Mathematics and Statistics 11(6): 960-964, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110611

Limit Theorems for Functionals of Random Convex Hulls in a Unit Disk

Isakjan Khamdamov[1], **Azam Imomov**[2,*]

[1]Department of Natural Sciences, Faculty of Economics, University of Public Safety of the Republic of Uzbekistan, Uzbekistan

[2]The Mathematics Department, Faculty of Mathematics and Computer Sciences, Karshi State University, Uzbekistan

Received March 3, 2023; Revised May 13, 2023; Accepted June 11, 2023

Cite This Paper in the following Citation Styles

***(a)**: [1] Isakjan Khamdamov, Azam Imomov, "Limit Theorems for Functionals of Random Convex Hulls in a Unit Disk," Mathematics and Statistics, Vol.11, No.6, pp. 960-964, 2023. DOI: 10.13189/ms.2023.110611*

***(b)**: Isakjan Khamdamov, Azam Imomov (2023). Limit Theorems for Functionals of Random Convex Hulls in a Unit Disk. Mathematics and Statistics, 11(6), 960-964. DOI: 10.13189/ms.2023.110611*

Abstract In this article, we study the functionals of the convex hull generated by independent observations over two-dimensional random points. When the random points are given in the polar coordinate system, their components are independent of each other, the angular coordinate is distributed uniformly, and the tail of the distribution of the radial coordinate is a regularly varying function near the circle of the unit disk – support. Here, with the approximation of the binomial point process by an inhomogeneous Poisson one, it is possible to study the asymptotic properties of the main functionals of the convex hull. Using the independence property of the increment of Poisson processes, we find an asymptotic expression for the mean values and variances for the main functionals of the convex hull. Uniform boundedness of exponential moments is proved for the same functionals, in the case when the convex hull is generated from an inhomogeneous Poisson point process inside the disk. The indicated independence property of the increment of the Poisson process allows us to express the area of the convex hull as a sum of independent identically distributed random variables, with which we prove the central limiting theorem for the number of vertices and the area of the convex hull. From the results obtained, we can conclude that if the tail of the distribution near the boundary is heavier, then there are many elements of the sample near the support boundary, and this means that there are many vertices of the convex hull, but the area bounded by the perimeter of the convex hull and the circle, as well as the difference between the perimeter of the convex hull and the circle, becomes negligible.

Keywords Convex Hull, Poisson Point Process, Binomial Point Process

1 Introduction

H. Carnal [1] studied the asymptotic expressions for the mean values of the main functionals, such as the number of vertices, the area and perimeter of the convex hull generated by independent observations over two-dimensional random points, when random points are given in polar coordinates, and their components are independent of each other, the angular coordinate is distributed uniformly, and the tail of the distribution of the radial coordinate is a regularly varying function at infinity (if the carrier is R^2) or near the circle (if the carrier is a unit disk). The second case involves a uniform distribution in a unit disk. The results of [1] generalized the results previously obtained by B.Efron [2], H.Raynaud [3], and A.Reny and R.Sulanke [4].

This article is devoted to the study of limit distributions of the main functionals of the vertex process for the cases considered in [1]. We use a modified technique proposed by P.Groeneboom [5], adapted to a wider class of initial distributions. The centering constants in the limit theorems used are the same as in the asymptotic expressions for the mean value of the corresponding convex hull functionals obtained in [1]. Along with the main theorems, a number of auxiliary lemmas are proved, which are of independent interest.

2 Statement of the problem and main results

Let A be a unit disk with the center at $(0, 1)$.

Let us assume that random points (r_i, α_i) are given in the polar coordinate system at A (the origin of coordinates is at $(0, 1)$), where r_i and α_i are independent (see Fig. 1), and α_i is uniformly distributed in $[-\pi,\pi]$ and

$$P\left(r_i > 1-x\right) = x^{\beta} L\left(\frac{1}{x}\right), \;\; 0<x<1, \;\; \beta \geq 1, \qquad (1)$$

where $L(x)$ is the slowly varying function in the Karamata sense given by

$$L(u) = \exp\left\{\int_1^u \frac{\varepsilon(t)}{t} dt\right\}, \quad \varepsilon(t) \to 0, \quad t \to \infty.$$

Next, assume that $X_i = r_i \sin\alpha_i, 1-Y_i = r_i \cos\alpha_i$ and the convex hulls generated by random points (X_1,Y_1); (X_2,Y_2); ...; (X_n,Y_n) we denote by C_n, and the number of vertices and the area of C_n we denote by ν_n and s_n, respectively.

Note that in the class of distributions (1), H. Carnal [1] stated the asymptotics for the average values of the above functionals

$$E\nu_n = \lambda_1(\beta) b_n^{\frac{1}{2}} \left(1+o(1)\right) \;\; \text{and}$$

$$Es_n = \pi - \lambda_2(\beta) b_n^{-1} \left(1+o(1)\right) \qquad (2)$$

as $n \to \infty$, where b_n - is the least root of equation

$$nx^{-\left(\beta+\frac{1}{2}\right)} L(x) = 1 \qquad (3)$$

and

$$\lambda_1(\beta) = \frac{\sqrt{2} B\left(2\beta+1; \frac{1}{2}\right)}{2\pi} \times \left(\frac{\sqrt{2}\pi}{B\left(\beta+1;\frac{1}{2}\right)}\right)^{2-\frac{1}{2\beta+1}} \Gamma\left(3 - \frac{1}{2\beta+1}\right), \qquad (4)$$

$$\lambda_2(\beta) = \frac{2\beta+1}{2\beta+3}\left(\frac{\sqrt{2}\pi}{B\left(\beta+1;\frac{1}{2}\right)}\right)^{\frac{1}{2\beta+1}} \times \Gamma\left(3+\frac{1}{2\beta+1}\right). \qquad (5)$$

In this article, the problem of finding limit distributions ν_n and s_n is solved for the class of distributions set by relation (1).

Let us present the main results of the study.

Theorem 1. As $n \to \infty$, the following relation is true

$$\frac{\nu_n - \lambda_1(\beta) b_n^{\frac{1}{2}}}{\sqrt{2\pi\lambda_3(\beta)} b_n^{\frac{1}{4}}} \stackrel{d}{\Rightarrow} N(0,1),$$

where $\lambda_3(\beta)$ is introduced in [6], $\stackrel{d}{\Rightarrow}$ means weak convergence, $N(0,1)$ – the normal r.v. with parameters (0,1).

Next, let $\Delta_n = \pi - s_n$.

Theorem 2. As $n \to \infty$, the following relation is true

$$\sqrt{\frac{\pi b_n^{\frac{5}{2}}}{2\lambda_4(\beta)}}\left(\frac{\Delta_n}{\pi} - \lambda_2(\beta) b_n^{-1}\right) \stackrel{d}{\Rightarrow} N(0,1),$$

and $\lambda_4(\beta) = c_1(\beta) + c_2(\beta) + c_3(\beta)$. Here $c_1(\beta), \; c_2(\beta)$ and $c_3(\beta)$ – are the constants introduced in [7].

Theorem 1 was proved in [6], in the particular case when $L(x) = 1$, so here we prove only Theorem 2 under conditions (1). And Theorem 1 is proved in a similar way.

3 Inhomogeneous Poisson approximation

Similar to [6] on the Poisson approximation of a binomial point process (b.p.p.) $B_n(A)$ generated by n independent observations over r.v., having distribution (1) with carrier A, we assume that

$$S_\varepsilon = \left\{(x,y) : \; 1-\varepsilon \leq \sqrt{x^2+(1-y)^2} \leq 1\right\},$$

$$\Lambda_\beta(\cdot) = P\left((X_1,Y_1) \in \cdot\right).$$

Let us introduce an event

$$E_\varepsilon = \{\text{at least one of the vertices } \; C_n \; \text{ does not lie in } S_\varepsilon\}$$

The meaning of the lemma below is that C_n lies in S_ε.

Lemma 1. There is a constant $c > 0$ such that

$$P(E_\varepsilon) \leq cn^2\left(1-\Lambda_\beta(A_\varepsilon)\right)^n \sim cn^2 \exp\left(-n\Lambda_\beta(A_\varepsilon)\right),$$

where A_ε is the segment formed by the chord of the unit disk tangent to the circle of radius $1-\varepsilon$.

Proof. Let Z_n be the set of vertices of the convex hull C_n generated by $B_n(A)$, $B_\varepsilon = S_1 - S_\varepsilon$. Then

$$E_\varepsilon = \{Z_n \cap B_\varepsilon \neq \oslash\}$$

Hence, we get

$$P(E_\varepsilon) \leq n^2\left(1-\Lambda_\beta(A_\varepsilon)\right)^{n-2} \sim cn^2 \exp\left\{-n\Lambda_\beta(A_\varepsilon)\right\}.$$

Lemma 1 is proved.

Now consider the convex hull C_n' generated by the realization of $(r_k', \alpha_k'), \;\; k = 1,2,...$ of the restriction of the i.p.p.p. $\Pi_{n,\beta}(A)$ in disk with intensity $\Lambda_{n,\beta}(\cdot) = n\Lambda_\beta(\cdot)$, while the center of disk A is at the point $(0,1)$ and the intensive measure $\Lambda_{n,\beta}(B)$ has the following form

$$\Lambda_{n,\beta}(B) = \begin{cases} \frac{n}{2\pi}\iint_B \frac{\partial}{\partial r}\left((1-r)^\beta L\left(\frac{1}{1-r}\right)\right) d\alpha\, dr, \\ \qquad\qquad \text{at} \quad B \subset A; \\ 0, \qquad\qquad , \;\; \text{at} \;\; B \not\subset A. \end{cases}$$

Let $X_i' = r_i' \sin\alpha_i', \;\; 1-Y_i' = r_i'\cos\alpha_i'$ and the set of vertices of the convex hull C_n' generated by random points $(X_1',Y_1'), (X_2',Y_2'), ...$ are denoted by Z_n'

Let

$$E_\varepsilon' = \{Z_n' \cap B_\varepsilon \neq \oslash\}.$$

Then, the following lemma is true

Lemma 2. There are positive constants $c_1, \; c_2, \; c_3$ and c_4 such that

$$P\left(E_\varepsilon'\right) \leq c_1\left(n^{-1} + n^2\left(1-\Lambda_\beta(A_\varepsilon)\right)^{c_2 n}\right)$$

$$\leq c_3\left(n^{-1} + n^2 \exp\left\{-c_4 n\Lambda_\beta(A_\varepsilon)\right\}\right)$$

The proof of this lemma is similar to the proof of Lemma 1.

The following lemma shows the closeness of b.p.p. and i.p.p.p. and is proved in the same way as in [6].

Lemma 3. Let $B_n(A)$ be a b.p.p. with parameter $(n, \Lambda_\beta(A))$. Then there exists an inhomogeneous Poisson point process (i.p.p.p.) $\Pi_{n,\beta}(A)$ with intensity $\Lambda_{n,\beta}(A)$ such that

$$P\left(B_n(A) \neq \Pi_{n,\beta}(A)\right) \leq 2\Lambda_\beta(S_\varepsilon).$$

It follows from Lemmas 1 – 3 that as $n \to \infty$

$$P\left(C_n \neq C'_n\right) \to 0.$$

Let us turn to the analysis of the estimates given in Lemmas 1–3. Similar to [5], for small $\varepsilon > 0$

$$\Lambda_\beta(A_\varepsilon) \approx \frac{B\left(\beta+1, \frac{1}{2}\right)}{\sqrt{2}\pi} \varepsilon^{\beta+\frac{1}{2}} L\left(\frac{1}{\varepsilon}\right).$$

Let $\varepsilon \geq \varepsilon_n = c_0 b_n^{-1} \log^{1+\tau} n$, which is defined in (3), c_0 and τ - are some positive constants. From this and from the definition of b_n it follows that there are positive constants c_1, c_2 and τ^* such that as $n \to \infty$

$$c_1 \log^{\beta+\frac{1}{2}-\tau^*} n \leq n\Lambda_\beta(A_{\varepsilon_n}) \leq c_2 \log^{\beta+\frac{1}{2}+\tau^*} n,$$

where $\beta + \frac{1}{2} - \tau^* > 1$. Therefore, choosing c_0, we have

$$P\left(E_\varepsilon\right) = O\left(n^{-1}\right), \quad P\left(E'_\varepsilon\right) = O\left(n^{-1}\right),$$

$$P\left(B_n(A) \neq \Pi_{n,\beta}(A)\right) = O\left(n^{-1} \log^{\beta+1} n\right).$$

Let $W'_n(a) = (X'_n(a), Y'_n(a))$ for any $a \in [-\pi, \pi]$ is such a point $(X'_n(a), Y'_n(a)) = (b_n X'_k, b_n Y'_k)$ for which $r'_k \cos(\alpha'_k - a)$ takes the maximum value.

It follows from the definition that $W'_n(a)$ forms a jump-like steady Markov process (see [9],[10]), generated by a restriction in $b_n A$ of the implementation of the i.p.p.p. $\Pi_{n,\beta}(\cdot)$ with intensity $\Lambda'_{n,\beta}(\cdot) = \Lambda_{n,\beta}(b_n \cdot)$. From this and from (3) it is easy to see that

$$\Lambda'_{n,\beta}(D) = \begin{cases} \frac{\sqrt{b_n}}{2\pi L(b_n)} \iint_D \frac{\partial}{\partial r}\left((b_n - r)^\beta L\left(\frac{b_n}{b_n - r}\right)\right) d\alpha\, dr, & \text{at } D \subset b_n A; \\ 0, & \text{at } D \not\subset b_n A, \end{cases} \tag{6}$$

where $b_n A$ is a circle with radius b_n, with center in $(0, b_n)$.

Let $0 = a_0 < a_1 < \ldots < a_k \leq a$ be times of jumps of process $W'_n(c), 0 \leq c \leq a$. Then $s'_n(a_{i-1}, a_i),\ i = 1, 2, \ldots, k$ is the area of the figure bounded by lines $(e_{i-1}, w - W'_n(a_{i-1})) = 0,\ (e_i, w - W'_n(a_{i-1})) = 0$ and circles $x^2 + (b_n - y)^2 = b_n^2$. The area of the figure is bounded by lines $(e_k, w - W'_n(a_k)) = 0,\ (e_a, w - W'_n(a_k)) = 0$ and circles $x^2 + (b_n - y)^2 = b_n^2$ we denote by $s'_n(a_k, a)$, where $w = (x, y),\ e_0 = (0, 1),\ e_i = (-\tan a_i, 1),\ i = 1, 2, \ldots, k,\ \ e_a = (-\tan a, 1)$. Then, if we assume that (see Figure 1) $s'_n(0, a) = s'_n(a_0, a_1) + s'_n(a_1, a_2) + \ldots + s'_n(a_k, a)$, than $s'_n(0, a)$ (similar to [8]) behaves like a sum of a random number of independent identically distributed random variables.

By construction, the vertex process $W'_n(a)$ is stationary.

We denote

$$E(\alpha_0, \delta) = \{(r, \alpha):\ r \leq b_n,\ r\cos(\alpha_0 - \alpha)$$

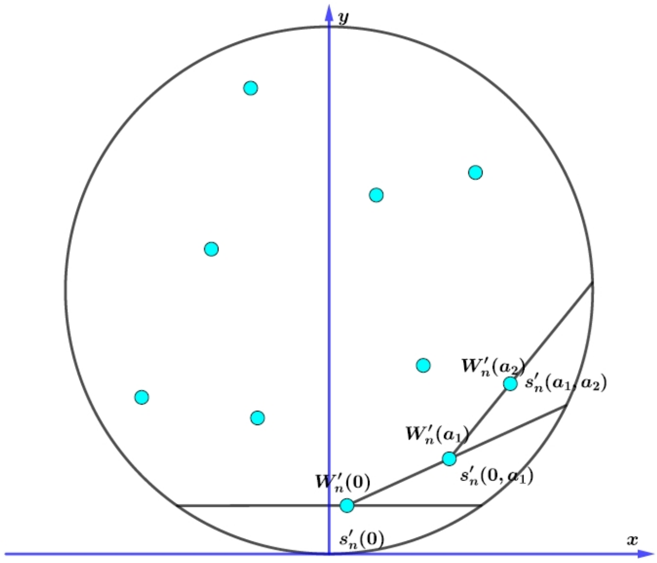

Figure 1. Illustrations of $W'_n(a_k)$ and $s'_n(a_{k-1}, a_k)$.

$$\geq b_n \cos\left(b_n^{-1/2} \log^\delta n\right)\Big\}.$$

Lemma 4. If $\delta > \frac{1}{2\beta+1}$, then

$$P\left(W'_n(\alpha_0) \notin E(\alpha_0, \delta)\right) = o\left(n^{-\tau}\right) \quad \text{for any} \quad \tau > 0.$$

Proof. By definition, we have

$$P\left(W'_n(\alpha_0) \notin E(\alpha_0, \delta)\right) = \exp\left(-\Lambda_n\left(E(\alpha_0, \delta)\right)\right). \tag{7}$$

Let $\alpha^* = b_n^{-1/2} \log^\delta n$, then calculating by formula (6), we obtain

$$\Lambda_n\left(E(\alpha_0, \delta)\right) = \frac{\sqrt{b_n}}{\pi L(b_n)} \int_0^{\alpha^*} \left(b_n - \frac{b_n \cos\alpha^*}{\cos\alpha}\right)^\beta$$

$$\times L\left(b_n - \frac{b_n \cos\alpha^*}{\cos\alpha}\right) d\alpha$$

$$\sim \frac{b_n^{\beta+\frac{1}{2}}}{\pi} \int_0^{\alpha^*} \left(\frac{\cos\alpha - \cos\alpha^*}{\cos\alpha}\right) d\alpha$$

$$\sim \frac{b_n^{\beta+\frac{1}{2}}}{\pi} \int_0^{\alpha^*} \left(2\sin\frac{\alpha^* - \alpha}{2} \sin\frac{\alpha^* + \alpha}{2}\right)^\beta d\alpha$$

$$\sim \frac{b_n^{\beta+\frac{1}{2}}}{\pi} \int_0^{\alpha^*} \left((\alpha^*)^2 - \alpha^2\right)^\beta d\alpha$$

$$= \frac{b_n^{\beta+\frac{1}{2}} (\alpha^*)^{2\beta+1}}{2^\beta \pi} \int_0^1 (1 - t^2)^\beta dt =$$

$$= \frac{b_n^{\beta+\frac{1}{2}} (\alpha^*)^{2\beta+1}}{2^\beta \pi} B\left(\beta + 1, \frac{1}{2}\right)$$

$$= \frac{\log^{(2\beta+1)\delta} n}{2^{\beta+1}\pi(2\beta+1)} B\left(\beta + 1, \frac{1}{2}\right) > c \log^{1+\mu} n,$$

where $\mu = (2\beta + 1)\delta - 1 > 0$.

From that and from (7) the assertion of Lemma 4 immediately follows.

4 Proof of Theorem 2

Following [5], in a circle we consider non-intersecting segments (large blocks) with central angle $2\pi b_n^{-1/2}\log n$,

$$E_k = \{(r,\alpha) : r \le b_n,\ r\cos(\alpha_k - \alpha) \ge b_n \cos\left(\pi b_n^{-1/2}\log n\right)\right\}$$

(hence, $m_n = \sqrt{b_n}\,/\log n$, $\alpha_k = \{(2k-1)\pi \log n\}\,/\sqrt{b_n}$). Similarly, consider non-intersecting segments (small blocks) with central angle $2\left(\log^{\delta_0} n + \log^{\delta_0^*} n\right)/\sqrt{b_n}$,

$$E_k^* = \{(r,\alpha) : r \le b_n,\ r\cos(\alpha_k - \alpha) \ge b_n \cos(\xi_0 + \xi_0^*)\},$$

where $\xi_0 = b_n^{-1/2}\log^{\delta_0} n$, $\xi_0^* = b_n^{-1/2}\log^{\delta_0^*} n$, $1 > \delta_0 > \delta_0^* > 2/(2\beta+1)$.

Lemma 4 immediately implies that

$$P\left(\text{there is at least one } \alpha \text{ such that } |\alpha_k - \alpha| < \pi b_n^{-1/2}\log n - \xi_0,\ W_n'(\alpha) \notin E_k\right) = o\left(n^{-\tau}\right),$$

$$P\left(\text{there is at least one } \alpha \text{ such that } |\alpha_k^* - \alpha| < \xi_0,\ W_n'(\alpha) \notin E_k\right) = o\left(n^{-\tau}\right).$$

Now, let $W_n^{(k)}(\alpha)$ be a vertex process of the convex hull generated by realization $\Pi_n(E_k)$, the restriction of the i.p.p.p. in E_k. Following P.Groeneboom (see Theorem 3.1 [5]), we have

$$P\left(\text{there is at least one } \alpha \text{ such that } |\alpha_k - \alpha| < \frac{\pi\log n}{\sqrt{b_n}} -\xi_0,\ W_n^{(k)}(\alpha) \ne W_n'(\alpha)\right) = o\left(b_n^{\tau-1}\log^{\beta+2} n\right) \quad (8)$$

for any $\tau > 0$. A similar relation can be obtained for any $\tau > 0$,

$$P\left(\text{there is at least one } \alpha \text{ such that } |\alpha_k^* - \alpha| < \xi_0,\ W_n^{*(k)}(\alpha) \ne W_n'(\alpha)\right) = o\left(b_n^{\tau-1}\log^{\beta+2} n\right), \quad (9)$$

where $W_n^{*(k)}(\alpha)$ is the vertex process of the convex hull generated by implementation $\Pi_n(E_k^*)$, the restriction of the i.p.p.p. in E_k^*.

If we denote the area of the vertex process $W_n^{(k)}(\alpha)$ of the convex hull generated by realization $\Pi_n(E_k)$ of the i.p.p.p. for $|\alpha_k - \alpha| < \frac{\pi\log n}{\sqrt{b_n}} - \xi_0$ by $s_{n,k}^*(\alpha_k+\xi_0, \alpha_{k+1}-\xi_0)$, and the area of the vertex process $W_n^{*(k)}(\alpha)$ of the convex hull generated by realization $\Pi_n(E_k^*)$ of the i.p.p.p. for $|\alpha_k^* - \alpha| < \xi_0$ we denote by $s_{n,k}^{**}(\alpha_k^* - \xi_0, \alpha_k^* + \xi_0)$, then from (4) and (4) we obtain

$$P\left(A(\alpha_{k-1}+\xi_0, \alpha_k - \xi_0) \ne s_{n,k}^*(\alpha_{k-1}+\xi_0, \alpha_k-\xi_0)\right) = o\left(\log^{\beta+2} n\,/b_n^{1-\tau}\right),$$
$$P\left(A(\alpha_k - \xi_0, \alpha_k + \xi_0) \ne s_{n,k}^{**}(\alpha_k-\xi_0, \alpha_k+\xi_0)\right) = o\left(\log^{\beta+2} n\,/b_n^{1-\tau}\right), \quad (10)$$

Therefore, approximating the i.p.p.p. in each large block E_k and small block E_k^* separately, instead of the limit distributions s_n, it suffices to study the limit distributions of the sums of independent r.v. $\sum_{k=1}^{m_n} s_{n,k}^*(\alpha_{k-1}+\xi_0, \alpha_k - \xi_0)$, $\sum_{k=1}^{m_n} s_{n,k}^*(\alpha_k - \xi_0, \alpha_k + \xi_0)$, respectively.

Consider segment E_0 of circle $b_n A$.

We present the following main lemma, proved in the same way as in [7].

Lemma 5. If $a = k_n\,/\sqrt{b_n}$ as $k_n \to \infty$ and $k_n = O(\log n)$, then

$$Es_n'(0,a) = \lambda_2(\beta) a b_n + O\left(1\,/b_n\right) \text{ and}$$

$$Vars_n'(0,a) = \lambda_4(\beta) a b_n^{\frac{3}{2}}\left(1 + o(1)\right).$$

Lemma 6. If $a = k_n\,/\sqrt{b_n}$ as $k_n \to \infty$ and $k_n = O(\log n)$, then

$$\frac{\frac{s_n'(0,a)}{\sqrt{b_n}} - E\left(\frac{s_n'(0,a)}{\sqrt{b_n}}\right)}{\sqrt{Var\left(\frac{s_n'(0,a)}{\sqrt{b_n}}\right)}} \overset{d}{\Rightarrow} N(0,1) \text{ as } n \to \infty.$$

We now proceed to prove Theorem 2.

According to the principle of constructing segments E_k, E_k^*, and from (10) it follows that

$$P\left(s_n \ne \sum_{k=0}^{m_n-1} s_{n,k}^*(a_k+\xi_0, a_{k+1}-\xi_0) + \sum_{k=0}^{m_n-1} s_{n,k}^{**}(a_{k+1}-\xi_0, a_{k+1}+\xi_0)\right) = o(1).$$

Lemmas 5 and 6 for $s_{n,k}^*(a_k+\xi_0, a_{k+1}-\xi_0)\,/\sqrt{b_n}$ and $s_{n,k}^{**}(a_{k+1}-\xi_0, a_{k+1}+\xi_0)\,/\sqrt{b_n}$ imply the validity of the central limit theorem for the corresponding centering and normalizing constants.

On the other hand, the functionals corresponding to the large sector E_k, are independent of each other. Similarly, the functionals corresponding to the small sector E_k^*, are also independent of each other. Then again, from Lemmas 5 and 6 we have

$$\frac{\left(s_n\,/\sqrt{b_n}\right) - m_n\left(2\pi\lambda_3\log n\right)}{\sqrt{m_n\left(2\pi\lambda_4\log n\right)}} = \frac{1}{\sqrt{m_n}}\sum_0^{m_n-1}\left(\frac{\left(s_{n,k}^*(a_k+\xi_0, a_{k+1}-\xi_0)\,/\sqrt{b_n}\right)}{\sqrt{2\pi\lambda_4\log n}} - \frac{\left(2\pi\lambda_3\log n - 2\pi\lambda_3\log^{\delta_0} n\right)}{\sqrt{2\pi\lambda_4\log n}}\right) + \frac{1}{\sqrt{m_n}} \times \sum_0^{m_n-1}\frac{\left(s_{n,k}^{**}(a_{k+1}-\xi_0, a_{k+1}+\xi_0)\,/\sqrt{b_n}\right) - 2\pi\lambda_3\log^{\delta_0} n}{\sqrt{2\pi\lambda_4\log^{\delta_0} n}} \frac{\sqrt{\log^{\delta_0} n}}{\sqrt{\log n}} + o_p(1) \overset{d}{\Rightarrow} N(0,1).$$

This implies the proof of Theorem 2.

REFERENCES

[1] H. Carnal. Die konvexe Hulle von n rotationssymmetrisch verteilten Punkten, Z. Wahrscheinlichkeitstheorie verw.Geb., Vol.15, 168–176, 1970. doi.org/10.1007/BF00531885

[2] B. Efron. The convex hull of a random set of points, Biometrika, Vol.52, 331–343, 1965. doi.org/10.2307/2333687

[3] H. Raynaud. Sur l'enveloppe convexe des nuages de points aleatoires dans, J.Appl. Prob. Vol.7, 35–48, 1970. doi.org/10.2307/3212146

[4] A. Reny and R. Sulanke R. Uber die konvexe Hulle von n zufalling gewahlten Punkten. Z. Wahrscheinlichkeitstheorie verw. Geb., Vol.2, 75–84, 1963. doi.org/10.1007/BF00535300

[5] P. Groeneboom. Limit theorems for convex hull. Probab. Th. Rel. Fields, Vol.79(3), 327–368, 1988. doi.org/10.1007/BF00342231

[6] I. Khamdamov. On Limit Theorem for the Number of Vertices of the Convex Hulls in a Unit Disk. Journal of Siberian Federal University. Mathematics and Physics, Vol.13(3), 275–284, 2020. DOI:10.17516/1997-1397-2020-13-3-275-284

[7] I. Khamdamov. Properties of convex hull generated by inhomogeneous Poisson point process. Ufimsk. Math. J., Vol.12(3), 83–98, 2020. doi.org/10.13108/2020-12-3-81

[8] A. Nagaev. Some properties of convex hulls generated by homogeneous Poisson point processes in an unbounded convex domain. Ann. Inst. Statist. Math., Vol.47, 21–29, 1995. doi.org/10.1007/BF00773409

[9] A. Imomov. Limit properties of transition functions of continuous-time markov branching processes. International Journal of Stochastic Analysis, Volume 2014, Article ID 409345, 2014. doi.org/10.1155/2014/409345

[10] A. Imomov. On a limit structure of the galton-watson branching processes with regularly varying generating functions. Probability and Mathematical Statistics, Vol.39(1), 61–73, 2019, DOI: 10.19195/0208-4147.39.1.4