Mathematics and Statistics 11(6): 936-942, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110608

Generalized Half-Logistic Distribution Using Linear Regression Model

Ahmed Al-Adilee[1], Wasan Al-Shemmari[2,*]

[1]Department of Mathematics, Faculty of Education, University of Kufa, Iraq
[2]Department of Mathematics, Faculty of Education for Girls, University of Kufa, Iraq

Received July 7, 2023; Revised October 24, 2023; Accepted November 14, 2023

Cite This Paper in the Following Citation Styles

(a): *[1] Ahmed Al-Adilee, Wasan Al-Shemmari, "Generalized Half-Logistic Distribution Using Linear Regression Model," Mathematics and Statistics, Vol. 11, No. 6, pp. 936 - 942, 2023. DOI: 10.13189/ms.2023.110608.*

(b): *Ahmed Al-Adilee, Wasan Al-Shemmari (2023). Generalized Half-Logistic Distribution Using Linear Regression Model. Mathematics and Statistics, 11(6), 936 - 942. DOI: 10.13189/ms.2023.110608.*

Abstract In this study, the generalized half-logistic distribution (GHLD) was expanded by replacing the shape parameter with a linear model, denoted by the notation $\underline{\lambda}\ \underline{\acute{Z}}$. This model involves a vector of explanatory variables denoted by $\underline{\acute{Z}} = (z_{0i}, z_{1i}, \dots, z_{ki})$, where $z_{0i} = 1$ with a vector of coefficients of each one of those explanatory variables, denoted by $\underline{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_n)$. The linear model represents several explanatory variables with their coefficients that represent effects on some items. Briefly, the proposed distribution is denoted by, LM-GHLD. Afterward, by finding the pdf, and cdf of LM-GHLD, many mathematical and statistical characteristics were investigated, such as the survival function, the hazard function, the moments, the moment generating function, quantiles, the Rényi entropy, and the order statistic function. The unknown parameters of the modern distribution were estimated with the non-Bayesian method, which is known as the Maximum Likelihood Estimate (MLE). An important part of such a study is related to the simulation, which is shown within a generation of different sample sizes. A goodness-of-fit measure has been implemented on real data sets to compare the classical distribution (GHLD) and the proposed distribution (LM-GHLD) enabling us to determine which distribution is better. Eventually, we provide some conclusions and summarize our findings.

Keywords Continues Distributions, Generalized Half-Logistic Distribution, Linear Regression Model, Moments, Non-Bayesian Methods

1. Introduction

There has been a recent increase in the pursuit of developing new and more adaptable statistical distributions to fit the vastly expanding diversity of data collected in real life. Many researchers have shown interest in extending the producing family as a modern approach to making data analysis much better.

Balakrishnan [1] reasons half-logistic probability models gained as the models of the absolute value of the standard logistic models.

For the extended generalized half-logistic distribution, Torabi and Bagheri [2] gave the estimators of parameters for the expanded generalized half-logistic distribution based on full and censored data. The statistical research started registering values with Chandler [3]. Resnick [4] debates the asymptotic theory of records. The theory of register values and its distributional properties have been inclusively studied in good manners; for instance, Ahsanullah [5]. A GHLD is extended by the Kumaraswamy family [6].

This study focused on generating the generalized half-logistic distribution (GHLD) by replacing the shape parameter with a linear model $\underline{\lambda}\ \underline{\grave{z}}$. The linear model represents several explanatory variables with their coefficients that represent effects on some items. Briefly, our proposed distribution is denoted by LM-GHLD.

This paper consists of six sections. In the next section, we present the most basic concepts related to the GHLD and its statistical properties. Section three introduces the main structure of LM-GHLD with some important derivations and statistical properties of this distribution.

Section four is devoted to discussing the parameters model estimation of the new distribution by using a method of maximum likelihood estimate (MLE). Section five involves a simulation study to generate different sample sizes and an application using real complete data to compare the classical distributions (GHLD) and the generated LM-GHLD so that we can test the efficiency of that distribution to fit some real data set. Finally, we present several conclusions that highlight our work, which is presented in section six.

2. Preliminaries and Background

We review some essential statistical properties of the distribution that we are concerned with. We begin by presenting the probability density function (pdf), cumulative distribution function (cdf), survival function $S(x)$, and hazard function $h(x)$ of the GHLD.

$$f(x) = \frac{\alpha\gamma}{1+e^{-\alpha x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}, \quad x \geq 0, \alpha, \gamma > 0 \quad (1)$$

$$F(x) = 1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\gamma}, \; x \geq 0, \alpha, \gamma > 0 \quad (2)$$

$$S(x) = 1 - F(x) \quad (3)$$

$$h(x) = \frac{f(x)}{S(x)} \quad (4)$$

Also, we refer to the negative series expansion which is given by the following.

$$(1+s)^{-m} = \sum_{j=0}^{\infty}(-1)^j \binom{m+j-1}{j} s^j, \quad (5)$$

$$|s| < 0, m > 0$$

The formula in (5) is essential to achieve some proof of the statistical properties in the next parts.

Finally, we employ the measures of goodness forms and compare the created LM-GHLD against the GHLD. The Akanke information criterion (AIC), the consistent Akanke information criterion (CAIC), the Haman-Quinn information criterion (HQIC), and the Bayesian information criterion (BIC) are various ways to evaluate how well a model fits the data [7]. They are commonly used to determine the best-fitting distribution for a set of data, among other distributions. In addition, they are widely used in various statistical conclusions.

3. The LM-GHLD Model and its Statistical Properties

The GHLD can be modified to the LM-GHLD model by replacing the shape parameter (γ) throughout a linear model, $\underline{\lambda}\ \underline{\acute{Z}} = \sum_{k=0}^{n} \lambda_k z_{ki}$ for some integer n, where $\underline{\acute{Z}} = (1, z_{1i}, \dots, z_{ki})$ is the explanatory variables with their coefficients $\underline{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_k)$. We believe that such modification yields better results. This process enables us to write the pdf of GHLD from equation (1) in the following general form.

$$f(x;\alpha,\underline{\lambda}) = \frac{\alpha\ \sum_{k=0}^{n}\lambda_k z_{ki}}{1+e^{-\alpha x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\sum_{k=0}^{n}\lambda_k z_{ki}} \quad (6)$$

$$x \geq 0, \alpha > 0, -\infty < \underline{\lambda} < \infty$$

The modified pdf in (6) represents the LM-GHLD pdf. Similarly, we can present other functions like $F(x), S(x),$ and $h(x)$, that are:

$$F(x;\alpha,\underline{\lambda}) = 1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\sum_{k=0}^{n}\lambda_k z_{ki}}, \quad (7)$$

$$x \geq 0, \alpha > 0, -\infty < \underline{\lambda} < \infty$$

$$S(x;\alpha,\underline{\lambda}) = \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\sum_{k=0}^{n}\lambda_k z_{ki}} \quad (8)$$

$$h(x;\alpha,\underline{\lambda}) = \frac{\alpha\ \sum_{k=0}^{n}\lambda_k z_{ki}}{1+e^{-\alpha x}} \quad (9)$$

In particular, it is sufficient to study the status of LM-GHLD when $n = 1$. This yields the pdf, cdf, and other as follow

$$f(x;\alpha,\lambda_0,\lambda_1) = \frac{\alpha\ (\lambda_0+\lambda_1 z_1)}{1+e^{-\alpha x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\lambda_0+\lambda_1 z_1} \quad (10)$$

$$x \geq 0, \alpha > 0, -\infty < \lambda_0, \lambda_1 < \infty$$

$$F(x;\alpha,\lambda_0,\lambda_1) = 1 - \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\lambda_0+\lambda_1 z_1} \quad (11)$$

$$x \geq 0, \alpha > 0, -\infty < \lambda_0, \lambda_1 < \infty$$

$$S(x;\alpha,\lambda_0,\lambda_1) = \left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\lambda_0+\lambda_1 z_1} \quad (12)$$

$$h(x;\alpha,\lambda_0,\lambda_1) = \frac{\alpha\ (\lambda_0+\lambda_1 z_1)}{1+e^{-\alpha x}} \quad (13)$$

In the next part, we investigate the statistical properties of the LM-GHLD.

3.1. Moments

One of the main statistical properties that we need to discuss is 1^{st} moment, 2^{nd} moments, and variance. This can be achieved by the following theorem.

Theorem 3.1. let $\boldsymbol{X}$ be a random variable from LM-GHLD. The $\mathbf{1^{st}}$ moment, $\mathbf{2^{nd}}$ moments, and variance of $\boldsymbol{X}$, are given by

$$\mu = E(X) = \frac{1}{\alpha}(\lambda_0 + \lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}\sum_{j=0}^{\infty} h_j \ A^2 \quad (14)$$

$$E(X^2) = \frac{1}{\alpha^2}\ (\lambda_0 + \lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}\sum_{j=0}^{\infty} h_j \ \mathrm{A}^3 \quad (15)$$

$$var(X) = \mathrm{E}[(X-\mu)^2] = \frac{1}{\alpha^2}\left[(\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1+1}\sum_{j=0}^{\infty} h_j \ \mathrm{A}^3 - \left[\ (\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}\sum_{j=0}^{\infty} h_j \ \mathrm{A}^2\right]^2\right] \quad (16)$$

where $h_j = (-1)^j \ \binom{\lambda_0+\lambda_1 z_1+j}{j}$, and $A = \left(\frac{1}{\lambda_0+\lambda_1 z_1+j}\right)$

Proof: The first moment of a random variable $\boldsymbol{X}$ with respect to the pdf of LM-GHLD is defined by

$$\mu = E(X) = \int_0^{\infty} x \ \frac{\alpha\ (\lambda_0+\lambda_1 z_1)}{1+e^{-\alpha x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\lambda_0+\lambda_1 z_1} dx$$

by using the negative series expansion in equation (5), we obtain that

$$\mu = \ \alpha(\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}\sum_{j=0}^{\infty}(-1)^j \ \binom{\lambda_0+\lambda_1 z_1+j}{j}\int_0^{\infty} x \ e^{-(\alpha(\lambda_0+\lambda_1 z_1+j))x}dx$$

Let $p = (\alpha(\lambda_0+\lambda_1 z_1+j))x$

$$x = \frac{p}{(\alpha(\lambda_0+\lambda_1 z_1+j))} \text{ implies that } dx = \frac{1}{(\alpha(\lambda_0+\lambda_1 z_1+j))}dp$$

Thus

$$\mu = \ \frac{1}{\alpha}(\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}\times\sum_{j=0}^{\infty}(-1)^j \ \binom{\lambda_0+\lambda_1 z_1+j}{j} \ \left(\frac{1}{\lambda_0+\lambda_1 z_1+j}\right)^2 \times\int_0^{\infty} p \ e^{-p}\,dp$$

With the help of gamma distribution, it is well-known that the above integral is equal to one.
Therefore,

$$\mu = \ \frac{1}{\alpha}(\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}\sum_{j=0}^{\infty}(-1)^j \ \binom{\lambda_0+\lambda_1 z_1+j}{j} \ \left(\frac{1}{\lambda_0+\lambda_1 z_1+j}\right)^2$$

Similarly,

$$E(X^2) = \int_0^{\infty} x^2 \ \frac{\alpha\ (\lambda_0+\lambda_1 z_1)}{1+e^{-\alpha x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\lambda_0+\lambda_1 z_1} dx$$

By using the formula in equation (5), we obtain

$$E(X^2) = \ \alpha(\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}\sum_{j=0}^{\infty}(-1)^j \ \binom{\lambda_0+\lambda_1 z_1+j}{j}\int_0^{\infty} x^2 \ e^{-(\alpha(\lambda_0+\lambda_1 z_1+j))x}dx$$

By calculating the above integral, we yield

$$E(X^2) = \ \frac{1}{\alpha^2}(\lambda_0+\lambda_1 z_1)2^{1+\lambda_0+\lambda_1 z_1}\sum_{j=0}^{\infty}(-1)^j \ \binom{\lambda_0+\lambda_1 z_1+j}{j} \ \left(\frac{1}{\lambda_0+\lambda_1 z_1+j}\right)^3$$

We can easily obtain the variance in equation (16).

3.2. Moment Generating Function

The moment-generating function (mgf) of a random variable X from LM-GHLD can be derived by presenting its proof in the following theorem.

Theorem 3.2. If $\boldsymbol{X}$ is a random variable from the distribution LM-GHL, then the mgf is given by

$$M_X(t) = \alpha(\lambda_0+\lambda_1 z_1) \ 2^{\lambda_0+\lambda_1 z_1}\sum_{j=0}^{\infty}(-1)^j \ \binom{\lambda_0+\lambda_1 z_1+j}{j}\frac{1}{(\alpha(\lambda_0+\lambda_1 z_1+j)-t)} \quad (17)$$

$$-m < t < m \text{ where } m \ \in \ R$$

Proof: The definition of the mgf for X is given by

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \ f(x;\gamma_0,\gamma_1,\alpha)\,dx$$

Thus

$$M_x(t) = \ \alpha(\lambda_0+\lambda_1 z_1)\times\int_0^{\infty}\frac{e^{tx}}{1+e^{-\alpha x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\lambda_0+\lambda_1 z_1} dx$$

by recalling the form of the negative binomial series in equation (5), we obtain that

$$M_x(t) = \alpha(\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}\int_0^\infty \sum_{j=0}^{\infty}(-1)^j \binom{\lambda_0+\lambda_1 z_1+j}{j} e^{-(\alpha(\lambda_0+\lambda_1 z_1+j)-t)x}\, dx$$

$$M_x(t) = \frac{\alpha(\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}}{(\alpha(\lambda_0+\lambda_1 z_1+j)-t)}\sum_{j=0}^{\infty}(-1)^j \binom{\lambda_0+\lambda_1 z_1+j}{j}(\alpha(\lambda_0+\lambda_1 z_1+j)-t)\int_0^\infty e^{-(\alpha(\lambda_0+\lambda_1 z_1+j)-t)x} dx$$

$$M_x(t) = \sum_{j=0}^{\infty}(-1)^j \binom{\lambda_0+\lambda_1 z_1+j}{j} \frac{\alpha(\lambda_0+\lambda_1 z_1)2^{\lambda_0+\lambda_1 z_1}}{(\alpha(\lambda_0+\lambda_1 z_1+j)-t)}$$

3.3. Quantiles and Random Number Generator

To find an expression for the quantile function of the suggested model, we use the inverse transformation method. Quantiles are the points in a distribution that refer to the rank order of values. In the LM-GHLD, the q^{th} quantile x_q is defined as

$$q = P\ r(x \le x_q) = F(x_q)$$

Thus

$$q = 1 - \left(\frac{2e^{-\alpha x_q}}{1+e^{-\alpha x_q}}\right)^{\lambda_0+\lambda_1 z_1} \tag{18}$$

By solving equation (18) for x, we obtain that

$$x_q = \frac{-1}{\alpha}\left(\frac{1}{\lambda_0+\lambda_1 z_1}\log(1-q) - \log\left(2-(1-q)^{\frac{1}{\lambda_0+\lambda_1 z_1}}\right)\right) \tag{19}$$

Utilizing the expression in (19), we can generate a random variable X with the LM-GHLD, where $\alpha = 1$, $\lambda_0 = 1, \lambda_1 = 1$ and $q \in (0,1)$. Setting $q = 0.25, 0.50,$ and 0.75 in equation (19) yields the first quarter, median, and third quarter, respectively, as shown in Table (1).

Table 1. Quantiles with different values

q	**0.25**	**0.50**	**0.75**
x_q	0.1531522	0.3374197	0.6063474

3.4. Rényi Entropy

The Rényi entropy represents a measure of the variation of the uncertainty. Quantum information also relies heavily on the Rényi entropy. It's given by

$$h_\vartheta = \frac{1}{1-\vartheta}\log\left[\int_0^\infty f^\vartheta(x)\ dx\right] \tag{20}$$

Theorem 3.3. If X, is a random variable from the LM-GHLD, then the Rényi entropy of X is given by

$$h_\vartheta(X) = \frac{1}{1-\vartheta}\log\left[\sum_{i=0}^{\infty} y_{i,} \frac{2^{(\lambda_0+\lambda_1 z_1)\vartheta}\ [\alpha\ (\lambda_0+\lambda_1 z_1)]^\vartheta}{\alpha(\vartheta(\lambda_0+\lambda_1 z_1)+i)}\right] \tag{21}$$

where $y_i = (-1)^i\binom{\vartheta(1+\lambda_0+\lambda_1 z_1)+i-1}{i}$

Proof: By recalling the general formula in (20), we can show that the Rényi entropy is as follows.

$$h_\vartheta = \frac{1}{1-\vartheta}\log\left[\int_0^\infty \left[\frac{\alpha\ (\lambda_0+\lambda_1 z_1)}{1+e^{-\alpha x}}\left(\frac{2e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\lambda_0+\lambda_1 z_1}\right]^\vartheta dx\right]$$

Once again, and by recalling equation (5), we obtain that

$$h_\vartheta = \frac{1}{1-\vartheta}\log\left[\sum_0^\infty(-1)^j \binom{\lambda_0+\lambda_1 z_1+j}{j}2^{(\lambda_0+\lambda_1 z_1)\vartheta} \times [\alpha\ (\lambda_0 + \lambda_1 z_1)]^\vartheta\int_0^\infty e^{-\alpha(\vartheta(\lambda_0+\lambda_1 z_1)+i)x}\ dx\right]$$

$$h_\vartheta = \frac{1}{1-\vartheta}\log\left[\sum_{i=0}^{\infty} y_{i,} \frac{2^{(\lambda_0+\lambda_1 z_1)\vartheta}\ [\alpha\ (\lambda_0+\lambda_1 z_1)]^\vartheta}{\alpha(\vartheta(\lambda_0+\lambda_1 z_1)+i)}\right]$$

where $y_i = (-1)^i\binom{\vartheta(1+\lambda_0+\lambda_1 z_1)+i-1}{i}$

3.5. Order Statistic Function

The i^{th} order statistic will be denoted as $x_{i:n}$. Let $f_{i:n}(x)$ the pdf of the i^{th} order statistic for the LM-GHLD sample $X_1, X_2, \ldots, X_n$. In general, the i^{th}-The order statistics pdf is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\left(F(x_i)\right)^{i-1}\left(1-F(x_i)\right)^{n-i} f(x_i) \tag{22}$$

By substituting the $f(x)$, and $F(x)$, of the LM-GHLD in (22), we obtain the order statistic pdf of the LM-GHLD, that is

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\left[1-\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\lambda_0+\lambda_1 z_1}\right]^{i-1} \frac{\alpha\ (\lambda_0+\lambda_1 z_1)}{1+e^{-\alpha x_i}}\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{(\lambda_0+\lambda_1 z_1)(n-i+1)} \tag{23}$$

According to equation (22), the LMGHLD, smallest order statistics, is

$$f_1(x) = \frac{n\alpha\ (\lambda_0+\lambda_1 z_1)}{1+e^{-\alpha x_1}}\left(\frac{2e^{-\alpha x_1}}{1+e^{-\alpha x_1}}\right)^{(\lambda_0+\lambda_1 z_1)n}$$

While LM-GHLD greatest order statistics can be written as

$$f_n(x) = n\left[1-\left(\frac{2e^{-\alpha x_n}}{1+e^{-\alpha x_n}}\right)^{\lambda_0+\lambda_1 z_1}\right]^{n-1} \times \frac{\alpha\ (\lambda_0+\lambda_1 z_1)}{1+e^{-\alpha x_n}}\left(\frac{2e^{-\alpha x_n}}{1+e^{-\alpha x_n}}\right)^{\lambda_0+\lambda_1 z_1}$$

4. Parameters Estimation of the LM-GHLD

In this part, we go over the MLE method for estimating the LM-GHLD parameters. Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample from the LM-GHLD, and let $\Theta = (\alpha, \lambda_0, \lambda_1)^T$ represent the vector of parameters. For Θ, we can express the likelihood function as

$$L(\Theta; x_i) = \prod_{i=1}^{n} f(x)$$

$$L(\Theta; x_i) = \frac{\alpha^n(\lambda_0+\lambda_1 z_1)^n}{\prod_{i=1}^{n}(1+e^{-\alpha x_i})}\prod_{i=1}^{n}\left(\frac{2e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\lambda_0+\lambda_1 z_1}$$

Then the log-likelihood can be expressed as

$$\begin{aligned}\ell(\Theta; x) &= \log L(x_i; \alpha, \lambda_0, \lambda_1)\\ &= n\log\alpha + n(\log(\lambda_0+\lambda_1 z_1)) - \sum_{i=1}^{n} log(1+e^{-\alpha x_i}) + (\lambda_0+\lambda_1 z_1)\sum_{i=1}^{n}\log(2e^{-\alpha x_i}) - (\lambda_0\\ &+ \lambda_1 z_1)\sum_{i=1}^{n} log(1+e^{-\alpha x_i})\end{aligned}$$

By differentiating the log-likelihood function partially with respect to the unknown parameters $(\alpha, \gamma_0 \text{and } \gamma_1)$ and equate to zero, respectively, we obtain the following nonlinear equations

$$U_\alpha = \frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - (\lambda_0+\lambda_1 z_1)\sum_{i=1}^{n} x_i + (1+\lambda_0+\lambda_1 z_1)\sum_{i=1}^{n}\frac{x_i e^{-\alpha x_i}}{(1+e^{-\alpha x_i})} = 0$$

$$U_{\lambda_0} = \frac{\partial\ell}{\partial\lambda_0} = \frac{n}{\lambda_0+\lambda_1 z_1} - \sum_{i=1}^{n} log(2e^{-\alpha x_i}) - \sum_{i=1}^{n} log(1+e^{-\alpha x_i}) = 0$$

$$U_{\lambda_1} = \frac{\partial\ell}{\partial\lambda_1} = \frac{n z_1}{\lambda_0+\lambda_1 z_1} - z_1\sum_{i=1}^{n} log(2e^{-\alpha x_i}) - z_1\sum_{i=1}^{n} log(1+e^{-\alpha x_i}) = 0$$

The roots of the equations above correspond to the MLEs. However, owing to the nonlinearity of those equations, we cannot obtain explicit expressions. Thus, the Newton–Raphson method is employed to solve them, [8].

$$\begin{bmatrix} \hat{\alpha}_{i+1} \\ \hat{\lambda}_{0_{i+1}} \\ \hat{\lambda}_{1_{i+1}} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{i} \\ \hat{\lambda}_{0_{i}} \\ \hat{\lambda}_{1_{i}} \end{bmatrix} - J^{-1} \begin{bmatrix} U_{\alpha} \\ U_{\lambda_0} \\ U_{\lambda_1} \end{bmatrix} \quad (24)$$

where the Jacobean matrix is defined as

$$J = \begin{bmatrix} U_{\alpha\alpha} & U_{\alpha\lambda_0} & U_{\alpha\lambda_1} \\ U_{\lambda_0\alpha} & U_{\lambda_0\lambda_0} & U_{\lambda_0\lambda_1} \\ U_{\lambda_1\alpha} & U_{\lambda_1\lambda_0} & U_{\lambda_1\lambda_1} \end{bmatrix},$$

where

$$U_{\alpha\alpha} = \frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2} - (1 + \lambda_0 + \lambda_1 z_1) \sum_{i=1}^{n} \frac{x_i^2 e^{-\alpha x_i}}{(1 + e^{-\alpha x_i})^2}$$

$$U_{\alpha\lambda_0} = \frac{\partial^2 \ell}{\partial \alpha \lambda_0} = -\sum_{i=0}^{n} x_i + \sum_{i=1}^{n} \frac{x_i e^{-\alpha x_i}}{(1 + e^{-\alpha x_i})}$$

$$U_{\alpha\lambda_1} = \frac{\partial^2 \ell}{\partial \alpha \lambda_1} = -z_1 \sum_{i=0}^{n} x_i + z_1 \sum_{i=1}^{n} \frac{x_i e^{-\alpha x_i}}{(1 + e^{-\alpha x_i})}$$

$$U_{\lambda_0\lambda_0} = \frac{\partial^2 \ell}{\partial {\lambda_0}^2} = \frac{-n}{(\lambda_0 + \lambda_1 z_1)^2}$$

$$U_{\lambda_0\lambda_1} = \frac{\partial^2 \ell}{\partial \lambda_0 \lambda_1} = \frac{-n z_1}{(\lambda_0 + \lambda_1 z_1)^2}$$

$$U_{\lambda_1\lambda_1} = \frac{\partial^2 \ell}{\partial {\lambda_1}^2} = \frac{-n z_1^2}{(\lambda_0 + \lambda_1 z_1)^2}$$

Since the Jacobean matrix must be a non–singular symmetric matrix, then $U_{\lambda_0\alpha} = U_{\alpha\lambda_0}, U_{\lambda_1\alpha} = U_{\alpha\lambda_1}$, $U_{\lambda_0\lambda_1} = U_{\lambda_1\lambda_0}$

5. Data Analysis

This section contains two parts. First of all, we discuss the simulation part, and then we show an application of a real data set.

5.1. The Simulation Study

In this part, we present a simulation technique to generate random samples of different sizes for LM-GHLD. A simulation with an inverse approach that depends on the cdf of LM-GHLD can accomplish our goal. The algorithm of the proposed simulation can be written as follows.

Step1. Generate random numbers from a uniform distribution, denoted by U(0,1).

$$x = \frac{-1}{\alpha}\left(\frac{1}{\lambda_0+\lambda_1 z_1}\log(1-u) - \log\left(2-(1-u)^{\frac{1}{\lambda_0+\lambda_1 z_1}}\right)\right) \quad (25)$$

Step2. Substitute the generated numbers from step 1 in the inverse formula of the cdf in equation (25) to obtain the numbers that belong to LM-GHLD.

Step3. Set initial values for each parameter of LM-GHLD and employ the iteration method (Newton-Raphson) to estimate the values of the desired parameters.

Step4. Setting a stop condition $|\hat{\Theta}_{i+1} - \hat{\Theta}_i| < \epsilon$ where ϵ is the proportion of error and $i = 1, \ldots, n$, to perform the solution of Newton-Raphson method.

By considering the sample sizes n = 20, 50, 100, 150, 200, we estimate the unknown parameters of LM-GHLD; by using the MLE method to find out the best estimator value of LM-GHLD comparing with the values of parameters of GHLD. We utilize the Mean Square Error (MSE) around the desired LM-GHLD to measure the average squared difference between the estimated values and actual value [9]. The simulation procedure is based on 1000 repetitions. This estimation has been accomplished by using R programming. The following values of the estimated parameters and MSE are shown in Table 2.

Table 2. MLEs of parameters and MSEs

sample sizes	Parameter estimate$(\hat{\alpha}, \hat{\lambda}_0, \hat{\lambda}_1)$	$\hat{\ell}$	MSE
	1.22517		0.5258715
n=20	8.51364	-36.70469	0.236546
	0.86521		0.01816834
	0.96950		0.2329674
n=50	9.03524	-75.50356	0.008326198
	0.36848		0.3965397
	1.51619		1.07226
n=100	8.68731	-194.4492	0.00407044
	0.17445		0.6732202
	1.1706020		0.449707
n=150	8.6999113	-259.7808	0.09005323
	0.1155471		0.7822569
	1.094160		0.3530261
n=200	8.529716	-316.5722	0.221167
	0.085608		0.8361127

5.2. Real Data Analysis

To demonstrate the usefulness of the suggested model, we analyzed a real data set that was a report in [10]. 72 people were infected with very infectious tubercle bacilli, and the data shows how long they lived. The data are as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1.00, 1.00, 1.02, 1.05, 1.07, 0.7, 0.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.20, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.60, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.30, 2.31, 2.40, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 3. Log-likelihood, AIC, AICC, BIC, and HQIC values of models fitted

model	$\hat{\ell}$	AIC	BIC	CAIC	HQIC
$GHLD$	-104.7105	213.421	217.9743	213.5949	215.2337
$LM-GHLD$	- 87.41754	180.8351	187.6651	178.8351	183.5541

We evaluate the maximum log-likelihood functions with the desired data set so that we can compute the values of AIC, BIC, CACI, and HQIC. This provides the required comparison between the two distributions (GHLD and LM-GHLD). The results of those measures show the enhancement of data set analysis within our distribution as shown in Table 3.

Also, the results in Table 3 show that the log-likelihood of the LM-GHLD is better than that of the GHLD. As a result of these considerations, it was decided that a linear model would be a good technique to employ for the sake of improving the analysis of the data set.

6. Conclusions

In this study, we introduce a new expansion of the generalized half-logistic distribution with a linear model, and this extension is namely by LM-GHLD. Several new statistical and mathematical properties of the LM-GHLD were found with the help of the expansion of the GHD. The non-linear equations that we obtained from the MLE method were solved numerically by employing the Newton-Raphson method. The simulation part and the real data application show that the LM-GHLD is quite good at fitting a data set. This was explained in Table 3, and the results of AIC, AICC, BIC, and HQIC, as shown in that table, are better for the LM-GHLD. Also, in Table 2, the results of the MSE within different sample sizes show that the average squared difference is somehow small that we can accept the estimated values of the LM-GHLD.

REFERENCES

[1] Balakrishnan N. and R. Aggarwala, "Relationships for moments of order statistics from the right-truncated generalized half logistic distribution," Annals of the Institute of Statistical Mathematics, vol. 48, pp. 519-534, 1996.

[2] H. Torabi and F. Bagheri, "Estimation of Parameters for an Extended Generalized Half Logistic Distribution Based on Complete and Censored Data," 2010.

[3] K. Chandler, "The distribution and frequency of record values," Journal of the Royal Statistical Society: Series B (Methodological), vol. 14, no. 2, pp. 220-228, 1952.

[4] S. I. Resnick, *Heavy-tail phenomena: probabilistic and statistical modeling*: Springer Science & Business Media, 2007.

[5] M. Ahsanullah, "Record Statistics, Nova Science, publishers," ed: Inc, 1995.

[6] Wasan AL Shemmari, Ahmed AL Adilee, "Kumaraswamy Generalized Half-Logistic Distribution," Mathematics and Statistics, Vol. 11, No. 4, pp. 625-633, 2023. DOI: 10.13189/ms.2023.110402.

[7] Ahmed A.-A., "Estimation of parameters model based on a modification of Newton-Raphson method," Turkish Journal of Computer and Mathematics Education (TURCOMAT), vol. 12, no. 13, pp. 1484-1491, 2021.

[8] Kalt H. G. and I. Al Kanani, "A comparison of MLE method and OLS for the estimation of modified Weibull distribution parameters by using the simulation," ARPN Journal of Engineering and Applied Sciences, vol. 13, no. 17, 2018.

[9] Arshad R. M. I., M. Tahir, C. Chesneau, S. Khan, and F. Jamal, "The gamma power half-logistic distribution: theory and applications," São Paulo Journal of Mathematical Sciences, pp. 1-28, 2022.

[10] R. M. Usman, M. Haq, and J. Talib, "Kumaraswamy half-logistic distribution: properties and applications," J Stat Appl Probab, vol. 6, pp. 597-609, 2017.