Mathematics and Statistics 11(6): 973-987, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110613

On Nash Equilibrium Solutions for Rough Differential Games

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Received September 15, 2023; Revised October 30, 2023; Accepted November 23, 2023

Cite This Paper in the following Citation Styles

(a)*: [1] Abd El-Monem A. Megahed, Mohamed R. Zeen El Deen, Asmaa A. Ahmed, "On Nash Equilibrium Solutions for Rough Differential Games," Mathematics and Statistics, Vol.11, No.6, pp. 973-987, 2023. DOI: 10.13189/ms.2023.110613*

(b)*: Abd El-Monem A. Megahed, Mohamed R. Zeen El Deen, Asmaa A. Ahmed (2023). On Nash Equilibrium Solutions for Rough Differential Games. Mathematics and Statistics, 11(6), 973-987. DOI: 10.13189/ms.2023.110613*

Abstract The purpose of this paper is to investigate the Nash equilibrium concept for differential games when there is uncertainty in the available information for the players. Our study involves examining the problem of uncertainty in player information during the game using the "rough sets" concept, which is widely used for many such problems. Furthermore, we also explore the possible alliance between continuous differential games and the rough programming approach. Our primary aim is to ascertain the Nash equilibrium for a differential game in situations where the players have uncertain information, so they are exerting rough control, along with the trajectory of the system state being rough as well. We derive the necessary and sufficient conditions for the open-loop Nash equilibrium of the rough differential game. Additionally, we make use of the expected value operator and trust measure of rough interval to convert the rough problem into a crisp problem, allowing us to calculate the expected Nash equilibrium strategies and α-trust Nash equilibrium strategies for the game. Finally, a numerical example that outlines the steps involved in producing the rough interval of the Nash equilibrium and system state trajectory for the rough differential game is given. Moreover, this example demonstrates how to obtain each crisp problem from a rough one and then determines its Nash equilibrium and the corresponding state trajectory.

Keywords Nash Differential Games, Rough Set, Rough Intervals, Trust Measure, Expected Value

1 Introduction

The game theory's role in many areas of life can not be ignored; it is a vital tool in many fields, including finance, politics, economics, and biology. The power of game theory comes from its ability to mimic every competition or cooperation between different organizations or between individuals within the same organization as a game. Still, more intriguingly, is its ability to determine the optimal decision to be made in any circumstance [1, 2, 3, 4, 5]. In the realm of entrepreneurship and other fields where every company or person seeks to make the best possible decision, Nash equilibrium is a brilliant concept. The foundation of decision-making lies in the information accessible to the parties involved. Nonetheless, with uncertainty accompanying this information, it can pose a challenge to the decision-makers. Therefore, the Nash equilibrium in a rough space was deemed an appropriate research subject, as decision-making under uncertain information is typically encountered in this scenario. A variety of attempts have been made by researchers to address the challenge of uncertainty by adopting fuzzy sets. Megahed et al. [6] during their research on the min-max zero-sum two-player continuous differential games with fuzzy control, developed a nonlinear membership function for fuzzy control. As well as, Seikh et al. [7] introduced a mathematical model to overcome the problem of minor changes in strategy in matrix games. Such changes could cause a change in the payoff matrix, and to solve this problem, the team proposed using triangular dense fuzzy lock sets to represent the game's payoffs. Besides, Jana et al. [8] outlined the axiomatic definition of a similarity measure between dual hesitant fuzzy sets. For hesitant fuzzy sets, they had developed a new measure of similarity. Dual hesitant fuzzy set's similarity measure and distance measure connection has been clarified. Furthermore, Seikh et al. [9] introduced a new methodology to solve matrix games that include triangular hesitant fuzzy elements (THFEs) in their payoffs. The approach was based on the concepts of the weighted average operator and score function, resulting in an innovative solution for such games. Hegazy et al. [10] studied the min-max zero-sum two-person fuzzy continuous differential games, and they made use of the α -level set concept to split the original problem into two problems (the lower and upper problems). Besides, Seikh et al. [11] in order to effectively solve interval-valued matrix games, utilized an approach that incorporated intuitionistic fuzzy optimization techniques along with non-linear membership and non-membership functions.

Rough sets present an alternative method to handle uncertainty, which is useful when conventional approaches may not be adequate or appropriate. The extensive utilization of rough set theory in various fields, such as artificial intelligence, decision-making, and machine learning, demonstrates its versatile applicability and effectiveness. Pawlak [12, 13] invented the "rough set" theory in order to cope with vague information by constructing two exact sets known as the "upper approximation set" and the "lower approximation set". Rebolledo [14] introduced the rough interval concept to apply all the principles and fundamental concepts of the rough set, including the definitions of upper and lower approximations to represent continuous variables. The approaches were utilized to transform the rough problem into a crisp problem debated [16, 15]. Ruidas et al. [17] utilized rough intervals to represent the demand for the product and the defective rate, enabling them to account for their inherent imprecision due to the fact that they are not fixed in reality, which makes a RI a more suitable approach. As well as, the utilization of RIs to represent uncertain payoffs was incorporated in the study conducted by Seikh et al. [18], who also presented two approaches for resolving a new matrix game where payoffs were depicted as RIs. Ammar et al. [19] proposed the two-person zero-sum continuous differential game in which all controls and state trajectories are regarded as rough intervals. Roy et al. [20], in a challenging set setting, looked into the bi-matrix game solution process. The elements of the bi-matrix game are not exact numbers since real-world practical problems are not either. They, therefore, think elements of the bi-matrix game are rough variables to overcome this problem. Since the elements of the pay-off matrix might not be known with absolute certainty, just as they are in real-world applications, due to uncertain factors, and to provide more information to make decisions. Rough variables were used to express the pay-off matrix, as opposed to crisp variables [21]. Roy et al. [22] proposed the idea of random rough variables for a multi-objective fixed-charge transportation problem, and they transformed the multi-objective fixed-charge transportation problem (MOFCTP) with rough and random rough parameters into a deterministic MOFCTP using the "expected value operator." To propose the optimal compromise solutions, they use a variety of techniques to solve the deterministic MOFCTP, including fuzzy programming, global criterion, and ϵ-constrained methods. Ghosh et al. [23] created a multi-objective transportation problem (MOTP), at which point they took into account two opposing scenarios for low vehicle capacity, such as truck load constraints and type-II fixed-charge. They had added the product blending constraints for transporting raw materials with varying purity levels that meet the consumer with a minimum purity level in the presence of transfer station (TS). For solving their multi-objective model, they used fuzzy programming (FP), neutrosophic linear programming (NLP), and global criteria method (GCM) methods. A rough programming problem (RPP), which is a non-linear programming problem with a rough set of constraints, was developed by Youness [24]. He also defined a rough convex set. The Nash equilibrium concept has been utilized by the authors in multiple problems, such as Kassam et al. [25] in their study of how governments cooperate to fight terrorism, used the Nash-collative method. Further more, Megahed [26] investigated the problem of countering terrorism; thus, he constructed the problem and used the Nash approach to a differential game to determine the optimal strategies for combating terrorism.

In this paper, we are treating the gap which results in the differential games due to the existence of uncertainty in the players' information by applying the concept of rough set. We find the optimal control for player i in a Nash differential game with uncertainty in the player's information and the optimal state trajectory for the problem in that case.

The rest of the paper is presented in the subsequent sections: Section 2 is crucial for understanding the paper, as we provide important definitions and preliminary information. In Section 3 , we present the formulation of a continuous differential game, where the control and system state are considered as rough intervals. Additionally, we describe our approach to solving the problem. In Sections 4 and 5, we present the necessary and sufficient conditions for an open loop Nash-equilibrium, respectively. Section 6 contains numerical simulations that showcase the effectiveness of our approach. Finally, Section 7 concludes the paper and provides a summary of our research findings.

2 Preliminaries

In this section, we provide the definitions that are essential for comprehending the content of our paper.

Definition 2.1. Continuous Differential Games [1]

Differential games among n players can be formulated as:

$$J_i(\nu_1,\nu_2,\dots,\nu_i,\dots,\nu_n) \;=\; \phi_i(x(t_f)) + \int_{t_\circ}^{t_f} L_i(t,x(t),\nu_1(t),\nu_2(t),\dots,\nu_i(t),\dots,\nu_n(t))dt, \qquad i \;=\; 1,2,3,\dots,n, \tag{2.1}$$

subject to:

$$\dot{x}(t) = f(t,x(t),\nu_1(t),\nu_2(t),\dots,\nu_i(t),\dots,\nu_n(t)), \tag{2.2}$$

$$x(t_0) = x_0. \tag{2.3}$$

Where $x(t)$ is the system's state trajectory at a particular time t, $x_\circ$ is the initial state, J_i is the objective function to be maximized or minimized, $\phi_i(x(t_f))$ is the terminal payoff, and $L_i(t,x(t),\nu_1(t),\nu_2(t),\dots,\nu_i(t),\dots,\nu_n(t))$ is the running payoff.

Definition 2.2. Nash Equilibrium [2]

For the differential game, the Nash equilibrium solution is defined as an n-tuple of strategies $(\nu_1^*,\nu_2^*,...,\nu_n^*)$ such that

$$\begin{aligned} &J_1(\nu_1^*,\nu_2^*,\dots,\nu_n^*) \le J_1(\nu_1,\nu_2^*,\dots,\nu_n^*)\\ &\vdots\\ &J_n(\nu_1^*,\nu_2^*,\dots,\nu_n^*) \le J_n(\nu_1^*,\nu_2^*,\dots,\nu_n). \end{aligned}$$

The Nash equilibrium concept means that if one person attempts to unilaterally change his strategy, he will not be able to improve his optimization criterion.

Definition 2.3. Admissible Situation [5]

A situation $(\nu_1,\nu_2,...,\nu_n)$ is called admissible for player i if for any other strategy $\nu_i^{'}$ we have

$$J_i(\nu_1,\nu_2,...,\nu_i,...,\nu_n) \le J_i(\nu_1,\nu_2,...,\nu_i^{'},...,\nu_n).$$

Definition 2.4. Rough Set [12, 13]

Let U be the universal set, X a nonempty set such that X $\subseteq$ U and R be an equivalence relation on U, then X is a rough set if it can't be expressed as a union of some R-basic categories. In this case, X can be approximately defined by two R-exact sets termed as the upper approximation set $\overline{R}X$ of X which includes objects that may belong to the set X as determined by the equivalence relation R, and it is represented by

$$\overline{R}X = \{\, u \in U \mid [u]_R \cap X \neq \phi\}, \tag{2.4}$$

and the lower approximation set $\underline{R}X$ of X which includes objects that belong to the set X as determined by the equivalence relation R, and it is represented by

$$\underline{R}X = \{\, u \in U \mid [u]_R \subseteq X\}. \tag{2.5}$$

As a consequence the borderline region $BN_R(X)$ of X includes objects that can't be classified with certainty into X or -X as determined by the equivalence relation R, and it is represented by

$$BN_R(X) = \overline{R}X - \underline{R}X. \tag{2.6}$$

Definition 2.5. Rough Interval [14]

A rough interval (RI) is a specific type of rough set that satisfies all properties and core concepts of the rough set including the upper and lower approximation definitions. A Lower Approximation Interval (LAI) and an Upper Approximation Interval (UAI) form a Rough Interval (RI).
For instance: $([a_1, b_1] : [a_2, b_2])$ with $a_2 \leq a_1 < b_1 \leq b_2$.

Definition 2.6. Rough Space [15]

Let Λ be a nonempty set, A is a σ-algebra of subsets of Λ, Δ is an element in A, and π is a set function that fulfills the four axioms listed below

1. $\pi(\Lambda) < \infty$,
2. $\pi(\Delta) > 0$,
3. $\pi(B) \geqslant 0$, for any B $\in A$,
4. for every countable sequence of mutually disjoint events $\{B_i\}_{i=1}^{\infty}$ then

$$\pi\{\bigcup_{i=1}^{\infty} B_i\} = \sum_{i=1}^{\infty} \pi\{B_i\}.$$

Then (Λ, Δ, A, π) is referred to as a rough space.

Definition 2.7. Rough Variable [15]

A rough variable $\eta : (\Lambda, \Delta, A, \pi) \to \Re$ is a measurable function from the rough space (Λ, Δ ,A, π) to the set of real numbers. That is, for every Borel set B of $\Re$, we have

$$\{\kappa \in \Lambda \mid \eta(\kappa) \in B\} \in A, \tag{2.7}$$

the upper and the lower approximations of the rough variable η are defined as follows, respectively:

$$\begin{aligned} \overline{\eta} &= \{\eta(\kappa) \mid \kappa \in \Lambda\}, \\ \underline{\eta} &= \{\eta(\kappa) \mid \kappa \in \Delta\}. \end{aligned} \tag{2.8}$$

Definition 2.8. Trust Measure [15]

Let (Λ, Δ, A, π) be a rough space, then the upper trust measure of an event B is defined as

$$T\overline{r} = \frac{\pi\{B\}}{\pi\{\Lambda\}}, \tag{2.9}$$

the lower trust measure of event B is defined as

$$T\underline{r} = \frac{\pi\{B \cap \Delta\}}{\pi\{\Delta\}}, \tag{2.10}$$

as dictated by *Remark 13.3* in [16], the trust may be defined as any convex combination of the lower and upper trusts according to different management purposes. Then the trust of event B is defined as

$$Tr\{B\} = (\mu T\overline{r}\{B\} + (1-\mu)T\underline{r}\{B\}),\ \mu \in [0,1]. \tag{2.11}$$

The trust measure of rough variable η^R is defined as

$$Tr\{\eta^R \leq \beta\} = \begin{cases} 0, & if \quad \beta \leq \eta^{-(UAI)}, \\ \frac{1}{2}(\frac{\eta^{-(UAI)}-\beta}{\eta^{-(UAI)}-\eta^{+(UAI)}}), & if \quad \eta^{-(UAI)} \leq \beta \leq \eta^{-(LAI)}, \\ \frac{1}{2}(\frac{\eta^{-(UAI)}-\beta}{\eta^{-(UAI)}-\eta^{+(UAI)}} + \frac{\eta^{-(LAI)}-\beta}{\eta^{-(LAI)}-\eta^{+(LAI)}}), & if \quad \eta^{-(LAI)} \leq \beta \leq \eta^{+(LAI)}, \\ \frac{1}{2}(\frac{\eta^{-(UAI)}-\beta}{\eta^{-(UAI)}-\eta^{+(UAI)}} + 1), & if \quad \eta^{+(UAI)} \leq \beta \leq \eta^{+(UAI)}, \\ 1, & if \quad \beta \geq \eta^{+(UAI)}. \end{cases}$$

Definition 2.9. α-Pessimistic Value [16, 15]

The α-Pessimistic value of rough variable η^R is defined as

$$\eta^R_{inf} = inf\{\beta : Tr\{\eta^R \leq \beta\} \geq \alpha\}$$

$$= \begin{cases} (1-2\alpha)\eta^{-(UAI)} + 2\alpha\eta^{+(UAI)}, & if \quad \alpha \leq \frac{\eta^{-(LAI)} - \eta^{-(UAI)}}{2(\eta^{+(UAI)} - \eta^{-(UAI)})}, \\ 2(1-\alpha)\eta^{-(UAI)} + (2\alpha-1)\eta^{+(UAI)}, & if \quad \alpha \geq \frac{\eta^{+(LAI)} + \eta^{+(UAI)} - 2\eta^{-(UAI)}}{2(\eta^{+(UAI)} - \eta^{-(UAI)})}, \\ \frac{\eta^{-(UAI)}(\eta^{+(LAI)} - \eta^{-(LAI)})}{(\eta^{+(LAI)} - \eta^{-(LAI)}) + (\eta^{+(UAI)} - \eta^{-(UAI)})} \\ + \frac{\eta^{-(LAI)}(\eta^{+(UAI)} - \eta^{-(UAI)})}{(\eta^{+(LAI)} - \eta^{-(LAI)}) + (\eta^{+(UAI)} - \eta^{-(UAI)})} & otherwise. \\ + \frac{2\alpha(\eta^{+(LAI)} - \eta^{-(LAI)})(\eta^{+(UAI)} - \eta^{-(UAI)})}{(\eta^{+(LAI)} - \eta^{-(LAI)}) + (\eta^{+(UAI)} - \eta^{-(UAI)})}, \end{cases}$$

Definition 2.10. Expected Value [16, 15]

Let η be a rough variable, the expected value of η is defined as

$$E(\eta) = \int_0^{+\infty} Tr\{\eta \geqslant r\}dr - \int_{-\infty}^{0} Tr\{\eta \leqslant r\}dr, \tag{2.12}$$

following *Theorem 1.5* in [27]: the expected value of a rough variable can be represented as

$$E(\eta) = \frac{1}{2}(\mu(\eta^{-(LAI)} + \eta^{+(LAI)}) + (1-\mu)(\eta^{-(UAI)} + \eta^{+(UAI)}), \ \mu \in [0,1]. \tag{2.13}$$

Definition 2.11. Convex Rough Set [24]

Let R be an equivalence relation on a non-empty finite set U, U $\in R^n$. If the upper approximation set $\overline{R}X$ of a rough set $X \subseteq U$ with regard to equivalence relation R is convex, the rough set is said to be convex.

3 Problem Formulation

In this section, we present a formulation of a rough differential game of n players, taking into account the control exerted by the players and the system state trajectory as rough intervals. The problem is outlined, and it is shown that to solve this problem, it must be divided into four subproblems. These subproblems are commonly referred to the lower lower problem (LLP), upper lower problem (ULP), lower upper problem (LUP), and the upper upper problem (UUP) in order to identify the Nash equilibrium strategies and the corresponding state trajectory.

Let us have n-players with rough control $\nu_i^R(t)$, $i = 1, 2, 3, \ldots, n$ and rough state trajectory $x^R(t)$, where the objective function for the player $i,\ i=1,2,3,\dots,n$ is given by

$$\underset{\nu_i^R}{min}\, J_i^R(\nu_1^R(t),\nu_2^R(t),\dots,\nu_i^R(t),\dots,\nu_n^R(t))=\phi_i^R(x^R(t_f))$$
$$+\int_{t_\circ}^{t_f} L_i^R(t,x^R(t),\nu_1^R(t),\nu_2^R(t),\dots,\nu_i^R(t),\dots,\nu_n^R(t))dt, \tag{3.14}$$

with the state trajectory:

$$\dot{x}^R(t)=f^R(t,x^R(t),\nu_1^R(t),\nu_2^R(t),\dots,\nu_i^R(t),\dots,\nu_n^R(t)), \tag{3.15}$$

and the initial condition

$$x^R(t_0)=x_0^R. \tag{3.16}$$

We can now present the formulations for the four subproblems. Hint: $\nu(t)$ will be abbreviated to ν, $x(t)$ will be abbreviated to x and $\chi(t)$ will be abbreviated to χ.

3.1 The Lower Lower Problem (LLP).

The formulation for the LLP takes the following structure:

$$\underset{\nu_i^{-(LAI)}}{min}\, J_i^{-(LAI)}(\nu_1^{-(LAI)},\nu_2^{-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{-(LAI)})=\phi_i^{-(LAI)}(x^{-(LAI)}(t_f))+$$
$$\int_{t_\circ}^{t_f} L_i^{-(LAI)}(t,x^{-(LAI)},\nu_1^{-(LAI)},\nu_2^{-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{-(LAI)})dt,\ i=1,2,3,\dots,n, \tag{3.17}$$

subject to:

$$\dot{x}^{-(LAI)}=f^{-(LAI)}(t,x^{-(LAI)},\nu_1^{-(LAI)},\nu_2^{-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{-(LAI)}), \tag{3.18}$$
$$x^{-(LAI)}(t_0)=x_0^{-(LAI)}. \tag{3.19}$$

3.2 The Upper Lower Problem (ULP).

The formulation for the ULP takes the following structure:

$$\underset{\nu_i^{+(LAI)}}{min}\, J_i^{+(LAI)}(\nu_1^{+(LAI)},\nu_2^{+(LAI)},\dots,\nu_i^{+(LAI)},\dots,\nu_n^{+(LAI)})=\phi_i^{+(LAI)}(x^{+(LAI)}(t_f))+$$
$$\int_{t_\circ}^{t_f} L_i^{+(LAI)}(t,x^{+(LAI)},\nu_1^{+(LAI)},\nu_2^{+(LAI)},\dots,\nu_i^{+(LAI)},\dots,\nu_n^{+(LAI)})dt,\ i=1,2,3,\dots,n, \tag{3.20}$$

subject to:

$$\dot{x}^{+(LAI)}=f^{+(LAI)}(t,x^{+(LAI)},\nu_1^{+(LAI)},\nu_2^{+(LAI)},\dots,\nu_i^{+(LAI)},\dots,\nu_n^{+(LAI)}), \tag{3.21}$$
$$x^{+(LAI)}(t_0)=x_0^{+(LAI)}. \tag{3.22}$$

3.3 The Lower Upper Problem (LUP).

The formulation for the LUP takes the following structure:

$$\underset{\nu_i^{-(UAI)}}{min}\, J_i^{-(UAI)}(\nu_1^{-(UAI)},\nu_2^{-(UAI)},\dots,\nu_i^{-(UAI)},\dots,\nu_n^{-(UAI)})=\phi_i^{-(UAI)}(x^{-(UAI)}(t_f))+$$
$$\int_{t_\circ}^{t_f} L_i^{-(UAI)}(t,x^{-(UAI)},\nu_1^{-(UAI)},\nu_2^{-(UAI)},\dots,\nu_i^{-(UAI)},\dots,\nu_n^{-(UAI)})dt,\ i=1,2,3,\dots,n, \tag{3.23}$$

subject to:

$$\dot{x}^{-(UAI)}=f^{-(UAI)}(t,x^{-(UAI)},\nu_1^{-(UAI)},\nu_2^{-(UAI)},\dots,\nu_i^{-(UAI)},\dots,\nu_n^{-(UAI)}), \tag{3.24}$$
$$x^{-(UAI)}(t_0)=x_0^{-(UAI)}. \tag{3.25}$$

3.4 The Upper Upper Problem (UUP).

The formulation for the UUP takes the following structure:

$$\min_{\nu_i^{+(UAI)}} J_i^{+(UAI)}(\nu_1^{+(UAI)}, \nu_2^{+(UAI)}, \ldots, \nu_i^{+(UAI)}, \ldots, \nu_n^{+(UAI)}) = \phi_i^{+(UAI)}(x^{+(UAI)}(t_f)) + \int_{t_0}^{t_f} L_i^{+(UAI)}(t, x^{+(UAI)}, \nu_1^{+(UAI)}, \nu_2^{+(UAI)}, \ldots, \nu_i^{+(UAI)}, \ldots, \nu_n^{+(UAI)})dt,\ i = 1, 2, 3, \ldots, n, \quad (3.26)$$

subject to:

$$\dot{x}^{+(UAI)} = f^{+(UAI)}(t, x^{+(UAI)}, \nu_1^{+(UAI)}, \nu_2^{+(UAI)}, \ldots, \nu_i^{+(UAI)}, \ldots, \nu_n^{+(UAI)}), \quad (3.27)$$

$$x^{+(UAI)}(t_0) = x_0^{+(UAI)}. \quad (3.28)$$

4 Necessary Conditions

In this section, we will work towards achieving the necessary conditions that are satisfied by an open-loop Nash equilibrium solution for a rough interval differential game. We will establish the necessary conditions for LLP (3.17)-3.19) and similar to how we can demonstrate the necessary conditions for the ULP (3.20-3.22), LUP (3.23-3.25), and UUP (3.26-3.28).

Theorem 4.12. *Let* $L_i^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)})$ *and* $f^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)})$ *be continuously differentiable functions and U is a convex rough set, if* $(\nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{*-(LAI)}, \ldots, \nu_n^{*-(LAI)})$ *is an open loop Nash equilibrium with state trajectory* $x^{*-(LAI)}$ *,* $t\epsilon[t_0, t_f]$*, for the LLP (3.17-3.19), then there exist* n*-costate vectors* $\chi_i^{-(LAI)}(t), \chi_i^{-(LAI)}(t) : [t_0, t_f] \to R^n$ *and* n*-Hamiltonian functions* $H_i^{-(LAI)}$ *,* $i = 1, 2, \ldots, n$ *defined as*

$$H_i^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)}, \chi_i^{-(LAI)}) = L^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)}) + [\chi_i^{-(LAI)}]^T f^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)}), \quad (4.29)$$

such that:

$$\dot{x}^{*-(LAI)} = f^{-(LAI)}(t, x^{*-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{*-(LAI)}, \ldots, \nu_n^{*-(LAI)}),\ x^{*-(LAI)}(t_0) = x_0, \quad (4.30)$$

$$\dot{\chi}_i^{-(LAI)} = -\frac{\partial H_i^{-(LAI)}(t, x^{*-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{*-(LAI)}, \ldots, \nu_n^{*-(LAI)}, \chi_i^{-(LAI)})}{\partial x^{-(LAI)}}, \quad (4.31)$$

$$\chi_i^{-(LAI)}(t_f) = \frac{\partial \phi_i^{-(LAI)}(x^{-(LAI)}(t_f))}{\partial x^{-(LAI)}(t_f)}, \quad (4.32)$$

$$\frac{\partial H_i^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)}, \chi_i^{-(LAI)})}{\partial \nu_i^{-(LAI)}} = 0, \quad (4.33)$$

$$\frac{\partial^2 H_i^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)}, \chi_i^{-(LAI)})}{\partial^2 \nu_i^{-(LAI)}} \geq 0. \quad (4.34)$$

Proof:
Since $L_i^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)})$ and $f^{-(LAI)}(t, x^{-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{-(LAI)}, \ldots, \nu_n^{*-(LAI)})$ are continuous and differentiable functions, then $\chi_i^{-(LAI)}$ is a solution of the differential equation below

$$\dot{\chi}_i^{-(LAI)} = -\chi_i^{-(LAI)} \frac{\partial f^{-(LAI)}(t, x^{*-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{*-(LAI)}, \ldots, \nu_n^{*-(LAI)})}{\partial x^{-(LAI)}} - \frac{\partial L_i^{-(LAI)}(t, x^{*-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \ldots, \nu_i^{*-(LAI)}, \ldots, \nu_n^{*-(LAI)})}{\partial x^{-(LAI)}}, \quad (4.35)$$

with

$$\chi_i^{-(LAI)}(t_f) = \frac{\partial \phi_i^{-(LAI)}(x^{-(LAI)}(t_f))}{\partial x^{-(LAI)}(t_f)}. \quad (4.36)$$

The adjoint equation of (4.35) is

$$\begin{aligned}\chi_i^{-(LAI)}\delta\dot{x}^{-(LAI)} &= [\chi_i^{-(LAI)}\frac{\partial f^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)})}{\partial x^{-(LAI)}}\\ &+\frac{\partial L_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)})}{\partial x^{-(LAI)}}]\delta x^{-(LAI)}\\ &+H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})\\ &-H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)}).\end{aligned} \tag{4.37}$$

The solution of (4.37) is $\delta x^{-(LAI)}(t)$ with the initial condition $\delta x^{-(LAI)}(t_0)=0$.
From Theorem 10.1 in [28] we get

$$\begin{aligned}\frac{d[\chi_i^{-(LAI)}\delta\dot{x}^{-(LAI)}]}{dt} &= H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})\\ &-H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)}).\end{aligned} \tag{4.38}$$

By integrating from t_0 to t_f

$$\begin{aligned}\chi_i^{-(LAI)}(t_f)\delta x^{-(LAI)}(t_f) &= \int_{t_0}^{t_f} H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})\\ &- H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})dt.\end{aligned} \tag{4.39}$$

From Theorem 11.1 in [28]

$$\begin{aligned}\delta J_i(u^*,\nu_i) &= \int_{t_0}^{t_f} H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})\\ &-H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})dt.\end{aligned} \tag{4.40}$$

Where u^* denotes the composite optimal control for the remaining players.
From (4.40)

$$\begin{aligned}\int_{t_0}^{t_f}\chi(t)f_u(t,x,u)v(t)dt &= \int_{t_0}^{t_f} H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})\\ &-H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})dt.\end{aligned} \tag{4.41}$$

From Theorem 11.2 in [28], we have

$$\chi(t)f_u(t,x,u)v(t)\le 0. \tag{4.42}$$

From (4.41) and (4.42)

$$\begin{aligned}&H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})\\ &-H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})\le 0,\end{aligned} \tag{4.43}$$

and

$$\begin{aligned}&H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})\le\\ &H_i^{-(LAI)}(t,x^{*-(LAI)},\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)}).\end{aligned} \tag{4.44}$$

□

5 Sufficient Conditions

In this section, we will drive the sufficient conditions that must be fulfilled by an open-loop Nash equilibrium solution for a rough interval differential game. We will establish the sufficient conditions for LLP (3.17-3.19) and similar to how we can demonstrate the sufficient conditions for the ULP (3.20-3.22), LUP (3.23-3.25), and UUP (3.26-3.28).

Theorem 5.13. *For the LLP (3.17-3.19), let* $H_i^{-(LAI)}(t,x^{-(LAI)},\nu_1^{*-(LAI)},\dots,\nu_i^{-(LAI)},\dots,\nu_n^{*-(LAI)},\chi_i^{-(LAI)})$ *be twice differentiable on* R^n*, then the* n*-dimensional vector* $(\nu_1^{*-(LAI)},\nu_2^{*-(LAI)},\dots,\nu_i^{*-(LAI)},\dots,\nu_n^{*-(LAI)})$ *is an open-loop Nash equilibrium if an* n*-dimensional co-state vector* χ_i *exists and satisfies the conditions (4.30-4.34)*

Proof:
Since

$$H_i^{-(LAI)}(t, x^{*-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \dots, \nu_i^{-(LAI)}, \dots, \nu_n^{*-(LAI)}, \chi_i^{-(LAI)}) = \\ H_i^{-(LAI)}(t, x^{*-(LAI)}(t), \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \dots, \nu_i^{*-(LAI)}, \dots, \nu_n^{*-(LAI)}, \chi_i^{-(LAI)}) + \frac{\partial H_i^{-(LAI)}}{\partial \nu_i^{-(LAI)}} \delta \nu_i^{*-(LAI)} \\ + \frac{1}{2!} \frac{\partial^2 H_i^{-(LAI)}}{\partial^2 \nu_i^{-(LAI)}} \delta^2 ||\nu_i^{*-(LAI)}(t)||^2. \tag{5.45}$$

From (4.33) and (4.34), we obtain

$$H_i^{-(LAI)}(t, x^{*-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \dots, \nu_i^{-(LAI)}, \dots, \nu_n^{*-(LAI)}, \chi_i^{-(LAI)}) \geq \\ H_i^{-(LAI)}(t, x^{*-(LAI)}, \nu_1^{*-(LAI)}, \nu_2^{*-(LAI)}, \dots, \nu_i^{*-(LAI)}, \dots, \nu_n^{*-(LAI)}, \chi_i^{-(LAI)}). \tag{5.46}$$

□

6 Example

In this section, we present numerical examples to obtain the rough interval for the Nash equilibrium of the rough differential games utilizing the provided methodology, we assume that all controls and state trajectories are rough intervals.
Assuming that only two players are involved in the game, (player 1 and player 2), where the payoff functions of player 1 and player 2 are given by

$$\min_{\nu_1^R} J_1^R(\nu_1^R(t), \nu_2^R(t)) = a^R x_1^R(t_f) + \int_{t_0}^{t_f} (a^R \nu_1^R(t) - a^R)^2 dt, \\ \min_{\nu_2^R} J_2^R(\nu_1^R(t), \nu_2^R(t)) = b^R x_2^R(t_f) + \int_{t_0}^{t_f} (b^R \nu_2^R(t) - b^R)^2 dt, \tag{6.47}$$

with the state trajectories:

$$\dot{x}_1^R(t) = a^R \nu_1^R(t) + a^R \nu_2^R(t), \\ \dot{x}_2^R(t) = -a^R \nu_1^R(t) - a^R \nu_2^R(t). \tag{6.48}$$

Where $a^R = [(2.5, 3) : (2, 3.5)]$ and $b^R = [(1.5, 2) : (1, 3)]$ are the rough intervals for the two players.
The approach to tackling this problem involves dividing it into LLP, ULP, LUP, and UUP problems described previously.

6.1 The Solution of the LLP for Players 1 and 2

The LLP for the two players (Players 1 and 2) can be structured in the following way

$$\min_{\nu_1^{-(LAI)}} J_1^{-(LAI)} = 2.5 x_1^{-(lAI)}(t_f) + \int_{t_0}^{t_f} (3\nu_1^{-(LAI)} - 2.5)^2 dt, \\ \min_{\nu_2^{-(LAI)}} J_2^{-(LAI)} = 1.5 x_2^{-(LAI)}(t_f) + \int_{t_0}^{t_f} (2\nu_2^{-(LAI)} - 1.5)^2 dt, \tag{6.49}$$

subject to:

$$\dot{x}_1^{-(LAI)} = 2.5\nu_1^{-(LAI)} + 2.5\nu_2^{-(LAI)}, \\ \dot{x}_2^{-(LAI)} = -3\nu_1^{-(LAI)}(t) - 2.5\nu_2^{-(LAI)}. \tag{6.50}$$

The Hamiltonian functions are

$$H_1^{-(LAI)}(t, x_1^{-(LAI)}, x_2^{-(LAI)}, \nu_1^{-(LAI)}, \nu_2^{-(LAI)}, \chi_1^{-(LAI)}, \chi_2^{-(LAI)}) = (3\nu_1^{-(LAI)} - 2.5)^2 \\ + \chi_1^{-(LAI)}(2.5\nu_1^{-(LAI)} + 2.5\nu_2^{-(LAI)}) + \chi_2^{-(LAI)}(-3\nu_1^{-(LAI)} - 2.5\nu_2^{-(LAI)}), \tag{6.51}$$

$$H_2^{-(LAI)}(t, x_1^{-(LAI)}, x_2^{-(LAI)}, \nu_1^{-(LAI)}, \nu_2^{-(LAI)}, \chi_1^{-(LAI)}, \chi_2^{-(LAI)}) = (2\nu_2^{-(LAI)} - 1.5)^2 \\ + \chi_1^{-(LAI)}(2.5\nu_1^{-(LAI)} + 2.5\nu_2^{-(LAI)}) + \chi_2^{-(LAI)}(-3\nu_1^{-(LAI)} - 2.5\nu_2^{-(LAI)}). \tag{6.52}$$

From (4.31) and (4.32), we obtain $\chi_1^{-(LAI)}(t) = c_1$ and $\chi_1^{-(LAI)}(t_f) = 2.5$,
similarly, we get $\chi_2^{-(LAI)}(t) = c_2$ and $\chi_2^{-(LAI)}(t_f) = 1.5$.
According to (4.33) and from the previous knowledge of $\chi_1^{-(LAI)}$ and $\chi_2^{-(LAI)}$, we have
$\nu_1^{-(LAI)} = 0.7361$, $\nu_2^{-(LAI)} = 0.4375$, $x_1^{-(LAI)} = 2.934(t_f - t_0)$, and $x_2^{-(LAI)} = -3.3021(t_f - t_0)$.

6.2 The Solution of the ULP for Players 1 and 2

The ULP for the two players (Players 1 and 2) can be structured in the following way

$$\begin{aligned} \min_{\nu_1^{+(LAI)}} J_1^{+(LAI)} &= 3x_1^{+(lAI)}(t_f) + \int_{t_0}^{t_f} (2.5\nu_1^{+(LAI)} - 3)^2 dt, \\ \min_{\nu_2^{+(LAI)}} J_2^{+(LAI)} &= 2x_2^{+(LAI)}(t_f) + \int_{t_0}^{t_f} (1.5\nu_2^{+(LAI)} - 2)^2 dt, \end{aligned} \tag{6.53}$$

subject to:

$$\begin{aligned} \dot{x}_1^{+(LAI)} &= 3\nu_1^{+(LAI)} + 3\nu_2^{+(LAI)}, \\ \dot{x}_2^{+(LAI)} &= -2.5\nu_1^{+(LAI)} - 3\nu_2^{+(LAI)}. \end{aligned} \tag{6.54}$$

The Hamiltonian functions are

$$\begin{aligned} H_1^{+(LAI)}(t, x_1^{+(LAI)}, x_2^{+(LAI)}, \nu_1^{+(LAI)}, \nu_2^{+(LAI)}, \chi_1^{+(LAI)}, \chi_2^{+(LAI)}) &= (2.5\nu_1^{+(LAI)} - 3)^2 \\ &+ \chi_1^{+(LAI)}(3\nu_1^{+(LAI)} + 3\nu_2^{+(LAI)}) + \chi_2^{+(LAI)}(-2.5\nu_1^{+(LAI)} - 3\nu_2^{+(LAI)}), \end{aligned} \tag{6.55}$$

$$\begin{aligned} H_2^{+(LAI)}(t, x_1^{+(LAI)}, x_2^{+(LAI)}, \nu_1^{+(LAI)}, \nu_2^{+(LAI)}, \chi_1^{+(LAI)}, \chi_2^{+(LAI)}) &= (1.5\nu_2^{+(LAI)} - 2)^2 \\ &+ \chi_1^{+(LAI)}(3\nu_1^{+(LAI)} + 3\nu_2^{+(LAI)}) + \chi_2^{+(LAI)}(-2.5\nu_1^{+(LAI)} - 3\nu_2^{+(LAI)}). \end{aligned} \tag{6.56}$$

From (4.31) and (4.32), we have $\chi_1^{+(LAI)}(t) = c_1$ and $\chi_1^{+(LAI)}(t_f) = 3$,
similarly, we get $\chi_2^{+(LAI)}(t) = c_2$ and $\chi_2^{+(LAI)}(t_f) = 2$.
According to (4.33) and from the previous knowledge of $\chi_1^{+(LAI)}(t)$ and $\chi_2^{+(LAI)}(t)$, we have

$\nu_1^{+(LAI)} = 0.88$, $\nu_2^{+(LAI)} = 0.6667$, $x_1^{+(LAI)} = 4.6401(t_f - t_\circ)$ and $x_2^{+(LAI)} = -4.2001(t_f - t_\circ)$.

6.3 The Solution of the LUP For Players 1 and 2

The LUP for the two players (Players 1 and 2) can be structured in the following way

$$\begin{aligned} \min_{\nu_1^{-(UAI)}} J_1^{-(UAI)} &= 2x_1^{-(UAI)}(t_f) + \int_{t_0}^{t_f} (3.5\nu_1^{-(UAI)} - 2)^2 dt, \\ \min_{\nu_2^{-(UAI)}} J_2^{-(UAI)} &= x_2^{-(UAI)}(t_f) + \int_{t_0}^{t_f} (3\nu_2^{-(UAI)} - 1)^2 dt, \end{aligned} \tag{6.57}$$

subject to:

$$\begin{aligned} \dot{x}_1^{-(UAI)} &= 2\nu_1^{-(UAI)} + 2\nu_2^{-(UAI)}, \\ \dot{x}_2^{-(UAI)} &= -3.5\nu_1^{-(UAI)} - 2\nu_2^{-(UAI)}. \end{aligned} \tag{6.58}$$

The Hamiltonian functions are

$$\begin{aligned} H_1^{-(UAI)}(t, x_1^{-(UAI)}, x_2^{-(UAI)}, \nu_1^{-(UAI)}, \nu_2^{-(UAI)}, \chi_1^{-(UAI)}, \chi_2^{-(UAI)}) &= (3.5\nu_1^{-(UAI)} - 2)^2 \\ &+ \chi_1^{-(UAI)}(2\nu_1^{-(UAI)} + 2\nu_2^{-(UAI)}) + \chi_2^{-(UAI)}(-3.5\nu_1^{-(UAI)} - 2\nu_2^{-(UAI)}), \end{aligned} \tag{6.59}$$

$$\begin{aligned} H_2^{-(UAI)}(t, x_1^{-(UAI)}, x_2^{-(UAI)}, \nu_1^{-(UAI)}, \nu_2^{-(UAI)}, \chi_1^{-(UAI)}, \chi_2^{-(UAI)}) &= (3\nu_2^{-(UAI)} - 1)^2 \\ &+ \chi_1^{-(UAI)}(2\nu_1^{-(UAI)} + 2\nu_2^{-(UAI)}) + \chi_2^{-(UAI)}(-3.5\nu_1^{-(UAI)} - 2\nu_2^{-(UAI)}). \end{aligned} \tag{6.60}$$

From (4.31) and (4.32) $\chi_1^{-(UAI)}(t) = c_1$ and $\chi_1^{-(UAI)}(t_f) = 2$,
similarly, we get $\chi_2^{-(UAI)}(t) = c_2$ and $\chi_2^{-(UAI)}(t_f) = 1$.
According to (4.33) and from the previous knowledge of $\chi_1^{-(UAI)}$ and $\chi_2^{-(UAI)}$, we have
$\nu_1^{-(UAI)} = 0.5510$, $\nu_2^{-(UAI)} = 0.2222$, $x_1^{-(UAI)} = 1.5464(t_f - t_0)$ and $x_2^{-(UAI)} = -2.3729(t_f - t_0)$.

6.4 The Solution of the UUP for Players 1 and 2

The UUP for the two players (Players 1 and 2) can be structured in the following way

$$\begin{aligned} &\min_{\nu_1^{+(UAI)}} J_1^{+(UAI)} = 3.5x_1^{+(UAI)}(t_f) + \int_{t_0}^{t_f} (2\nu_1^{+(UAI)} - 3.5)^2, dt \\ &\min_{\nu_2^{+(UAI)}} J_2^{+(UAI)} = 3x_2^{+(UAI)}(t_f) + \int_{t_0}^{t_f} (\nu_2^{+(UAI)} - 3)^2 dt, \end{aligned} \tag{6.61}$$

subject to:

$$\begin{aligned} \dot{x}_1^{+(UAI)} &= 3.5\nu_1^{+(UAI)} + 3.5\nu_2^{+(UAI)}, \\ \dot{x}_2^{+(UAI)} &= -2\nu_1^{+(UAI)} - 3.5\nu_2^{+(UAI)}. \end{aligned} \tag{6.62}$$

The Hamiltonian functions are

$$\begin{aligned} H_1^{+(UAI)}(t, x_1^{+(UAI)}, x_2^{+(UAI)}, \nu_1^{+(UAI)}, \nu_2^{+(UAI)}, \chi_1^{+(UAI)}, \chi_2^{+(UAI)}) &= (2\nu_1^{+(UAI)} - 3.5)^2 \\ + \chi_1^{+(UAI)}(3.5\nu_1^{+(UAI)} + 3.5\nu_2^{+(UAI)}) &+ \chi_2^{+(UAI)}(-2\nu_1^{+(UAI)} - 3.5\nu_2^{+(UAI)}), \end{aligned} \tag{6.63}$$

$$\begin{aligned} H_2^{+(UAI)}(t, x_1^{+(UAI)}, x_2^{+(UAI)}, \nu_1^{+(UAI)}, \nu_2^{+(UAI)}, \chi_1^{+(UAI)}, \chi_2^{+(UAI)}) &= (\nu_2^{+(UAI)} - 3)^2 \\ + \chi_1^{+(UAI)}(3.5\nu_1^{+(UAI)} + 3.5\nu_2^{+(UAI)}) &+ \chi_2^{+(UAI)}(-2\nu_1^{+(UAI)} - 3.5\nu_2^{+(UAI)}). \end{aligned} \tag{6.64}$$

From (4.31) and(4.32), we obtain $\chi_1^{+(UAI)}(t) = c_1$ and $\chi_1^{+(UAI)}(t_f) = 3.5$,
similarly, we get $\chi_2^{+(UAI)}(t) = c_2$ and $\chi_2^{+(UAI)}(t_f) = 3$.
According to (4.33)and from the previous knowledge of $\chi_1^{+(UAI)}$ and $\chi_2^{+(UAI)}$, we have
$\nu_1^{+(UAI)} = 0.9688$, $\nu_2^{+(UAI)} = 2.125$, $x_1^{+(UAI)} = 10.8283(t_f - t_0)$, and $x_2^{+(UAI)} = -9.3751(t_f - t_0)$.

Through the process of solving the four subproblems, one can obtain the following rough intervals of the Nash equilibrium and the corresponding state trajectories for players 1 and 2 :

$\nu_1^R = [(0.7361, 0.88) : (0.5510, 0.9688)]$
$\nu_2^R = [(0.4375, 0.6667) : (0.2222, 2.125)]$

$x_1^R = [(2.934, 4.6401) : (1.5464, 10.8283)](t_f - t_0)$
$x_2^R = [(-4.2001, -3.3021) : (-9.3751, -2.3729)](t_f - t_0)$

6.5 Expected Value

By applying the expected value operator, the rough problem can be converted into crisp one. Subsequently, the Nash equilibrium and its corresponding state trajectories for players 1 and 2 can be determined.

When $\mu = 0.1$, the problem is.

$$\begin{aligned} &\min_{\nu_1} J_1 = 2.75x_1(t_f) + \int_{t_0}^{t_f} (2.75\nu_1 - 2.75)^2 dt, \\ &\min_{\nu_2} J_2 = 1.975x_2(t_f) + \int_{t_0}^{t_f} (1.975\nu_2 - 1.975)^2 dt, \end{aligned} \tag{6.65}$$

subject to:

$$\begin{aligned} \dot{x}_1 &= 2.75\nu_1 + 2.75\nu_2, \\ \dot{x}_2 &= -2.75\nu_1 - 2.75\nu_2. \end{aligned} \tag{6.66}$$

The Hamiltonian functions are

$$H_1(t, x_1, x_2, \nu_1, \nu_2, \chi_1, \chi_2) = (2.75\nu_1 - 2.75)^2 + \chi_1(2.75\nu_1 + 2.75\nu_2) + \chi_2(-2.75\nu_1 - 2.75\nu_2), \tag{6.67}$$

$$H_2(t, x_1, x_2, \nu_1, \nu_2, \chi_1, \chi_2) = (1.975\nu_2 - 1.975)^2 + \chi_1(2.75\nu_1 + 2.75\nu_2) + \chi_2(-2.75\nu_1 - 2.75\nu_2). \tag{6.68}$$

Based on (4.31) and (4.32), we obtain $\chi_1(t) = c_1$ and $\chi_1(t_f) = 2.75$,
consistently, the following result can be easily derived for the co-state vector of the second player:
$\chi_2(t) = c_2$ and $\chi_2(t_f) = 1.975$.
In response to (4.33) and from previous knowledge of $\chi_1(t)$ and $\chi_2(t)$, we get
$\nu_1 = 0.8591$, $\nu_2 = 0.7268$ $x_1 = 4.3612(t_f - t_0)$, and $x_2 = -4.3612(t_f - t_0)$.

When $\mu = 0.2$ the problem takes the following form

$$\begin{aligned} \min_{\nu_1} J_1 &= 2.75x_1(t_f) + \int_{t_0}^{t_f} (2.75\nu_1 - 2.75)^2 dt, \\ \min_{\nu_2} J_2 &= 1.95x_2(t_f) + \int_{t_0}^{t_f} (1.95\nu_2 - 1.95)^2 dt, \end{aligned} \tag{6.69}$$

subject to:

$$\begin{aligned} \dot{x}_1 &= 2.75\nu_1 + 2.75\nu_2, \\ \dot{x}_2 &= -2.75\nu_1 - 2.75\nu_2. \end{aligned} \tag{6.70}$$

The Hamiltonian functions are

$$H_1(t, x_1, x_2, \nu_1, \nu_2, \chi_1, \chi_2) = (2.75\nu_1 - 2.75)^2 + \chi_1(2.75\nu_1 + 2.75\nu_2) + \chi_2(-2.75\nu_1 - 2.75u_2), \tag{6.71}$$

$$H_2(t, x_1, x_2, \nu_1, \nu_2, \chi_1, \chi_2) = (1.95\nu_2 - 1.95)^2 + \chi_1(2.75\nu_1 + 2.75\nu_2) + \chi_2(-2.75\nu_1 - 2.75\nu_2). \tag{6.72}$$

Based on (4.31) and (4.32), we obtain $\chi_1(t) = c_1$ and $\chi_1(t_f) = 2.75$,
Consistently, the following result can be easily derived for the co-state vector of the second player:
$\chi_2(t) = c_2$ and $\chi_2(t_f) = 1.95$.
In response to (4.33) and from previous knowledge of $\chi_1(t)$ and $\chi_2(t)$, we get
$\nu_1 = 0.8545$, $\nu_2 = 0.7107$, $x_1 = 4.3043(t_f - t_0)$, and $x_2 = -4.3043(t_f - t_0)$.

6.6 Trust Measure Method

Using the trust measure approach, the rough problem can be transformed into a crisp ones, as demonstrated below.

When $\alpha = 0.1$, the problem turns into the following crisp problem:

$$\begin{aligned} \min_{\nu_1} J_1 &= 2.3x_1(t_f) + \int_{t_0}^{t_f} (2.3\nu_1 - 2.3)^2 dt, \\ \min_{\nu_2} J_2 &= 1.4x_2(t_f) + \int_{t_0}^{t_f} (1.4\nu_2 - 1.4)^2 dt, \end{aligned} \tag{6.73}$$

subject to:

$$\begin{aligned} \dot{x}_1 &= 2.3\nu_1(t) + 2.3\nu_2, \\ \dot{x}_2 &= -2.3\nu_1 - 2.3\nu_2. \end{aligned} \tag{6.74}$$

The Hamiltonian functions are

$$H_1(t, x_1, x_2, \nu_1, \nu_2, \chi_1, \chi_2) = (2.3\nu_1 - 2.3)^2 + \chi_1(2.3\nu_1 + 2.3\nu_2) + \chi_2(-2.3\nu_1 - 2.3\nu_2), \tag{6.75}$$

$$H_2(t, x_1, x_2, \nu_1, \nu_2, \chi_1, \chi_2) = (1.4\nu_2 - 1.4)^2 + \chi_1(2.3\nu_1 + 2.3\nu_2) + \chi_2(-2.3\nu_1 - 2.3\nu_2). \tag{6.76}$$

Based on (4.31) and (4.32), we obtain $\chi_1(t) = c_1$ and $\chi_1(t_f) = 2.3$,
consistently, the following result can be easily derived for the co-state vector of the second player:
$\chi_2(t) = c_2$ and $\chi_2(t_f) = 1.4$.
In response to (4.33) and from previous knowledge of $\chi_1(t)$ and $\chi_2(t)$, we get
$\nu_1 = 0.8043$, $\nu_2 = 0.4719$, $x_1 = 2.9353(t_f - t_0)$, and $x_2 = -2.9353(t_f - t_0)$.

When $\alpha = 0.2$, the problem turns into the following crisp problem:

$$\begin{aligned} \min_{\nu_1} J_1 &= 2.525x_1(t_f) + \int_{t_0}^{t_f} (2.525\nu_1 - 2.525)^2 dt, \\ \min_{\nu_2} J_2 &= 1.56x_2(t_f) + \int_{t_0}^{t_f} (1.56\nu_2 - 1.56)^2 dt, \end{aligned} \tag{6.77}$$

subject to:

$$\dot{x}_1 = 2.525\nu_1 + 2.525\nu_2,$$
$$\dot{x}_2 = -2.525\nu_1 - 2.525\nu_2. \tag{6.78}$$

The Hamiltonian functions are

$$H_1(t, x_1, x_2, \nu_1, \nu_2, \chi_1, \chi_2) = (2.525\nu_1 - 2.525)^2 + \chi_1(2.525\nu_1 + 2.525\nu_2) + \chi_2(-2.525\nu_1 - 2.525\nu_2), \tag{6.79}$$

$$H_2(t, x_1, x_2, \nu_1, \nu_2, \chi_1, \chi_2) = (1.56\nu_2 - 1.56)^2 + \chi_1(2.525\nu_1 + 2.525\nu_2) + \chi_2(-2.525\nu_1 - 2.525\nu_2). \tag{6.80}$$

Based on (4.31) and (4.32), we obtain $\chi_1(t) = c_1$ and $\chi_1(t_f) = 2.525$,
consistently,, the following result can be easily derived for the co-state vector of the second player:
$\chi_2(t) = c_2$ and $\chi_2(t_f) = 1.56$
In response to (4.33) and from previous knowledge of $\chi_1(t)$ and $\chi_2(t)$, we get
$\nu_1 = 0.8089$, $\nu_2 = 0.4994$, $x_1 = 3.3035(t_f - t_0)$ and $x_2 = -3.3035(t_f - t_0)$.

The Nash equilibrium and state trajectories for the players using the expected value method and the trust measure method are presented in tables 1 and 2, respectively.

Table 1. Nash equilibrium Solutions and State Trajectories Using the Expected Value Method.

μ	ν_1	ν_2	x_1	x_2
0.1	0.8591	0.7268	$4.3612(t_f - t_0)$	$-4.3612(t_f - t_0)$
0.2	0.8545	0.7107	$4.3043(t_f - t_0)$	$-4.3043(t_f - t_0)$
0.3	0.85	0.6939	$4.2457(t_f - t_0)$	$-4.2457(t_f - t_0)$
0.4	0.8455	0.6762	$4.1847(t_f - t_0)$	$-4.1847(t_f - t_0)$
0.5	0.8409	0.6578	$4.1214(t_f - t_0)$	$-4.1214(t_f - t_0)$
0.6	0.8364	0.6384	$4.0557(t_f - t_0)$	$-4.0557(t_f - t_0)$
0.7	0.8318	0.6181	$3.9872(t_f - t_0)$	$-3.9872(t_f - t_0)$
0.8	0.8273	0.5968	$3.9163(t_f - t_0)$	$-3.9163(t_f - t_0)$
0.9	0.8227	0.5745	$3.8423(t_f - t_0)$	$-3.8423(t_f - t_0)$

Table 2. Nash equilibrium Solutions and State Trajectories Using the Trust Measure Method.

α	ν_1	ν_2	x_1	x_2
0.1	0.8043	0.4719	$2.9353(t_f - t_0)$	$-2.9353(t_f - t_0)$
0.2	0.8089	0.4994	$3.3035(t_f - t_0)$	$-3.3035(t_f - t_0)$
0.3	0.8154	0.536	$3.5136(t_f - t_0)$	$-3.5136(t_f - t_0)$
0.4	0.8215	0.5682	$3.7174(t_f - t_0)$	$-3.7174(t_f - t_0)$
0.5	0.8273	0.5968	$3.9163(t_f - t_0)$	$-3.9163(t_f - t_0)$
0.6	0.8327	0.6223	$4.1104(t_f - t_0)$	$-4.1104(t_f - t_0)$
0.7	0.8379	0.6452	$4.301(t_f - t_0)$	$-4.301(t_f - t_0)$
0.8	0.8697	0.7618	$4.8537(t_f - t_0)$	$-4.8537(t_f - t_0)$
0.9	0.9063	0.858	$5.6458(t_f - t_0)$	$-5.6458(t_f - t_0)$

7 Conclusions

This study looked at possible interactions between continuous differential games and the rough programming approach. This is because there was some kind of uncertainty in the information that the players had access to. For a rough differential game, the Nash equilibrium was examined. We regarded the control that a player exerted on the system at any point as being rough because the information provided carried some degree of uncertainty, and we regarded the system state trajectory as being rough because it depended on the control that players have over the system. We derived the necessary and sufficient conditions for the Nash equilibrium of a rough differential game. A numerical example was given, and in it, we demonstrated in detail how to build the rough interval for the Nash equilibrium and the rough interval for the system state trajectory. The rough problem was then converted into crisp problems using the expected value operator and the trust measure. The Nash equilibrium and state trajectory were then established for each crisp problem. According to tables 1 and 2, both the expected value operator and trust measure depend on parameters μ and α, respectively, that belong to the interval $[0, 1]$. From the original rough problem, new crisp problems were constructed by varying the parameter values then for each crisp problem, the Nash equilibrium and its associated state trajectory were then determined. A significant feature of this article is the use of the "rough set" concept to handle the uncertainty of the information available to the players. The first procedure involved presenting a solution range instead of a direct one to the problem. On the other hand, the second approach used the expected value operator and trust measure of the rough interval to yield a direct solution that depends on variable parameter values for μ and α during the solution process.

The techniques employed in this paper can be utilized to manage uncertainty in game theory (differential games), such as Nash-Collative and Stackelberg differential games.

REFERENCES

[1] Bagchi, A., "Stackelberg Differential Games in Economic Models," Berlin, Heidelberg, Springer Berlin Heidelberg, 1984.

[2] Başar, T., & Olsder, G. J., "Dynamic Noncooperative Game Theory," Society for Industrial and Applied Mathematics, 1998.

[3] Friedman, A., "Differential Games," John Wiley & Sons, New York, London, 1971.

[4] Isaacs, R.,"Differential Games: A Mathematical Theory with Application to Warfare and Pursuit, Control and Optimization," John Wiley & Sons, 1965.

[5] Vorob'ev, N. N., "Game Theory: Lectures for Economists and Systems Scientists," Springer-Verlag, 1977.

[6] Megahed, A. A., & Hegazy, S.,"Min–max zero sum two persons continuous differential game with fuzzy controls," Asian Journal of Current Engineering and Maths, 2(2), 86–98, 2013.

[7] Seikh, M. R., Karmakar, S., & Nayak, P. K.,"Matrix games with dense fuzzy payoffs," International Journal of Intelligent Systems, 36(4), 1770-1799, 2021.

[8] Jana, J., &Roy, S. K., "Dual hesitant fuzzy matrix games: based on new similarity measure," Soft Computing, 23, 8873-8886, 2019.

[9] Seikh, M. R., Karmakar, S., & Xia, M.,"Solving matrix games with hesitant fuzzy pay-offs," Iranian Journal of Fuzzy Systems, 17(4), 25-40, 2020.

[10] Hegazy, S., Megahed, A. A., Youness, E. A., & Elbanna, A., "Min-max zero-sum two persons fuzzy continuous differential games," International Journal of Applied Mathematics. 21(1), 1–16, 2008.

[11] Seikh, M. R., & Dutta, S., "Solution of interval-valued matrix games using intuitionistic fuzzy optimisation technique: an effective approach," International Journal of Mathematics in Operational Research, 20(3), 297-322, 2021.

[12] Pawlak, Z., "Rough sets: Theoretical Aspects of Reasoning about Data," Springer Science & Business Media, 1991.

[13] Pawlak, Z., & Skowron, A., "Rudiment of rough sets," Information sciences. 177(1), 3–27, 2007.

[14] Rebolledom M., "Rough intervals—enhancing intervals for qualitative modeling of technical systems," Artificial Intelligence, 170(8-9), 667-685, 2006.

[15] Liu, B., "Uncertainty Theory: An introduction to its Axiomatic Foundations," Physica-Verlag, Heidelberg, 2004.

[16] Liu, B., "Theory and Practice of Uncertain Programming," Physica-Verlag, Heidelberg, 2002.

[17] Ruidas, S., Seikh, M. R., & Nayak, P. K., "A production-repairing inventory model considering demand and the proportion of defective items as rough intervals," Operational Research, 22(3), 2803-2829, 2022.

[18] Seikh, M. R., Dutta, S., & Li, D. F., "Solution of matrix games with rough interval pay-offs and its application in the telecom market share problem," International Journal of Intelligent Systems, 36(10), 6066-6100, 2021.

[19] Ammar, E. S., Brikaa, M. G., & Abdel-Rehim, E., "A study on two-person zero-sum rough interval continuous differential games," OPSEARCH, 56(3), 689-716, 2019.

[20] Roy, S. K., & Mula, P., ”Rough set approach to bi-matrix game,” International Journal of Operational Research, 23(2), 229-244, 2015.

[21] Roy, S. K., & Mula, P., ”Solving matrix game with rough payoffs using genetic algorithm,” Operational Research: An International Journal, 16, 117-130, 2016.

[22] Roy, S. K., Midya, S., & Yu, V. F., ”Multi-objective fixed-charge transportation problem with random rough variables,” International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 26(6), 971-996, 2018.

[23] Ghosh S., Roy S. K., ”Fuzzy-rough multi-objective product blending fixed-charge transportation problem with truck load constraints through transfer statio,” RAIRO-Operations Research, Vol.55, pp.S2923-S2952, 2021. Ghosh, S., & Roy, S. K., ”Fuzzy-rough multi-objective product blending fixed-charge transportation problem with truck load constraints through transfer statio,” RAIRO-Operations Research, 55, S2923-S2952, 2021.

[24] Youness, E. A., ”Characterizing solutions of rough programming problems,” European Journal of Operational Research, 168(3), 1019-1029, 2006.

[25] Kassem, M. A. E. H., Megahed, A. A., & Arafat H. K., ”A Study on Nash-Collative Differential Game of N-Players for Counterterrorism,” Journal of Function Spaces. 2021, 2021.

[26] Megahed, A. A., ”A terrorism-based differential game: Nash differential game,” Advances in Difference Equations, 2021(1), 1-11, 2021.

[27] Xu, J., & Tao, Z., ”Rough Multiple Objective Decision Making,” Taylor and Francis Group, CRC Press, USA, 2012.

[28] Fleming, W. H., & Rishel, R. W., ”Deterministic and Stochastic Optimal Control,” Berlin, Heidelberg, New York, Springer-Verlag, 1975.