Mathematics and Statistics 11(6): 943-952, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110609

The Number of Games to Win by Two Points

Nahathai Rerkruthairat, Noppadon Wichitsongkram*

Department of Mathematics, Faculty of Science, Srinakharinwirot University, Thailand

Received August 27, 2023; Revised November 13, 2023; Accepted November 27, 2023

Cite This Paper in the Following Citation Styles

(a): *[1] Nahathai Rerkruthairat, Noppadon Wichitsongkram , "The Number of Games to Win by Two Points, "Mathematics and Statistics, Vol. 11, No. 6, pp. 943 - 952, 2023. DOI: 10.13189/ms.2023.110609.*

(b): *Nahathai Rerkruthairat, Noppadon Wichitsongkram (2023). The Number of Games to Win by Two Points. Mathematics and Statistics, 11(6), 943 - 952. DOI: 10.13189/ms.2023.110609.*

Abstract Sometimes draws or ties occur in sports. Tiebreakers are the forms of competition that break ties and decide the winner when a draw or a tie occurs. Depending on types of tiebreakers, some take shorter and some take longer to end the competition. In this article, we are interested in calculating the expectation and variance of the number of games that will continue after a draw from types of tiebreakers that require players to win by two points. We focus on three types of win by two points that are used in many popular sports, such as tennis, volleyball and racquetball. By calculating the expected number of games, we can compare the number of games in each type of tiebreakers that will approximately be taken to end the game. In these kinds of sports, the rules to gain each point are usually the same. This means that there are the same finite states that the players or teams can reach in each point and each possible state depends only on the previous state. Since we know that a Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event, we can use an application of Markov chains to solve the problems.

Keywords Markov Chain, Normal Matrix, Expectation

1. Introduction

A draw or tie occurs in a competitive sport when the results are identical or inconclusive. Such an outcome, sometimes referred to as deadlock, can also occur in other areas of life such as politics, business, and wherever there are different factions regarding an issue. A tiebreaker or sometimes called a deuce in sports is a form of competition that decides the winner. Due to variations of rules in sports, there are several types of tiebreakers. For example, in tennis, two or four players play on a rectangular court divided by a net. The players use rackets to hit a ball over the net and into the opponent's court. The objective of the game is to score more points than the opponent by making them unable to return the ball. The score of a tennis match is based on the number of games and sets that each player or team wins. A game is a series of points played with the same player serving, and a set is a series of games played with alternating service. A match is usually best of three or best of five sets, depending on the tournament. According to the international tennis official rule [1], the scoring system for each game is as follows: The first point won by a player is called 15, the second point 30, the third point 40 and the fourth point called game. If both players have 40 points, the score is called deuce, and one player must win by two points to win the game. The first point after deuce is called advantage for the player who won it. If the same player wins the next point, they win the game. If the other player wins the next point, the score returns to deuce. A player must win at least six games to win a set, and have a two-game lead over the opponent. For example, 6-4 or 7-5 are valid set scores, but 6-6 is not. If the score of a set reaches 6-6, a tiebreaker is played to decide the winner of the set. A tiebreaker in tennis is a special game that follows different rules than a normal game. A player wins the tiebreaker by reaching at least seven points and having a two-point lead over the other player. However, a deuce and a tiebreaker in tennis are two different types of tiebreakers. A deuce in tennis is a win by two-point type of tiebreaker and a tiebreaker in tennis is not. This is because a player can win, for example, by a score of 7-2 in a tiebreaker. Therefore, in the case of tennis, we are only interested in the number of games after a deuce not after a tiebreaker even though they share the same names.

In volleyball, two teams of six players play against each other on a rectangular court divided by a net. A point is awarded to the team that wins the rally, regardless of which team served the ball. The objective of the game is to score 25 points and win by a margin of two points [2]. For example, 25-23 or 26-24 are valid game scores, but 25-25 is not. If the score of a game reaches 24-24, the game continues until one team has a two-point lead which is similar to a deuce in tennis. However, the serving rule in volleyball is different from the serving rule in tennis, where there is no change in the server.

As we have seen, different sports may have different rules of serving. If there is no deuce or tie in a match the time that a match will take can be easier to predict. On the other hand, if there are some deuce or tie games in a match this may affect the length of the game greatly. Therefore, it is beneficial to know the average number of games for some popular types of tiebreakers.

The probability of winning a game or a match from sports has been computed in several works. For example, in 2007, Collings [3] presents the probability of winning a deuce game in tennis and volleyball, three years later, Newton and Keller [4] compute the probability of winning at tennis, in 2012, Brown and Pasko [5] compute the probability of winning a racquetball match, in 2018, Khotmongkon, Rerkruthairat and Suriwong [6] present the probability of winning a racquetball game with deuce. The length of games in volleyball has been analyzed in [7]. However, the average number of games that continues after a draw or tie is not seen in the literature. In this work, we calculate the expectation and variance of the number of games that continues after a draw from types of tiebreakers that require players to win by two points. Indeed, we will focus on three types of the most common forms of win by two-point games. The first type is similar to a deuce in tennis, the second type is similar to a deuce in volleyball and the third type is similar to a deuce in racquetball. The problems like these can be solved by using an application of finite absorbing Markov chains and their solution can be obtained from the elements of normal matrix of the Markov chains. In 2007, Wong [8] uses Markov chains to compute the n power of the transition matrix and find the probability of winning a game from tennis after a deuce. In this article, we use Markov chains and its normal matrix to compute the expectation and variance of the number of games that will continue after a draw in win-by-two points game.

2. Materials and Methods

First, we provide some mathematical background that is necessary for this work. The background related to Markov chains can be found in many undergraduate textbooks such as an undergraduate probability textbook written by Sheldon Ross [9]. Recall that a Markov chain or Markov process is a stochastic process that has the property of memorylessness, meaning that the future state depends only on the present state and not on the past states, that is, a Markov chain is a collection of random variables X_t where the index t run through the non-negative integers. We can interpret X_t as being a state of some system at time t. We say that the system is in state i at time t if $X_t = i$. The conditional probabilities

$$p_{ij} = P(X_{t+1} = j \mid X_t = i)$$

are called transition probabilities and we call $[p_{ij}]$ a transition matrix. A random walk satisfies this property. For example, suppose we have a coin that is biased such that it lands on heads with probability p. We start with \$1 and we flip the coin repeatedly. If we get heads, we win \$1. If we get tails, we lose \$1. We stop flipping the coin if we either reach \$4 or lose all our money. This is a Markov chain with five states: S_0, S_1, S_2, S_3, and S_4, where S_i represents having i dollars. Then we have

$p_{i,i+1} = P(X_{i+1} = i+1 \mid X_i = i) = p$ and

$p_{i,i-1} = P(X_{i+1} = i-1 \mid X_i = i) = 1-p$ for $i = 1,2,3$.

This gives the following transition matrix:

$$[p_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A state is called an absorbing state if we cannot leave it once we reach it. Hence the state S_0 and S_4 above are examples of absorbing states. A Markov chain is called an absorbing chain if it contains at least one absorbing state and it is possible to go from each non-absorbing state to at least one absorbing state in a finite number of steps. Also, for any non-absorbing state, we call it a transient state. From the example of coin flipping, the states S_1, S_2 and S_3 are transient states.

In the classical text [10], we know that for finite absorbing Markov chains with n absorbing states and m transient states, the $m \times m$ normal matrix can be used to compute the expected number of visits to a particular transient state given the system's initial state. Given an absorbing chain, we can compute its normal or fundamental matrix N. Indeed, we rearrange states in the transition matrix P as P' so that P' has the following form:

$$P' = \begin{bmatrix} Q & R \\ 0 & I_n \end{bmatrix},$$

where Q is the $m \times m$ matrix containing the transition probabilities among the non-absorbing states, R is the $m \times n$ matrix containing the probabilities of going from each of the non-absorbing states to one of the absorbing states, 0 is the $n \times m$ zero matrix and I_n is the $n \times n$ identity matrix representing the n absorbing states. Then the normal matrix N can be obtained from the sum of the powers of matrix Q:

$$N = \sum_{k=0}^{\infty} Q^k = (I_m - Q)^{-1}.$$

According to [10], the matrix entry $N_{i,j}$ is the expected number of visits to transient state j given that the system starts in transient state i.

Let $X_{i,j}$ be the random variable for the number of visits to transient state j given that the system starts in transient state i, then the expected value of $X_{i,j}$ is $E[X_{i,j}] = N_{i,j}$. Given that the system starts in transient state i, let

$$Y_i = \sum_{j=1}^{m} X_{i,j}$$

be the random variable for the total number of visits to all transient states (or the random variable of the number of games starting from state i), then the expected value of Y_i is

$$E[Y_i] = \sum_{j=1}^{m} E[X_{i,j}] = \sum_{j=1}^{m} N_{i,j}$$

This means that the expected number of games starting from a transient state i is the same as the summation of row i of the matrix N. According to Michael [11], we can calculate the variance of the number of games starting from a transient state i by using the summation of row i of the matrix $2N^2 - N$ then subtract from the corresponding row sum square of N, that is,

$$Var[Y_i] = \sum_{j=1}^{m} (2N^2 - N)_{i,j} - (\sum_{j=1}^{m} N_{i,j})^2.$$

From the example of coin flips, we have

$$N = \frac{1}{1-2p+2p^2} \begin{bmatrix} p^2 - p + 1 & p & p^2 \\ 1-p & 1 & p \\ (1-p)^2 & 1-p & p^2 - p + 1 \end{bmatrix}$$

Then the first-row sum is the expected number of coin flips until the game ends. That is

$$E[X] = \frac{2p^2 + 1}{1 - 2p + 2p^2}$$

and also, it is an easy exercise to show that

$$Var[X] = 4p \frac{1 + p - 2p^2}{1 - 2p + 2p^2},$$

where X is the number of coin flips until the game ends.

If, for example $p = 0.6$, then we have $E[X] \approx 3.31$ and $Var[X] \approx 4.06$. This means that the game will approximately end in 3.31 times of coin flips.

In this article, we only consider sports that have two sides competing against each other and each side has the same number of players. For simplicity, we assume there is only one player on each side, called player A and player B.

In win by two-point games, there are several states that players can reach depending on the types of sports.

Let p be the probability that player A wins the next point while serving and q be the probability that player B wins the next point while serving. In each scenario, we assume the probability of making the transition between states from point to point is constant.

In type 1 or in a tiebreaker that is similar to a deuce in tennis, we assume that player A always serves regardless of who wins a point. In each serve, only one point can be obtained by either one of the players. The game will continue until either player A or B scores two points more than the other. This results in the five states that the players can reach after a draw as follows:

(1) : *A* wins
(2) : *A* has an advantage and *A* serves
(3) : *A* draw
(4) : *B* has an advantage and *A* serves
(5) : *B* wins

Then we can write a diagram for the five states below:

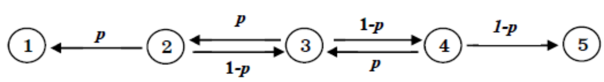

Figure 1. The diagram of type 1

This gives us the following transition matrix P_1:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In type 2 or in a tiebreaker that is similar to a 24-24 tie score in volleyball, we assume player A serves first. If A wins then A gains a point and A serves for the next point and if A loses B gains a point and B serves for the next point and so on. The game will continue until either player A or B scores two points more than the other. Therefore, there are six states that the players can reach after a draw as follows:

(1) : A wins
(2) : A has an advantage and A serves
(3) : A draw and A serves
(4) : A draw and B serves
(5) : B has an advantage and B serves
(6) : B wins

Then we have a diagram for the six states as follows:

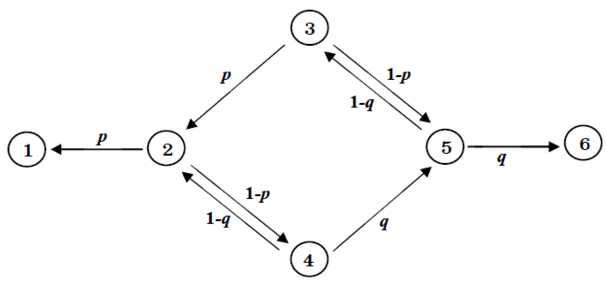

Figure 2. The diagram of type 2

This gives us the following transition matrix P_2:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ p & 0 & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 0 & 1-p & 0 \\ 0 & 1-q & 0 & 0 & q & 0 \\ 0 & 0 & 1-q & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, in type 3 or in a tiebreaker that is similar to a deuce in racquetball, we assume player A serves first. If A wins then A gains a point and A also serves for the next point and if A loses then no one gains a point and B will serve for the next point and so on. The game will continue until either player A or B scores two points more than the other. Thus, we have eight states that players can reach as follows:

(1) : A wins
(2) : A has an advantage and A serves
(3) : A has an advantage and B serves
(4) : A draw and A serves
(5) : A draw and B serves
(6) : B has an advantage and A serves
(7) : B has an advantage and B serves
(8) : B wins

The diagram of type 3 is shown below:

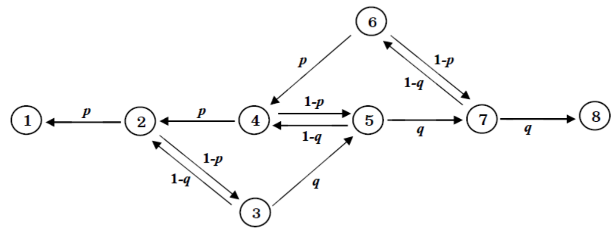

Figure 3. The diagram of type 3

This give us the following transition matrix P_3:

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 & q & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 1-p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q & 0 & 0 & q & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-q & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Results

In this section, we compute the expectation and variance of the number of games in each type of the tiebreakers. The expectation and variance can be obtained from the normal matrix that we mentioned in the previous section.

3.1. Type 1

In type 1, we first rearrange P_1 as

$$P_1' = \begin{bmatrix} 0 & 1-p & 0 & p & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 0 & 1-p \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then we have

$$Q_1 = \begin{bmatrix} 0 & 1-p & 0 \\ p & 0 & 1-p \\ 0 & p & 0 \end{bmatrix}.$$

By a routine calculation, we get

$$N_1 = (I_3 - Q_1)^{-1} = \begin{bmatrix} \frac{p^2-p+1}{2p^2-2p+1} & \frac{1-p}{2p^2-p+1} & \frac{(1-p)^2}{2p^2-p+1} \\ \frac{p}{2p^2-p+1} & \frac{1}{2p^2-p+1} & \frac{1-p}{2p^2-p+1} \\ \frac{p^2}{2p^2-p+1} & \frac{p}{2p^2-p+1} & \frac{p^2-p+1}{2p^2-p+1} \end{bmatrix}.$$

Since we are interested in the number of games that will continue after a draw, this corresponds to the second-row sum of the matrix N_1. That means $Y_2^{\text{type 1}}$ or simply Y_2^1 is the number of games that will continue until the game ends.

Then we have

$$E[Y_2^1] = \sum_{i=1}^{3}(N_1)_{2,i} = \frac{p}{2p^2-2p+1} + \frac{1}{2p^2-2p+1} + \frac{1-p}{2p^2-2p+1} = \frac{2}{2p^2-2p+1}$$

and

$$Var[Y_2^1] = \sum_{i=1}^{3}(2N_1^{\,2} - N_1)_{2,i} - (\sum_{i=1}^{3}(N_1)_{2,i})^2 = \frac{8p(1-p)}{(2p^2-2p+1)^2}.$$

The following figure shows the graphs of $E[Y_2^1]$ and $Var[Y_2^1]$.

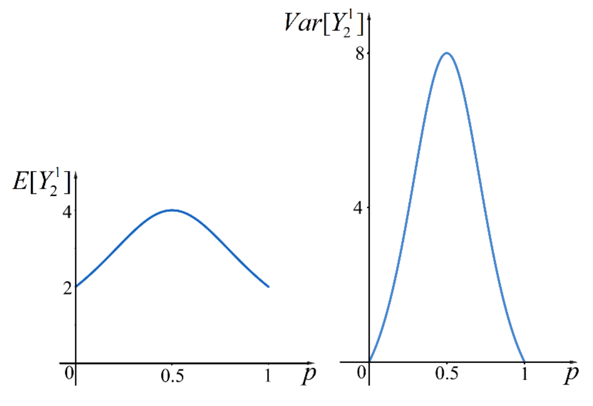

Figure 4. The expectation and variance of the number of games in type 1

From the graph, the expectation is between 2 and 4 and has the maximum at $p = 0.5$. This means that the average number of games is largest when $p = 0.5$ and the number of games will be 4 games with the variance 8. If the value of p is toward 0 or 1 then the average numbers of games will be close to 2. This makes sense because for p approaching zero, the server has a low percentage of winning and for p approaching one, the server has a very high probability of winning. In other words, the server will be most likely to win or lose in 2 consecutive games.

3.2. Type 2

In type 2, we have

$$P_2' = \begin{bmatrix} 0 & 0 & 1-p & 0 & p & 0 \\ p & 0 & 0 & 1-p & 0 & 0 \\ 1-q & 0 & 0 & q & 0 & 0 \\ 0 & 1-q & 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$Q_2 = \begin{bmatrix} 0 & 0 & 1-p & 0 \\ p & 0 & 0 & 1-p \\ 1-q & 0 & 0 & q \\ 0 & 1-q & 0 & 0 \end{bmatrix}.$$

So we have $N_2 = (I_4 - Q_2)^{-1}$ which is

$$N_2 = \frac{1}{p^2(1-q)+q^2(1-p)+pq}\begin{bmatrix} p+q-pq & q(1-p)(1-q) & (1-p)(p+q-pq) & q(1-p) \\ p & p+q-pq & p(1-p) & (p+q)(1-p) \\ (p+q)(1-q) & q(1-q) & p+q-pq & q \\ p(1-q) & (1-q)(p+q-pq) & p(1-p)(1-q) & p+q-pq \end{bmatrix}$$

Since we assume player A serves first, this corresponds to the second-row sum of the matrix N_2. This means that $Y_2^{\text{type 2}}$ or simply Y_2^2 is the number of games that will continue until the game ends.

Hence, $E[Y_2^2]=\sum_{i=1}^{4}(N_2)_{2,i} = \frac{p+(p+q-pq)+p(1-p)+(p+q)(1-p)}{p^2(1-q)+q^2(1-p)+pq} = \frac{2(2p+q-pq-p^2)}{p^2(1-q)+q^2(1-p)+pq}$.

The table below shows the values of expectation for some values of p and q.

Table 1. Some values of the expected number of games in type 2

p \ q	0	0.2	0.4	0.5	0.6	0.8	1.0
0	∞	10	5	4	3.33	2.50	2
0.2	18	10	5.86	4.75	3.96	2.94	2.32
0.4	8	6.55	5	4.37	3.84	3.04	2.48
0.5	6	5.31	4.42	4	3.62	2.99	2.5
0.6	4.67	4.34	3.84	3.59	3.33	2.87	2.48
0.8	3	2.94	2.83	2.75	2.67	2.50	2.32
1.0	2	2	2	2	2	2	2

The following figure shows the graph of $E[Y_2^2]$.

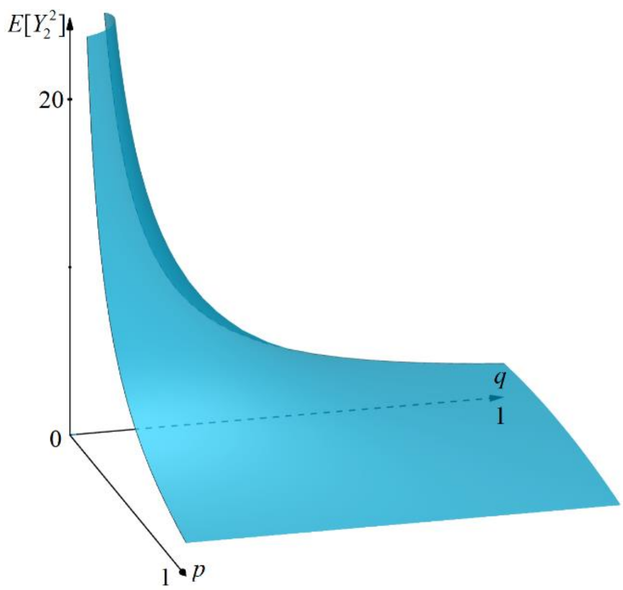

Figure 5. The graph of the expected number of games in type 2

Next, we compute the variance of the number of games $Var[Y_2^2]$.

$$Var[Y_2^2]=\sum_{i=1}^{4}(2N_2^{\,2}-N_2)_{2,i}-(\sum_{i=1}^{4}(N_2)_{2,i})^2 = \frac{4(1-p)(p^3q+2p^2q^2-5p^2q+2p^2+pq^3-6pq^2+6pq-q^3+q^2)}{(p^2(1-q)+pq+q^2(1-p))^2}.$$

The table below shows the values of variance for some values of p and q.

Table 2. Some values of the variance of the number of games in type 2

p \ q	0	0.2	0.4	0.5	0.6	0.8	1.0
0	∞	80	15	8	4.44	1.25	0
0.2	160	80	24.54	14.19	8.34	2.77	0.54
0.4	30	26.97	15	10.47	7.1	2.89	0.73
0.5	16	15.84	10.58	8	5.81	2.64	0.75
0.6	8.89	9.30	7.10	5.75	4.44	2.27	0.73
0.8	2.5	2.77	2.52	2.27	1.96	1.25	0.54
1.0	0	0	0	0	0	0	0

The graph of variance $Var[Y_2^2]$ is shown below.

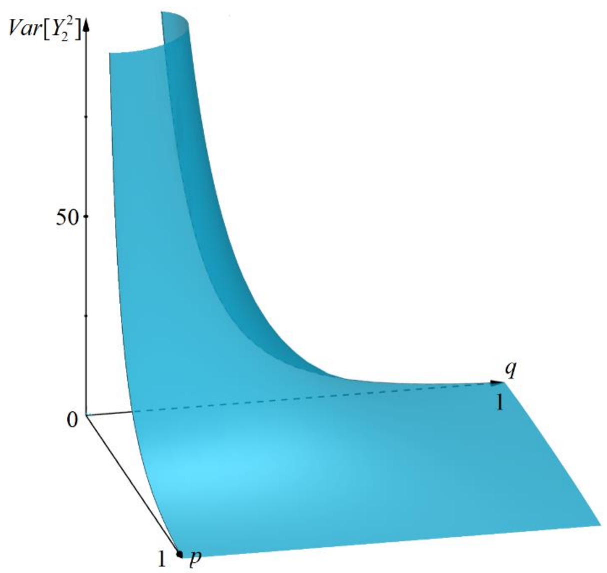

Figure 6. The graph of variance in type 2

From figure 5 and 6, if p and q are approaching zero then the expectation and variance of the number of games will be very large. This means that if player A and B have very low chances of winning from their serving then the game will be likely to continue for a large number of games. When p is close to 1 regardless of the value of q, the average number of games will be very close to 2 and the variance will also be very close to zero. This means that the games will likely be played for no more than 2 games when player A has a very high chance of winning. On the other hand, if the value of q is close to 1 then the average number of games will be between 2 and 2.5. This is because player B has a very high chance of winning from serving so that the games will end in 2 games in the case where A has a low or high chance of winning from serving and the game will approximately end in 2.5 games when $p = 0.5$. If we assume both players have almost the same skill level, then the average number of games will be approximately between 3 and 5 games with a variance approximately between 4 and 15. If the two players have much different skills, that is, p is much greater than q or vice versa, the average number of games will be approximately between 2 and 4 games. This means that the game will end sooner in favor of different skill levels. In addition, for $p > q$, the average number of games tends to be higher than the number of games in the case of $q > p$. This is because if player A loses the first serving game to B then player A loses a point and B will only need one more point to win in B serving game.

3.3. Type 3

Finally, in type 3, we have

$$P_3' = \begin{bmatrix} 0 & 1-p & 0 & 0 & 0 & 0 & p & 0 \\ 1-q & 0 & 0 & q & 0 & 0 & 0 & 0 \\ p & 0 & 0 & 1-p & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & 0 & 0 & q & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 1-p & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$Q_3 = \begin{bmatrix} 0 & 1-p & 0 & 0 & 0 & 0 \\ 1-q & 0 & 0 & q & 0 & 0 \\ p & 0 & 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-q & 0 & 0 & q \\ 0 & 0 & p & 0 & 0 & 1-p \\ 0 & 0 & 0 & 0 & 1-q & 0 \end{bmatrix}.$$

Since we assume player A serves first, this corresponds to the third-row sum of the matrix $N_3 = (I_6 - Q_3)^{-1}$.

This means $Y_3^{\text{type 3}}$ or simply Y_3^3 is the number of games that is required to continue until the game ends.

Due to very long formulae of the entries in the matrix N_3, we do not print the form of the matrix in this case.

We get the expectation of Y_3^3 as follows:

$$E[Y_3^3]=\sum_{i=1}^{6}(N_3)_{3,i}=\frac{2p^3q-2p^3+3p^2q^2-10p^2q+4p^2+pq^3-7pq^2+8pq-q^3+4q^2}{p^3(1-q)+p^2q-p^2q^2+pq^2+q^3(1-p)}.$$

The table below shows the values of expectation for some values of p and q.

Table 3. Some values of the expected number of games in type 3

p \ q	0	0.2	0.4	0.5	0.6	0.8	1.0
0	∞	19	9	7	5.67	4	3
0.2	18	18	10.39	8.12	6.52	4.49	3.28
0.4	8	9.43	8	6.97	6	4.43	3.32
0.5	6	6.99	6.55	6	5.38	4.20	3.25
0.6	4.67	5.30	5.26	5.01	4.67	3.87	3.12
0.8	3	3.21	3.30	3.28	3.22	3	2.68
1.0	2	2	2	2	2	2	2

The following figure shows the graph of $E[Y_3^3]$.

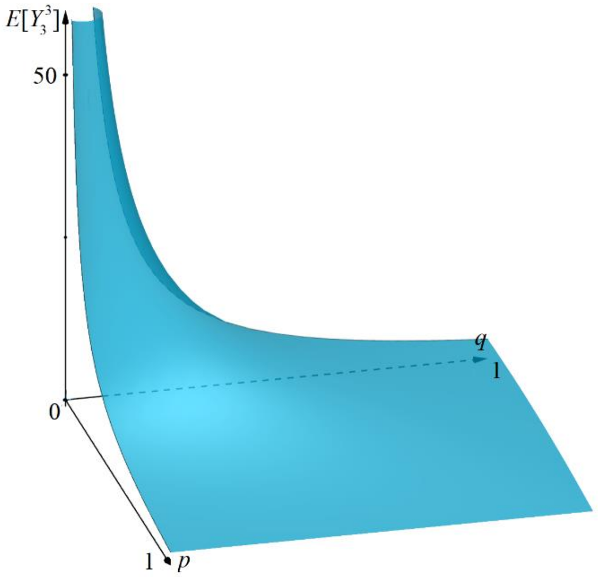

Figure 7. The graph of the expected number of games in type 3

The variance $Var[Y_3^3]$ can be obtained from

$$Var[Y_3^3]=\sum_{i=1}^{6}(2N_3{}^2-N_3)_{3,i}-(\sum_{i=1}^{6}(N_3)_{3,i})^2$$

$$=\frac{(1-p)}{(p^3(1-q)+p^2q-p^2q^2+pq^2+q^3(1-p))^2}(-5p^4q^3+21p^4q^2-24p^4q+8p^4-9p^3q^4$$

$$+66p^3q^3-96p^3q^2+48p^3q-4p^2q^5+57p^2q^4-136p^2q^3+80p^2q^2+12pq^5-56pq^4$$

$$+48pq^3-8q^5+8q^4).$$

The table below shows the values of variance for some values of p and q.

Table 4. Some values of the variance of number of games in type 3

p \ q	0	0.2	0.4	0.5	0.6	0.8	1.0
0	∞	160	30	16	8.89	2.50	0
0.2	160	232	63.88	34.06	18.59	5.29	0.60
0.4	30	62.09	42	28.89	18.65	6.47	1.02
0.5	16	32.42	28.28	22	15.73	6.33	1.19
0.6	8.89	17.21	17.70	15.24	12	5.70	1.31
0.8	2.50	4.37	5.34	5.29	4.88	3.25	1.17
1.0	0	0	0	0	0	0	0

The graph of $Var[Y_3^3]$ is shown below.

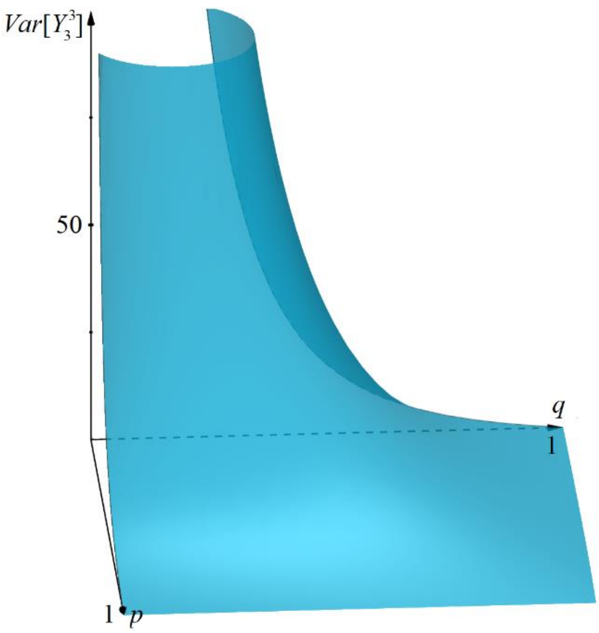

Figure 8. The graph of variance in type 3

From figure 7 and 8, the graph of the expectation and variance in type 3 look similar to the graph in type 2 but the values of expectation and variance in type 3 are higher for the same values of p and q. That is, if p and q are close to zero the expectation and variance of the number of games will be very large. If both players have almost the same skill levels the average number of games will be approximately between 4 and 8 games with the variance approximately between 12 and 42. In the case of different skill levels, the average number of games will be approximately between 2 and 6 games. This means that the game will end sooner in favor of different skill levels. For $p > q$, the average number of games tends to be lower than the number of games in the case of $q > p$. This is because in order for the game to end by the winning of player B, player A must lose a serving game first. This means player B needs at least one additional more game than player A to win from a draw.

4. Conclusions

Because the servers in type 1 and 2 can lose a point when serving and there is no changing the server in type 1, the average of the number of games in type 1 tends to be shorter than the others. On the other hand, due to the servers in type 3 not losing a point in their serving, the average number of games in type 3 tends to be higher than the others. In fact, in type 1, a draw usually ends between 2 and 4 games with a variance between 0 and 8. However, for type 2 and 3, the number of games can be very large. This means a draw will end sooner in the case of the servers being able to lose a point in their serving and a draw tends to be longer in the case of changing the server. This is why there was a change from type 3 to type 2 of tiebreaker rule in some sports such as volleyball so that the playing time can be controlled easier. In addition, a tiebreaker in the case of a competition between players who have high probabilities of winning in their own serving will end sooner than a tiebreaker between players who have low chances of winning in their own serving. In addition, due to player A in type 2 being able to lose a point in serving, the average number of games for $p > q$ tends to be higher than the number of games for $q > p$. This is opposite to the type 3, that is, in case of the servers not losing a point in their serving, the average number of games for $p > q$ tends to be lower than the number of games for $q > p$. This is because in type2, if player A loses the first serving game then player B only need one more point to win the game. In type 3, however, B still needs two point to win the game.

Acknowledgements

This research is supported by the SWU Endowment Fund Year 2021. In addition, we wish to thank the editor and reviewers for their comments and suggestions.

REFERENCES

[1] International Tennis Federation, "Tennis Rules and Regulations," International Tennis Federation, https://www.itftennis.com/en/about-us/governance/rules-and-regulations (accessed Nov. 1, 2023).

[2] International Volleyball Federation, "Official Volleyball Rules of the Games," Volleyball, https://www.fivb.com/en/volleyball/thegame_glossary/officialrulesofthegames

(accessed Nov. 1, 2023).

[3] Collings B., "Tennis (and volleyball) without geometric series," The College Mathematics Journal, vol. 38, no. 1, pp. 55-57, 2007. URL: https://maa.org/sites/default/files/Collings-1-0731820.pdf

[4] Newton P. and Keller J., "Probability of winning at tennis I. Theory and data," Studies in Applied Mathematics, vol. 114, no. 3, pp. 241-269, 2005. DOI: 10.1111/j.0022-2526.2005.01547

[5] Brown T., Pasko B., "Winning a racquetball match," The College Mathematics Journal, vol. 43, no. 5, pp. 395-400, 2012. DOI: 10.4169/college.math.j.43.5.395

[6] Khotmongkon P., Rerkruthairat N., Suriwong S., Watcharakarn K., "The Probability of Winning a Racquetball Game with Deuce," The College Mathematics Journal, vol. 49, no. 5, pp. 353-358, 2018. DOI: 10.1080/07468342.2018.1526018

[7] Kovacs B., "The Effect of the Scoring System Changes in Volleyball: A Model and an Empirical Test," Journal of Quantitative Analysis in Sports, vol. 5, no. 3, pp. 1-12, 2009. DOI: 10.2202/1559-0410.1182

[8] Wong R., Zigarovich M., "Tennis with Markov," The College Mathematics Journal, vol. 38, no. 1, pp. 53-54, 2007. URL: https://www.maa.org/sites/default/files/Wong-1-0733353.pdf

[9] Ross S., "Additional Topics in Probability," in A First Course in Probability, 8th ed, Pearson, 2010, pp. 419-425.

[10] John G., Kemeny and J. Laurie Snell, "ABSORBING MARKOV CHAINS," in Finite Markov Chains, 2nd ed, Springer-Verlag, 1976, pp. 43-68.

[11] Michael A. Carchidi, Robert L. Higgins, "Covariances Between Transient States in Finite Absorbing Markov Chains," The College Mathematics Journal, vol. 48, no. 1, pp. 42-50, 2017. DOI: 10.4169/college.math.j.48.1.42.