f(\omega_1) = \sigma_1$ and $f(\omega_2) = \sigma_2$. Since f is a group homomorphism.
$\vec{\mathcal{J}}\rho((\sigma_1 \in \mathcal{B}) \wedge (\sigma_2 \in \mathcal{B}) \rightarrow (\sigma_1\sigma_2 \in \mathcal{B})$ $\mathcal{B}$y theorem 3.3.
i.e.

$$\vec{\mathcal{J}}\rho((\sigma_1 \in \mathcal{B}_{\succ}) \wedge (\sigma_2 \in \mathcal{B}_{\succ})) \rightarrow (\sigma_1\sigma_2^{-1} \in \mathcal{B}_{\succ})$$

$$\vec{\mathcal{J}}\rho((\sigma_1 \in \mathcal{B}_{\bowtie}) \wedge (\sigma_2 \in \mathcal{B}_{\bowtie})) \rightarrow (\sigma_1\sigma_2^{-1} \in \mathcal{B}_{\bowtie})$$

$$\vec{\mathcal{J}}\rho(\sigma_1\sigma_2^{-1} \in \mathcal{B}_{\lhd}) \rightarrow ((\sigma_1 \in \mathcal{B}_{\lhd}) \vee (\sigma_2 \in \mathcal{B}_{\lhd}))$$

(i) $\vec{J}\rho((\sigma_1 \in \mathcal{B}_{>}) \wedge (\sigma_2 \in \mathcal{B}_{>}))$

$\rightarrow (\exists\omega_1 \{(\omega_1 \in \mathcal{N}_{>})\}; \omega_1 \in f^{-1}(\sigma_1)) \wedge$

$(\exists\omega_2 \{(\omega_2 \in \mathcal{N}_{>}); \omega_2 \in f^{-1}(\sigma_2))$

$\rightarrow (\exists\omega_1\omega_2 \{(\omega_1 \in \mathcal{N}_{>}) \wedge (\omega_2 \in \mathcal{N}_{>})\}; \omega_1 \in f^{-1}(\sigma_1)\ \omega_2 \in f^{-1}(\sigma_2))$

$\rightarrow (\exists\omega_1\omega_2 \{(\omega_1\omega_2^{-1} \in \mathcal{N}_{>})\}; \omega_1\omega_2^{-1} \in f^{-1}(\sigma_1)\ [f^{-1}(\sigma_2)]^{-1})$

$\rightarrow (\exists\omega_1\omega_2 \{(\omega_1\omega_2^{-1} \in \mathcal{N}_{>})\}; \omega_1\omega_2^{-1} \in f^{-1}(\sigma_1) \cdot f^{-1}(\sigma_2^{-1}))$

$\rightarrow (\exists\omega_1\omega_2 \{(\omega_1\omega_2^{-1} \in \mathcal{N}_{>})\}; \omega_1\omega_2^{-1} \in f^{-1}(\sigma_1\sigma_2^{-1}))$

$\rightarrow (\sigma_1\sigma_2^{-1} \in \mathcal{B}_{>})$

Therefore

$$\vec{J}\rho((\sigma_1 \in \mathcal{B}_{>}) \wedge (\sigma_2 \in \mathcal{B}_{>})) \rightarrow (\sigma_1\sigma_2^{-1} \in \mathcal{B}_{>})$$

(ii) $\vec{J}\rho(\sigma_1 \in \mathcal{B}_{\bowtie}) \wedge (\sigma_2 \in \mathcal{B}_{\bowtie})) \rightarrow (\exists\omega_1\{(\omega_1 \in \mathcal{N}_{\bowtie})\}; \omega_1 \in f^{-1}(\sigma_1)) \wedge (\exists\omega_2\{(\omega_2 \in \mathcal{N}_{\bowtie})\}; \omega_2 \in f^{-1}(\sigma_2))$

$\rightarrow (\exists\omega_1\omega_2\{(\omega_1 \in \mathcal{N}_{\bowtie}) \wedge (\omega_2 \in \mathcal{N}_{\bowtie})\}; \omega_1 \in f^{-1}(\sigma_1)\ \omega_2 \in f^{-1}(\sigma_2))$

$\rightarrow (\exists\omega_1\omega_2 \{(\omega_1\omega_2^{-1} \in \mathcal{N}_{\bowtie})\}; \omega_1\omega_2^{-1} \in f(\sigma_1) \cdot [f^{-1}(\sigma_2)]^{-1})$

$\rightarrow (\exists\omega_1\omega_2 \{(\omega_1\omega_2^{-1} \in \mathcal{N}_{\bowtie})\}; \omega_1\omega_2^{-1} \in f^{-1}(\sigma_1) \cdot f^{-1}(\sigma_2^{-1}))$

$\rightarrow (\exists\omega_1\ \omega_2 \{(\omega_1\omega_2^{-1} \in \mathcal{N}_{\bowtie})\}; \omega_1\omega_2^{-1} \in f^{-1}(\sigma_1\sigma_2^{-1}))$

$\rightarrow (\sigma_1\sigma_2^{-1} \in \mathcal{B}_{\bowtie})$

Therefore

$$\vec{J}\rho((\sigma_1 \in \mathcal{B}_{\bowtie}) \wedge (\sigma_2 \in \mathcal{B}_{\bowtie})) \rightarrow (\sigma_1\sigma_2^{-1} \in \mathcal{B}_{\bowtie})$$

(iii) $\vec{J}\rho(\sigma_1\sigma_2^{-1} \in \mathcal{B}_{\lhd}) \rightarrow (\forall\omega_1\omega_2 \{(\omega_1\omega_2^{-1} \in \mathcal{N}_{\lhd})\}; \omega_1\omega_2^{-1} \in f^{-1}(\sigma_1\sigma_2^{-1}))$

$\rightarrow (\forall\omega_1\omega_2 \{(\omega_1\omega_2^{-1} \in \mathcal{N}_{\lhd})\}; \omega_1\ \omega_2^{-1} \in f^{-1}(\sigma_1) \cdot f^{-1}(\sigma_2^{-1}))$

$\rightarrow (\forall\omega_1\omega_2\{(\omega_1\omega_2^{-1} \in \mathcal{N}_{\lhd})\}; \omega_1\ \omega_2^{-1} \in f^{-1}(\sigma_1) \cdot [f^{-1}(\sigma_2)]^{-1})$

$\rightarrow (\forall\omega_1\omega_2\{(\omega_1 \in \mathcal{N}_{\lhd}) \vee (\omega_2 \in \mathcal{N}_{\lhd})\}; \omega_1 \in f^{-1}(\sigma)\omega_2 \in f^{-1}(\sigma_2)$

$\rightarrow (\forall\omega_1 \{(\omega_1 \in \mathcal{N}_{\lhd})\}; \omega_1 \in f^{-1}(\sigma_1)) \vee (\forall\omega_2 \{(\omega_2 \in \mathcal{N}_{\lhd})\}; \omega_2 \in f^{-1}(\sigma_2))$

$\rightarrow ((\sigma_1 \in \mathcal{B}_{\lhd}) \vee (\sigma_2 \in \mathcal{B}_{\lhd}))$

Therefore

$$\vec{J}\rho(\sigma_1\sigma_2^{-1} \in \mathcal{B}_{\lhd}) \rightarrow ((\sigma_1 \in \mathcal{B}_{\lhd}) \vee (\sigma_2 \in \mathcal{B}_{\lhd}))$$

Theorem 3.8.

Let $\Im_1$ and $\Im_2$ be two finite groups and f: $\Im_1 \rightarrow \Im_2$ be a group homomorphism. If $\mathcal{B}$ is a $\vec{J}\rho$ neutrosophic fuzzy subgroup over a finite group of $\Im_2$, then $\mathcal{N}$, the preimage of $\mathcal{B}$ is also $\vec{J}\rho$ neutrosophic fuzzy subgroup over a finite group of $\Im_1$.

Proof:

$\mathcal{N}$ is a $\vec{J}\rho$ neutrosophic fuzzy subgroup over a finite group $\Im_1$.

$$\vec{J}\rho(f(\omega) \in \mathcal{B}) \rightarrow (\omega \in \mathcal{N})$$

i.e.

$$\vec{J}\rho(f(\omega) \in \mathcal{B}_{>}) \rightarrow (\omega \in \mathcal{N}_{>})$$

$$\vec{J}\rho(f(\omega) \in \mathcal{B}_{\bowtie}) \rightarrow (\omega \in \mathcal{N}_{\bowtie})$$

$$\vec{J}\rho(f(\omega) \in \mathcal{B}_{\lhd}) \rightarrow (\omega \in \mathcal{N}_{\lhd})$$

Let $\omega \in \Im_1$

$$\vec{\mathfrak{J}}\rho((\omega_1 \in \mathcal{N}_{>}) \wedge (\omega_2 \in \mathcal{N}_{>})) \rightarrow ((f(\omega_1) \in \mathcal{B}_{>}) \wedge (f(\omega_2) \in \mathcal{B}_{>}))$$
$$\rightarrow (f(\omega_1) \cdot [f(\omega_2)]^{-1} \in \mathcal{B}_{>})$$
$$\rightarrow (f(\omega_{01}) \cdot f(\omega_2^{-1}) \in \mathcal{B}_{>})$$
$$\rightarrow (f(\omega_1\omega_2^{-1}) \in \mathcal{B}_{>})$$
$$\rightarrow (\omega_1\omega_2^{-1} \in \mathcal{N}_{>})$$

Therefore

$$\vec{\mathfrak{J}}\rho((\omega_1 \in \mathcal{N}_{>}) \wedge (\omega_2 \in \mathcal{N}_{>})) \rightarrow (\omega_1\omega_2^{-1} \in \mathcal{N}_{>})$$

$$\vec{\mathfrak{J}}\rho((\omega_1 \in \mathcal{N}_{\bowtie}) \wedge (\omega_2 \in \mathcal{N}_{\bowtie})) \rightarrow ((f(\omega_1) \in \mathcal{B}_{\bowtie}) \wedge (f(\omega_2) \in \mathcal{B}_{\bowtie}))$$
$$\rightarrow (f(\omega_1) \cdot [f(\omega_2)]^{-1} \in \mathcal{B}_{\bowtie})$$
$$\rightarrow (f(\omega_1) \cdot f(\omega_2^{-1}) \in \mathcal{B}_{\bowtie})$$
$$\rightarrow (f(\omega_1\omega_2^{-1}) \in \mathcal{B}_{\bowtie})$$
$$\rightarrow (\omega_1\omega_2^{-1} \in \mathcal{N}_{\bowtie})$$

Therefore

$$\vec{\mathfrak{J}}\rho((\omega_1 \in \mathcal{N}_{\bowtie}) \wedge (\omega_2 \in \mathcal{N}_{\bowtie})) \rightarrow (\omega_1\omega_2^{-1} \in \mathcal{N}_{\bowtie})$$

$$\vec{\mathfrak{J}}\rho(\omega_1\omega_2^{-1} \in \mathcal{N}_{\triangleleft}) \rightarrow (f(\omega_1) \cdot [f(\omega_2)]^{-1} \in \mathcal{B}_{\triangleleft})$$
$$\rightarrow (f(\omega_1) \cdot f(\omega_2^{-1}) \in \mathcal{B}_{\triangleleft})$$
$$\rightarrow (f(\omega_1\omega_2^{-1}) \in \mathcal{B}_{\triangleleft})$$
$$\rightarrow f((\omega_1 \in \mathcal{B}_{\triangleleft}) \vee (\omega_2 \in \mathcal{B}_{\triangleleft}))$$
$$\rightarrow (f(\omega_1) \in \mathcal{B}_{\triangleleft}) \vee (f(\omega_2) \in \mathcal{B}_{\triangleleft})$$
$$\rightarrow (\omega_1 \in \mathcal{N}_{\triangleleft}) \vee (\omega_2 \in \mathcal{N}_{\triangleleft})$$

Therefore

$$\vec{\mathfrak{J}}\rho(\omega_1\omega_2^{-1} \in \mathcal{N}_{\triangleleft}) \rightarrow (\omega_1 \in \mathcal{N}_{\triangleleft}) \vee (\omega_2 \in \mathcal{N}_{\triangleleft})$$

Thus, $\mathcal{N}$ is a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy subgroup over a finite group $\mathfrak{I}_1$.

Theorem 3.9.

Let f: $\mathfrak{I}_1 \rightarrow \mathfrak{I}_2$ be a group homomorphism, where $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are two finite subgroups and $\mathcal{B}$ be a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy normal subgroup over a finite group of $\mathfrak{I}_2$. Then the preimage of $\mathcal{N} = f \circ \mathcal{B}$ be a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy normal subgroup over a finite group of $\mathfrak{I}_1$.

Proof:

To prove: $\mathcal{N}$ be a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy normal subgroup over a finite group of $\mathfrak{I}_1$

By theorem 3.5, if $\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}) \rightarrow (\sigma\omega \in \mathcal{N})$ for all ω, σ ∈ $\mathfrak{I}$
i.e.

$$\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{>}) \rightarrow (\sigma\omega \in \mathcal{N}_{>})$$
$$\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma\omega \in \mathcal{N}_{\bowtie})$$
$$\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{\triangleleft}) \rightarrow (\sigma\omega \in \mathcal{N}_{\triangleleft})$$

(i) $\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{>})$
$\rightarrow (f(\omega\sigma) \in \mathcal{B}_{>})$ since, f is homomorphism
$\rightarrow (f(\omega)\, f(\sigma) \in \mathcal{B}_{>})$ since, $\mathcal{B}$ is normal
$\rightarrow (f(\sigma)\, f(\omega) \in \mathcal{B}_{>})$
$\rightarrow (f(\sigma\omega) \in \mathcal{B}_{>})$
$\rightarrow (\sigma\omega \in \mathcal{N}_{>})$

Therefore

$$\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{>}) \rightarrow (\sigma\omega \in \mathcal{N}_{>})$$

(ii) $\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{\bowtie})$
$\rightarrow (f(\omega\sigma) \in \mathcal{B}_{\bowtie})$ since, f is homomorphism
$\rightarrow (f(\omega)\, f(\sigma) \in \mathcal{B}_{\bowtie})$ since, $\mathcal{B}$ is normal
$\rightarrow (f(\sigma)\, f(\omega) \in \mathcal{B}_{\bowtie})$
$\rightarrow (f(\sigma\omega) \in \mathcal{B}_{\bowtie})$
$\rightarrow (\sigma\omega \in \mathcal{N}_{\bowtie})$

Therefore

$$\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{\bowtie}) \rightarrow (\sigma\omega \in \mathcal{N}_{\bowtie})$$

(iii) $\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{\triangleleft})$
$\rightarrow (f(\omega\sigma) \in \mathcal{B}_{\triangleleft})$ since, f is homomorphism
$\rightarrow (f(\omega)\, f(\sigma) \in \mathcal{B}_{\triangleleft})$ since $\mathcal{B}$ is normal
$\rightarrow (f(\sigma)\, f(\omega) \in \mathcal{B}_{\triangleleft})$
$\rightarrow (f(\sigma\omega) \in \mathcal{B}_{\triangleleft})$
$\rightarrow (\sigma\omega \in \mathcal{N}_{\triangleleft})$

Therefore

$$\vec{\mathfrak{J}}\rho(\omega\sigma \in \mathcal{N}_{\triangleleft}) \rightarrow (\sigma\omega \in \mathcal{N}_{\triangleleft})$$

Hence $\mathcal{N}$ is a $\vec{\mathfrak{J}}\rho$ neutrosophic fuzzy normal subgroup over a finite group of $\mathfrak{I}_1$

Definition 3.10.

Let $\mathcal{N}_1 = (\mathcal{N}_{1\succ}, \mathcal{N}_{1\bowtie}, \mathcal{N}_{1\lhd})$ and $\mathcal{N}_2 = (\mathcal{N}_{2\succ}, \mathcal{N}_{2\bowtie}, \mathcal{N}_{2\lhd})$ be a $\vec{J}\rho$ neutrosophic fuzzy subgroups over a finite group $\Im$. Then $\mathcal{N}_1$ is said to be a conjugate to $\mathcal{N}_2$ if there exists ω, σ ∈ $\Im$. such that

$$\vec{J}\rho(\sigma^{-1}\omega\sigma \in \mathcal{N}_2) \to (\omega \in \mathcal{N}_1)$$

i.e.

$$\vec{J}\rho(\sigma^{-1}\omega\sigma \in \mathcal{N}_{2\succ}) \to (\omega \in \mathcal{N}_{1\succ})$$

$$\vec{J}\rho(\sigma^{-1}\omega\sigma \in \mathcal{N}_{2\bowtie}) \to (\omega \in \mathcal{N}_{1\bowtie})$$

$$\vec{J}\rho(\omega \in \mathcal{N}_{1\lhd}) \to (\sigma^{-1}\omega\sigma \in \mathcal{N}_{2\lhd})$$

Theorem 3.11.

Let f: $\Im_1 \to \Im_2$ be a homomorphism. Let $\mathcal{B}_1 = (\mathcal{B}_{1\succ}, \mathcal{B}_{1\bowtie}, \mathcal{B}_{1\lhd})$ and $\mathcal{B}_2 = (\mathcal{B}_{2\succ}, \mathcal{B}_{2\bowtie}, \mathcal{B}_{2\lhd})$ be two $\vec{J}\rho$ neutrosophic fuzzy subgroups which are conjugate to each other in $\Im_2$. Let $\mathcal{N}_1 = f \circ \mathcal{B}_1$ and $\mathcal{N}_2 = f \circ \mathcal{B}_2$. Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\vec{J}\rho$ neutrosophic fuzzy subgroups which are conjugate to each other in $\Im_1$.

Proof:

By theorem 3.8, $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\vec{J}\rho$ neutrosophic fuzzy subgroups over a finite group of $\Im_1$.

Let m, n ∈ $\Im_1$ and ω, σ ∈ $\Im_2$ such that $f(m) = \omega$, $f(n) = \sigma$ and $f(n^{-1}) = \sigma^{-1}$

$$\vec{J}\rho(n^{-1}mn \in \mathcal{N}_2) \to (m \in \mathcal{N}_1)$$

i.e.

$$\vec{J}\rho(n^{-1}mn \in \mathcal{N}_{2\succ}) \to (m \in \mathcal{N}_{1\succ})$$

$$\vec{J}\rho(n^{-1}mn \in \mathcal{N}_{2\bowtie}) \to (m \in \mathcal{N}_{1\bowtie})$$

$$\vec{J}\rho(m \in \mathcal{N}_{1\lhd}) \to (n^{-1}mn \in \mathcal{N}_{2\lhd})$$

$$\begin{aligned}\vec{J}\rho(n^{-1}mn \in \mathcal{N}_{2\succ}) &\to (f(n^{-1}mn) \in \mathcal{B}_{2\succ})\\ &\to (f(n^{-1})\, f(m)\, f(n) \in \mathcal{B}_{2\succ})\\ &\to (\sigma^{-1}\omega\sigma \in \mathcal{B}_{2\succ})\\ &\to (\omega \in \mathcal{B}_{2\succ})\\ &\to (m \in \mathcal{N}_{1\succ})\end{aligned}$$

$$\begin{aligned}\vec{J}\rho(n^{-1}mn \in \mathcal{N}_{2\bowtie}) &\to (f(n^{-1}mn) \in \mathcal{B}_{2\bowtie})\\ &\to (f(n^{-1})\, f(m)\, f(n) \in \mathcal{B}_{2\bowtie})\\ &\to (\sigma^{-1}\omega\sigma \in \mathcal{B}_{2\bowtie})\\ &\to (\omega \in \mathcal{B}_{2\bowtie})\\ &\to (m \in \mathcal{N}_{1\bowtie})\end{aligned}$$

$$\begin{aligned}\vec{J}\rho(m \in \mathcal{N}_{1\lhd}) &\to (f(m) \in \mathcal{B}_{1\lhd})\\ &\to (\omega \in \mathcal{B}_{1\lhd})\\ &\to (\sigma^{-1}\omega\sigma \in \mathcal{B}_{1\lhd})\\ &\to (f(n^{-1})\, f(m)\, f(n) \in \mathcal{B}_{1\lhd})\\ &\to (n^{-1}mn \in \mathcal{N}_{2\lhd})\end{aligned}$$

Therefore

$\mathcal{N}_1$ and $\mathcal{N}_2$ are $\vec{J}\rho$ neutrosophic fuzzy conjugate subgroups over a finite group of $\Im_1$

Definition 3.12.

Let f: $\Im_1 \to \Im_2$ be a homomorphism, where $\Im_1$ and $\Im_2$ are two finite groups and let $\mathcal{N} = (\mathcal{N}_{\succ}, \mathcal{N}_{\bowtie}, \mathcal{N}_{\lhd})$ be a $\vec{J}\rho$ neutrosophic fuzzy subgroup over a finite group of $\Im_1$, then $\mathcal{N}$ is said to be a $\vec{J}\rho$ neutrosophic fuzzy invariant subgroup over a finite group of $\Im_1$.

$$\vec{J}\rho(\omega \in \mathcal{N}) \to (\sigma \in \mathcal{N}) \text{ and } \vec{J}\rho(\sigma \in \mathcal{N}) \to (\omega \in \mathcal{N})$$

i.e.

$$\vec{J}\rho(\omega \in \mathcal{N}_{\succ}) \to (\sigma \in \mathcal{N}_{\succ}) \text{ and } \vec{J}\rho(\sigma \in \mathcal{N}_{\succ}) \to (\omega \in \mathcal{N}_{\succ})$$

$$\vec{J}\rho(\omega \in \mathcal{N}_{\bowtie}) \to (\sigma \in \mathcal{N}_{\bowtie}) \text{ and } \vec{J}\rho(\sigma \in \mathcal{N}_{\bowtie}) \to (\omega \in \mathcal{N}_{\bowtie})$$

$$\vec{J}\rho(\omega \in \mathcal{N}_{\lhd}) \to (\sigma \in \mathcal{N}_{\lhd}) \text{ and } \vec{J}\rho(\sigma \in \mathcal{N}_{\lhd}) \to (\omega \in \mathcal{N}_{\lhd}).$$

Theorem 3.13.

Let f: $\Im_1 \to \Im_2$ be a homomorphism, where $\Im_1$ and $\Im_2$ are two finite groups. Let $\mathcal{B} = (\mathcal{B}_{\succ}, \mathcal{B}_{\bowtie}, \mathcal{B}_{\lhd})$ be a $\vec{J}\rho$ neutrosophic fuzzy subgroup over a finite group which is invariant in $\Im_2$. Then $\mathcal{N} = f \circ \mathcal{B}$ is said to be a $\vec{J}\rho$ neutrosophic fuzzy subgroup over a finite group which is invariant in $\Im_1$.

Proof:

Let us assume that m, n ∈ $\Im_1$ and ω, σ ∈ $\Im_2$ such that

$f(m) = \omega$, $f(n) = \sigma$.

$$\vec{J}\rho(m \in \mathcal{N}) \to (n \in \mathcal{N}) \text{ and } \vec{J}\rho(n \in \mathcal{N}) \to (m \in \mathcal{N})$$

i.e.

$$\vec{J}\rho(m \in \mathcal{N}_{\succ}) \to (n \in \mathcal{N}_{\succ}) \text{ and } \vec{J}\rho(n \in \mathcal{N}_{\succ}) \to (m \in \mathcal{N}_{\succ})$$

$$\vec{J}\rho(m \in \mathcal{N}_{\bowtie}) \to (n \in \mathcal{N}_{\bowtie}) \text{ and } \vec{J}\rho(n \in \mathcal{N}_{\bowtie}) \to (m \in \mathcal{N}_{\bowtie})$$

$$\vec{J}\rho(m \in \mathcal{N}_{\lhd}) \to (n \in \mathcal{N}_{\lhd}) \text{ and } \vec{J}\rho(n \in \mathcal{N}_{\lhd}) \to (m \in \mathcal{N}_{\lhd})$$

The truth membership function is

$$\begin{aligned}\vec{J}\rho(m \in \mathcal{N}_{\succ}) &\to (f(m) \in \mathcal{B}_{\succ})\\ &\to (\omega \in \mathcal{B}_{\succ})\\ &\to (\sigma \in \mathcal{B}_{\succ})\\ &\to (f(n) \in \mathcal{B}_{\succ})\\ &\to (n \in \mathcal{N}_{\succ})\end{aligned}$$

and

$$\begin{aligned}\vec{J}\rho(n \in \mathcal{N}_{\succ}) &\to (f(n) \in \mathcal{B}_{\succ})\\ &\to (\sigma \in \mathcal{B}_{\succ})\\ &\to (\omega \in \mathcal{B}_{\succ})\\ &\to (f(m) \in \mathcal{B}_{\succ})\\ &\to (m \in \mathcal{N}_{\succ})\end{aligned}$$

Therefore

$\vec{\mathcal{J}}\rho(m \in \mathcal{N}_{\succ}) \rightarrow (n \in \mathcal{N}_{\succ})$ and $\vec{\mathcal{J}}\rho(n \in \mathcal{N}_{\succ}) \rightarrow (m \in \mathcal{N}_{\succ})$.

Similarly

$\vec{\mathcal{J}}\rho(m \in \mathcal{N}_{\bowtie}) \rightarrow (n \in \mathcal{N}_{\bowtie})$ and $\vec{\mathcal{J}}\rho(n \in \mathcal{N}_{\bowtie}) \rightarrow (m \in \mathcal{N}_{\bowtie})$

$\vec{\mathcal{J}}\rho(m \in \mathcal{N}_{\triangleleft}) \rightarrow (n \in \mathcal{N}_{\triangleleft})$ and $\vec{\mathcal{J}}\rho(n \in \mathcal{N}_{\triangleleft}) \rightarrow (m \in \mathcal{N}_{\triangleleft})$.

So, the conditions are satisfied by the same truth membership function.

4. Conclusions

The characteristics of neutrosophic sets are the truth membership function ($\succ$), indeterminate membership function ($\bowtie$), and falsity membership function ($\triangleleft$). In this study, the concept of homomorphism on $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup over a finite group is introduced. Then, we presented the definition of a homomorphism of $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy subgroup over a finite group and the $\vec{\mathcal{J}}\rho$ neutrosophic fuzzy normal subgroup over a finite group. Moreover, this research proved some of its basic properties.

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