Mathematics and Statistics 11(6): 923-935, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110607

Adomian Decomposition Method for Solving Fuzzy Hilfer Fractional Differential Equations

V. Padmapriya[1,2], **M. Kaliyappan**[1,*]

[1]Division of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai, India
[2]New Prince Shri Bhavani Arts and Science College, Chennai, India
*Corresponding Author: kaliyappan.m@vit.ac.in, v.padmapriya2015@vit.ac.in

Received June 22, 2023; Revised September 22, 2023; Accepted October 12, 2023

Cite This Paper in the following Citation Styles

***(a)**: [1] V. Padmapriya, M. Kaliyappan, "Adomian Decomposition Method for Solving Fuzzy Hilfer Fractional Differential Equations," Mathematics and Statistics, Vol.11, No.6, pp. 923-935, 2023. DOI: 10.13189/ms.2023.110607*

***(b)**: V. Padmapriya, M. Kaliyappan (2023). Adomian Decomposition Method for Solving Fuzzy Hilfer Fractional Differential Equations. Mathematics and Statistics, 11(6), 923-935. DOI: 10.13189/ms.2023.110607*

Abstract The field of fractional calculus is mainly concerned with the differentiation as well as integration of arbitrary orders. This concept is obviously present in various domains of science and engineering. Most people are familiar with the Caputo and Riemann-Liouville fractional definitions. Recently, Hilfer has related the Caputo and Riemann-Liouville derivatives by a general formula; this connection is referred to as the Hilfer or generalized Riemann-Liouville derivative. The Hilfer fractional derivative serves as an intermediary between the Riemann-Liouville and Caputo fractional derivatives, providing a means of interpolation. Parameters in the Hilfer derivative provide more degrees of freedom. Adomian decomposition method (ADM) is widely regarded as a highly effective mathematical technique for solving both linear and nonlinear differential equations. ADM provides an analytical solution in the form of a series solution. Motivated by the growing number of real-life applications for fractional calculus, the objective of this work is to explore the solutions of Hilfer fractional differential equations in a fuzzy sense using the ADM. The efficiency and accuracy of the proposed method are demonstrated by the solution of numerical examples. Graphical representations are provided to visualize the solutions' behavior. This shows that as the number of terms in the series goes up, the numerical results get closer and closer to the exact solutions.

Keywords Fuzzy Fractional Differential Equations, Riemann-Liouville Fractional Derivative, Caputo Fractional Derivative, Hilfer Fractional Derivative, Adomian Decomposition Method

1 Introduction

The topic of the fractional differential equation has received a lot of attention from authors interested in fractional calculus because of its essential application in the modeling of numerous occurrences in diverse fields of science, engineering, mathematics, bioengineering, and so on. Recent years have seen incredible progress in the study of ordinary, partial differential, and integral equations of fractional order. For more information, see Kilbas et al. [1], Lakshmikantham et al. [2], Miller and Rose [3], and Podlubny [4]. Fractional derivatives and integrals can be defined in a number of ways in the existing literature. Among these, the Riemann-Liouville and Caputo fractional definitions are the most prominent. Recently, the Riemann-Liouville fractional derivative has been generalized by Hilfer in [5]. Several authors refer to this type of derivative as the Hilfer derivative. The parameters in Hilfer fractional derivative add an additional degree of freedom to the initial conditions and generate a wider range of steady states. We refer the reader to a series of works [6]-[14] in which references to some recent results and applications of the Hilfer fractional derivative are provided.

However, researchers have proposed fuzzy fractional differential equations (FFDEs) to deal with uncertainty in various dynamical problems that exhibit imprecision, ambiguity, and non-normal dynamical behaviors with long memory or hereditary effects. The approaches of Caputo, Riemann-Liouville, Caputo-Katugampola, Hadamard, and Caputo-Hadamard are widely recognized in the field of fuzzy fractional derivatives and have been extensively studied and researched in the existing literature. More precisely, Agarwal et al. [15] and Allahviranloo et al. [16] investigated theoretical concerns such as the existence and uniqueness of results for FFDEs by employing the Riemann-Liouville derivative. Moreover, Mazandarani et al. [17] developed a new approach for researching fundamental concepts and computing the solutions of FFDEs with Caputo derivative. In the meantime, Lupulescu [18] established a broad theory of interval analysis for fractional cases, which is applied successfully to the study of FFDEs.

In addition, Allahviranloo et al. [19], Fard et al. [20], Dai and Chen [21], Hoa et al. [22, 23], and the references cited therein have offered various perspectives on the existence and stability of the solution to Caputo FFDEs. Further, methods for solving FFDEs using a shifted Chebyshev polynomials operational matrix and the spectral tau were proposed by Ahmadian et al. [24, 25] and Vinothkumar et al. [26] suggested the finite difference methodology to solve FFDEs. Moreover, Allahviranloo et al. [27] presented a new derivative concept known as the ABC generalized Hukuhara fractional derivative. Yang et al. [28] established the existence and stability of quaternion Hilfer fractional differential equations in the fuzzy sense. This paper is motivated by ongoing research in this area and presents the first step towards investigating the solution of Hilfer fractional differential equations with fuzzy initial conditions.

Since the majority of FFDEs cannot be solved analytically, finding effective approximation solutions using numerical approaches would be quite beneficial. Several scholars have recently focused their attention on exploring and investigating solutions to FFDEs using various numerical and analytical techniques, including the variational iteration method [29], Jacobi operational matrix method [30], fuzzy Laplace transform method [31], generalized differential transform method [32], modified fractional Runge-Kutta method [33], fractional Mellin transform method [34]. Recently, Johansyah et al. [42] presented a method for solving fractional-order differential equations. This approach integrates the Adomian decomposition method with the Kamal integral transformation. Nuruddeen [43] utilises a combination of Laplace transform and the Adomian decomposition method to address the linearized dynamical model with Hilfer fractional derivative.

For the reasons stated above, research into the FFDE to get a numerical solution has increased dramatically in the last decade as compared to analytical solutions. Therefore, this research focuses on the Adomian decomposition method (ADM) for reducing computational complexity in order to solve the fuzzy Hilfer fractional differential equations. ADM is proven to be more efficient than many other numerical approaches. It is an excellent technique to compute a fast convergent series solution with accuracy.

The main advantage of the proposed method is that it can be applied directly to integral and differential equations with variable or constant coefficients, regardless of whether they are homogeneous or nonhomogeneous, linear or nonlinear. Moreover, it decreases the significant computing work while maintaining the high accuracy of the numerical solutions. However, there are certain drawbacks to this approach as well: The ADM is based on the locally convergent Taylor expansion, allowing for the solution to be found as a finite series in a small region. This is one of its drawbacks. Furthermore, the series generated by the ADM may also converge to an exact solution if the closed form is well-known. However, among the current methods for dealing with nonlinearities, ADM is considered to be a highly effective approach. In recent years, the use of ADM in the solution of a wide class of differential equations has attracted significant attention. These equations might be linear, nonlinear, homogeneous, or inhomogeneous. Since Hilfer fractional derivative is more general than the Riemann-Liouville derivative, our obtained solutions are also more general than known solutions.

The next section provides some fundamental concepts of fuzzy Hilfer fractional derivatives. ADM algorithm is provided to solve Hilfer fractional differential equations in section 3. The discussion of the convergence of ADM can be found in section 4. Numerical examples are investigated that illustrate the efficiency of ADM in solving fuzzy Hilfer fractional differential equations in section 5 and the conclusion is provided in section 6.

2 Preliminaries

Several research papers have provided basic definitions of fuzzy and fractional calculus. Some of them were detailed further in [1, 35, 36] and the references therein. In this section, some fundamental definitions and results for fuzzy Hilfer fractional derivative are presented that will be used in the subsequent sections see [6, 8, 12, 28].

2.1 Definition [12, 32]

The Riemann-Liouville fractional integration of order α is defined as

$$(\mathcal{J}_{0^+}^{\alpha} x)(\mathrm{t}) = \frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\alpha-1} x(\mathrm{s}) d\mathrm{s}, \ \ \alpha > 0 \ \ \mathrm{t} > 0$$

2.2 Definition [12, 28]

The fuzzy Hilfer fractional derivative $\mathfrak{D}_{0^+}^{\alpha,\beta}$ of a function x with order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ is defined as follows:

$$\mathfrak{D}_{0^+}^{\alpha,\beta} x(\mathrm{t}) = \mathcal{J}_{0^+}^{\beta(1-\alpha)} \mathfrak{D} \mathcal{J}_{0^+}^{(1-\beta)(1-\alpha)} x(\mathrm{t})$$

where $\mathfrak{D} = \frac{d}{d\mathrm{t}}$
Then

$$\begin{aligned}\left[(\mathfrak{D}_{0^+}^{\alpha,\beta} x)(\mathrm{t})\right]^r &= \left[(\mathfrak{D}_{0^+}^{\alpha,\beta} \underline{x_r})(\mathrm{t}), (\mathfrak{D}_{0^+}^{\alpha,\beta} \overline{x_r})(\mathrm{t})\right] \\ &= \Big[(\mathcal{J}_{0^+}^{\beta(1-\alpha)} \mathfrak{D} \mathcal{J}_{0^+}^{(1-\beta)(1-\alpha)} \underline{x_r})(\mathrm{t}), \\ &\quad (\mathcal{J}_{0^+}^{\beta(1-\alpha)} \mathfrak{D} \mathcal{J}_{0^+}^{(1-\beta)(1-\alpha)} \overline{x_r})(\mathrm{t})\Big]\end{aligned}$$

Remark 1 *[8, 12, 28]*

(i) The Hilfer fractional derivative $\mathfrak{D}_{0^+}^{\alpha,\beta}$ can be expressed in the following form.
$\mathfrak{D}_{0^+}^{\alpha,\beta} = \mathcal{J}_{0^+}^{\beta(1-\alpha)}\mathfrak{D}\mathcal{J}_{0^+}^{(1-\beta)(1-\alpha)} = \mathcal{J}_{0^+}^{\beta(1-\alpha)}\mathfrak{D}_{0^+}^{\gamma}$
where $\gamma = \alpha + \beta - \alpha\beta$

(ii) Between the Riemann-Liouville and Caputo fractional derivatives, the Hilfer fractional derivative $\mathfrak{D}_{0^+}^{\alpha,\beta}$ is employed as an interpolator since

$$\mathfrak{D}_{0^+}^{\alpha,\beta} = \begin{cases} \mathfrak{D}\mathcal{J}_{0^+}^{1-\alpha} = ^{RL}\mathfrak{D}_{0^+}^{\alpha}, & if\ \beta = 0 \\ \mathcal{J}_{0^+}^{1-\alpha}\mathfrak{D} = ^{C}\mathfrak{D}_{0^+}^{\alpha}, & if\ \beta = 1 \end{cases}$$

We will require the following spaces for our upcoming analysis.

$$\mathfrak{C}_{1-\gamma}^{\alpha,\beta}[0,\ 1] = \left\{x \in \mathfrak{C}_{1-\gamma}[0,\ 1] : \mathfrak{D}_{0^+}^{\alpha,\beta}x \in \mathfrak{C}_{1-\gamma}[0,\ 1]\right\}$$

and

$$\mathfrak{C}_{1-\gamma}^{\gamma}[0,\ 1] = \left\{x \in \mathfrak{C}_{1-\gamma}[0,\ 1] : \mathfrak{D}_{0^+}^{\gamma}x \in \mathfrak{C}_{1-\gamma}[0,\ 1]\right\}$$

Since $\mathfrak{D}_{0^+}^{\alpha,\beta}x = \mathcal{J}_{0^+}^{\beta(1-\alpha)}\mathfrak{D}_{0^+}^{\gamma}x$, it is obvious that $\mathfrak{C}_{1-\gamma}^{\gamma}[0,\ 1] \subset \mathfrak{C}_{1-\gamma}^{\alpha,\beta}[0,\ 1]$.

Lemma 1 *[1, 12] Consider $\alpha > 0,\ \beta > 0$ and $\gamma = \alpha + \beta - \alpha\beta$. If $x \in \mathfrak{C}_{1-\gamma}^{\gamma}[0,\ 1]$, then*

$$\mathcal{J}_{0^+}^{\gamma}\mathfrak{D}_{0^+}^{\gamma}x = \mathcal{J}_{0^+}^{\alpha}\mathfrak{D}_{0^+}^{\alpha,\beta}x$$

and

$$\mathfrak{D}_{0^+}^{\gamma}\mathcal{J}_{0^+}^{\alpha}x = \mathfrak{D}_{0^+}^{\beta(1-\alpha)}x$$

2.3 Theorem [12]

Suppose that $x \in \mathfrak{C}_{\gamma}[0,\ 1],\ where\ 0 < \alpha < 1$ and $0 \le \gamma < 1$

$$\mathfrak{D}_{0^+}^{\alpha}\mathcal{J}_{0^+}^{\alpha}x(\mathrm{t}) = x(\mathrm{t}), \quad \forall\, \mathrm{t} \in (0,\ 1]$$

Further, if $x \in \mathfrak{C}_{\gamma}[0,\ 1]$ and $\mathcal{J}_{0^+}^{(1-\beta)(1-\alpha)}x \in \mathfrak{C}_{\gamma}^{1}[0,\ 1]$, then

$$\mathfrak{D}_{0^+}^{\alpha,\beta}\mathcal{J}_{0^+}^{\alpha}x(\mathrm{t}) = x(\mathrm{t}), \quad for\ a.e\ \mathrm{t} \in (0, 1]$$

2.4 Theorem [12]

Consider $\alpha, \beta \ge 0$ and $x \in \mathfrak{C}_{\gamma}^{1}[0,\ 1]$. Then

$$\mathcal{J}_{0^+}^{\alpha}\mathcal{J}_{0^+}^{\beta}x(\mathrm{t}) = \mathcal{J}_{0^+}^{\alpha+\beta}x(\mathrm{t})$$

Lemma 2 *[12] Consider $\alpha \ge 0$ and $\sigma > 0$. Then*

$$\mathcal{J}_{0^+}^{\alpha}\mathrm{t}^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)}\mathrm{t}^{\alpha+\sigma-1}, \quad \mathrm{t} > 0$$

and

$$\mathfrak{D}_{0^+}^{\alpha}\mathrm{t}^{\alpha-1} = 0, \quad 0 < \alpha < 1$$

Lemma 3 *[12, 16] If $x \in \mathfrak{C}_{\gamma}[0,\ 1]$ and $\mathcal{J}_{0^+}^{1-\alpha}x \in \mathfrak{C}_{\gamma}^{1}[0,\ 1]$ for $0 < \alpha < 1$ and $0 \le \gamma < 1$, then*

- $(\mathcal{J}_{0^+}^{\alpha}\mathfrak{D}_{0^+}^{\alpha}x)(\mathrm{t}) = x(\mathrm{t}) \ominus \frac{\mathcal{J}_{0^+}^{1-\alpha}x(0)}{\Gamma(\alpha)}\mathrm{t}^{\alpha-1}$, *for x is (1)-differentiable (gH differentiable)*
- $(\mathcal{J}_{0^+}^{\alpha}\mathfrak{D}_{0^+}^{\alpha}x)(\mathrm{t}) = (-1)\frac{\mathcal{J}_{0^+}^{1-\alpha}x(0)}{\Gamma(\alpha)}\mathrm{t}^{\alpha-1} \ominus (-1)x(\mathrm{t})$, *for x is (2)-differentiable (gH differentiable)*

where $\ominus$ is fuzzy subtraction.

3 Adomian Decomposition method

Consider the subsequent fractional order differential equation involving Hilfer derivative, where $0 < \alpha < 1, 0 \le \beta \le 1$ in [6]

$$\mathfrak{D}_{0^+}^{\alpha,\beta}x(\mathrm{t}) = \lambda x(\mathrm{t}) + f(\mathrm{t}) \tag{1}$$

with initial condition

$$\mathcal{J}_{0^+}^{1-\gamma}x(0) = b,\ \ \gamma = \alpha + \beta - \alpha\beta,\ \ b \in \mathbb{R} \tag{2}$$

Operating $\mathcal{J}_{0^+}^{\alpha}$ on both sides of Eqn. (1)

$$\mathcal{J}_{0^+}^{\alpha}\ \mathfrak{D}_{0^+}^{\alpha,\beta}x(\mathrm{t}) = \ \mathcal{J}_{0^+}^{\alpha}\lambda x(\mathrm{t}) + \ \mathcal{J}_{0^+}^{\alpha}f(\mathrm{t})$$

We have,

$$x(\mathrm{t}) = \frac{b\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + \lambda\mathcal{J}_{0^+}^{\alpha}x(\mathrm{t}) + \ \mathcal{J}_{0^+}^{\alpha}f(\mathrm{t}) \tag{3}$$

The ADM views the solution $x(\mathrm{t})$ as having several components, such

$$x(\mathrm{t}) = \sum_{n=0}^{\infty}x_n(\mathrm{t}) \tag{4}$$

Substituting the decomposition series (4) into both sides of (3) yields

$$\sum_{n=0}^{\infty}x_n(\mathrm{t}) = \frac{b\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + \mathcal{J}_{0^+}^{\alpha}f(\mathrm{t}) + \lambda\mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}x_n(\mathrm{t})$$

This equation allows us to derive the following recursive relation:

$$x_0(\mathrm{t}) = \frac{b\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + \mathcal{J}_{0^+}^{\alpha}f(\mathrm{t})$$
$$x_{k+1}(\mathrm{t}) = \lambda\mathcal{J}_{0^+}^{\alpha}x_k(\mathrm{t}), \quad k \ge 0$$

We find

$$\begin{aligned} x_1(\mathrm{t}) &= \lambda\mathcal{J}_{0^+}^{\lambda}x_0(\mathrm{t}) \\ &= \lambda\mathcal{J}_{0^+}^{\alpha}\left[\frac{b\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + \mathcal{J}_{0^+}^{\alpha}f(\mathrm{t})\right] \\ x_1(\mathrm{t}) &= \frac{\lambda b\mathrm{t}^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} + \lambda\mathcal{J}_{0^+}^{2\alpha}f(\mathrm{t}) \\ x_2(\mathrm{t}) &= \lambda\mathcal{J}_{0^+}^{\alpha}x_1(\mathrm{t}) \\ &= \lambda\mathcal{J}_{0^+}^{\alpha}\left[\frac{\lambda b\mathrm{t}^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} + \lambda\mathcal{J}_{0^+}^{2\alpha}f(\mathrm{t})\right] \\ x_2(\mathrm{t}) &= \frac{\lambda^2 b\mathrm{t}^{2\alpha+\gamma-1}}{\Gamma(2\alpha+\gamma)} + \lambda^2\mathcal{J}_{0^+}^{3\alpha}f(\mathrm{t}) \end{aligned}$$

Continuing this process, we have

$$x_n(\mathrm{t}) = \frac{\lambda^n b\mathrm{t}^{n\alpha+\gamma-1}}{\Gamma(n\alpha+\gamma)} + \lambda^n\mathcal{J}_{0^+}^{n\alpha+\alpha}f(\mathrm{t})$$

Then the solution $x(\mathrm{t})$ can be written by Eqn.(4) as follows.

$$\begin{aligned} x(\mathrm{t}) &= b\sum_{n=0}^{\infty}\frac{\lambda^n\mathrm{t}^{n\alpha+\gamma-1}}{\Gamma(n\alpha+\gamma)} + \int_0^{\mathrm{t}}\sum_{n=0}^{\infty}\frac{\lambda^n(\mathrm{t}-\mathrm{s})^{n\alpha+\alpha-1}}{\Gamma(n\alpha+\alpha)}f(\mathrm{s})d\mathrm{s} \\ &= b\mathrm{t}^{\gamma-1}\sum_{n=0}^{\infty}\frac{(\lambda\mathrm{t}^{\alpha})^n}{\Gamma(n\alpha+\gamma)} + \\ &\int_0^{\mathrm{t}}(\mathrm{t}-\mathrm{s})^{\alpha-1}\sum_{n=0}^{\infty}\frac{(\lambda(\mathrm{t}-\mathrm{s})^{\alpha})^n}{\Gamma(n\alpha+\alpha)}f(\mathrm{s})d\mathrm{s} \end{aligned}$$

Thus the solution of Hilfer fractional differential equation (1) - (2) is

$$x(\mathrm{t}) = b\mathrm{t}^{\gamma-1}E_{\alpha,\gamma}(\lambda\mathrm{t}^{\alpha}) + \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\alpha-1}E_{\alpha,\alpha}(\lambda(\mathrm{t}-\mathrm{s})^{\alpha})f(\mathrm{s})d\mathrm{s} \tag{5}$$

The result (5) is matched with the solution obtained in [6].

Remark 2 *1. If $\beta = 0$, then $\gamma = \alpha$ and*

$$x(\mathrm{t}) = b\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(\lambda\mathrm{t}^{\alpha}) + \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\alpha-1}E_{\alpha,\alpha}(\lambda(\mathrm{t}-\mathrm{s})^{\alpha})f(\mathrm{s})d\mathrm{s} \tag{6}$$

is the solution to the fractional differential equation involving the Riemann-Liouville derivative shown below.

$$\begin{cases} {}^{RL}\mathfrak{D}^{\alpha}_{0^+}x(\mathrm{t}) = \lambda x(\mathrm{t}) + f(\mathrm{t}) \\ \mathcal{J}^{1-\alpha}_{0^+}x(0) = b \end{cases}$$

2. If $\beta = 1$, then $\gamma = 1$ and

$$x(\mathrm{t}) = bE_{\alpha,1}(\lambda\mathrm{t}^{\alpha}) + \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\alpha-1}E_{\alpha,\alpha}(\lambda(\mathrm{t}-\mathrm{s})^{\alpha})f(\mathrm{s})d\mathrm{s} \tag{7}$$

is the solution to the fractional differential equation involving the Caputo derivative shown below.

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}_{0^+}x(\mathrm{t}) = \lambda x(\mathrm{t}) + f(\mathrm{t}) \\ x(0) = b \end{cases}$$

The results (6) and (7) coincide with the results in [7, 37].

4 Convergence of Adomian Decomposition method

Many researchers have been working on the issue of convergence for ADM involving Caputo fractional derivative [39, 40, 41]. In this section, through the reference [41], we present the convergence for the Adomian series involving Hilfer fractional derivative.

Consider the following FFDE with Hilfer derivative of order $0 < \alpha < 1, 0 \leq \beta \leq 1$

$$\mathfrak{D}^{\alpha,\beta}_{0^+}x(\mathrm{t}) = L_x(\mathrm{t}) + N_x(\mathrm{t}) \tag{8}$$

with initial condition

$$\mathcal{J}^{1-\gamma}_{0^+}x(0) = x_0, \;\; \gamma = \alpha + \beta - \alpha\beta, \;\; x_0 \in \mathbb{E} \tag{9}$$

Here $\mathbb{E}$ is the set of all fuzzy numbers, and L and N refer to the linear and nonlinear operators respectively.
In the view of the ADM, the solution $x(t)$ is decomposed into

$$x(\mathrm{t}) = \sum_{n=0}^{\infty} x_n(\mathrm{t})$$

And decomposing the nonlinear term N as

$$N_x(\mathrm{t}) = \sum_{n=0}^{\infty} A_n(\mathrm{t})$$

Adomian polynomials are denoted by A_n and are provided by

$$A_n = \frac{1}{n}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{j=0}^{\infty}\lambda^j x_j\right)\right]_{\lambda=0}$$

To prove the solution $x(\mathrm{t}) = \sum_{n=0}^{\infty} x_n(\mathrm{t})$ is uniformly convergent by ADM, we need some lemmas and theorem.

Lemma 4 *Let $x_i(\mathrm{t}), i = 0, 1, 2, ...n$ be fuzzy continuous functions. Then*

$$\mathfrak{D}^{\alpha,\beta}_{0^+}\sum_{i=0}^{n} x_i(\mathrm{t}) = \sum_{i=0}^{n}\mathfrak{D}^{\alpha,\beta}_{0^+}x_i(\mathrm{t})$$

Proof. Consider $x_i(\mathrm{t})$ is (1) - differentiable and we use mathematical induction to prove this lemma.

$$\begin{aligned}
\mathfrak{D}^{\alpha,\beta}_{0^+}\Big(x_0(\mathrm{t}) + x_1(\mathrm{t})\Big) &= \Big[\mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\underline{x_0}(\mathrm{t}) + \underline{x_1}(\mathrm{t})\Big), \\
&\quad \mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\overline{x_0}(\mathrm{t}) + \overline{x_1}(\mathrm{t})\Big)\Big] \\
&= \Big[\mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\underline{x_0}(\mathrm{t})\Big) + \mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\underline{x_1}(\mathrm{t})\Big), \\
&\quad \mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\overline{x_0}(\mathrm{t})\Big) + \mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\overline{x_1}(\mathrm{t})\Big)\Big] \\
&= \Big[\mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\underline{x_0}(\mathrm{t})\Big), \mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\overline{x_0}(\mathrm{t})\Big)\Big] \\
&\quad + \Big[\mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\underline{x_1}(\mathrm{t})\Big), \mathfrak{D}^{\alpha,\beta}_{0^+}\Big(\overline{x_1}(\mathrm{t})\Big)\Big] \\
&= \mathfrak{D}^{\alpha,\beta}_{0^+}x_0(\mathrm{t}) + \mathfrak{D}^{\alpha,\beta}_{0^+}x_1(\mathrm{t})
\end{aligned}$$

Suppose that

$$\mathfrak{D}^{\alpha,\beta}_{0^+}\sum_{i=0}^{n-1} x_i(\mathrm{t}) = \sum_{i=0}^{n-1}\mathfrak{D}^{\alpha,\beta}_{0^+}x_i(\mathrm{t})$$

Thus

$$\begin{aligned}
\mathfrak{D}^{\alpha,\beta}_{0^+}\sum_{i=0}^{n} x_i(\mathrm{t}) &= \mathfrak{D}^{\alpha,\beta}_{0^+}\Big[\sum_{i=0}^{n-1} x_i(\mathrm{t}) + x_n(\mathrm{t})\Big] \\
&= \mathfrak{D}^{\alpha,\beta}_{0^+}\sum_{i=0}^{n-1} x_i(\mathrm{t}) + \mathfrak{D}^{\alpha,\beta}_{0^+}x_n(\mathrm{t}) \\
&= \sum_{i=0}^{n-1}\mathfrak{D}^{\alpha,\beta}_{0^+}x_i(\mathrm{t}) + \mathfrak{D}^{\alpha,\beta}_{0^+}x_n(\mathrm{t}) \\
&= \sum_{i=0}^{n}\mathfrak{D}^{\alpha,\beta}_{0^+}x_i(\mathrm{t})
\end{aligned}$$

Theorem 4.1 *Let us consider $x(\mathrm{t}) = \sum_{i=0}^{\infty} x_i(\mathrm{t})$ in Eqn.(8) such that the terms of series might be of various differentiability types. This theorem discusses the following cases.*

Case (i) *Consider* $x_i(\mathrm{t}), i = 0,1,2,...$ *is (1) - differentiable, then*

$$x_0(\mathrm{t}) = \frac{x_0}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}$$
$$x_i(\mathrm{t}) = \tilde{0} \oplus \mathcal{J}^{\alpha}_{0^+} L_{x_{i-1}}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+} A_{i-1}(\mathrm{t}), \quad i = 1,2...$$

Case (ii) *Consider* $x_i(\mathrm{t}), i = 0,1,2,...$ *is (2) - differentiable, then*

$$x_0(\mathrm{t}) = \frac{x_0}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}$$
$$x_i(\mathrm{t}) = \tilde{0} \ominus (-1)\mathcal{J}^{\alpha}_{0^+} L_{x_{i-1}}(\mathrm{t}) \ominus (-1)\mathcal{J}^{\alpha}_{0^+} A_{i-1}(\mathrm{t}), i = 1,2...$$

Case (iii) *Let* i *is even,* $x_i(\mathrm{t})$ *is (1) - differentiable and* j *is odd,* $x_j(\mathrm{t})$ *is (2) - differentiable, then*

$$x_0(\mathrm{t}) = \frac{x_0}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}$$
$$x_i(\mathrm{t}) = \tilde{0} \oplus J^{\alpha}_{0^+} L_{x_{i-1}}(\mathrm{t}) \oplus J^{\alpha}_{0^+} A_{i-1}(\mathrm{t}),\ i\ is\ even$$
$$x_j(\mathrm{t}) = \tilde{0} \ominus (-1)J^{\alpha}_{0^+} L_{x_{j-1}}(\mathrm{t}) \ominus (-1)J^{\alpha}_{0^+} A_{j-1}(\mathrm{t}),\ j\ is\ odd$$

Proof. Assume that $x_i(\mathrm{t}), i = 0,1,2,...$ is continuous function.
(i) Consider $x_i(\mathrm{t})$ is (1) - differentiable. On both sides of Eqn. (8), we employ the Riemann-Liouville integral.

$$\mathcal{J}^{\alpha}_{0^+}\mathfrak{D}^{\alpha,\beta}_{0^+}x(\mathrm{t}) = \mathcal{J}^{\alpha}_{0^+}L_x(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}N_x(\mathrm{t})$$

$$\mathcal{J}^{\alpha}_{0^+}\mathfrak{D}^{\alpha,\beta}_{0^+}\sum_{i=0}^{\infty}x_i(\mathrm{t}) = \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$

$$\mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}\left(\mathfrak{D}^{\alpha,\beta}_{0^+}x_i(\mathrm{t})\right) = \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$

$$\sum_{i=0}^{\infty}\mathcal{J}^{\alpha}_{0^+}\left(\mathfrak{D}^{\alpha,\beta}_{0^+}x_i(\mathrm{t})\right) = \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$

$$\sum_{i=0}^{\infty}\mathcal{J}^{\gamma}_{0^+}\mathfrak{D}^{\gamma}_{0^+}x_i(\mathrm{t}) = \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t})\oplus\mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})\ (by\ lemma\ 1)$$

$$\mathcal{J}^{\gamma}_{0^+}\left(\mathfrak{D}^{\gamma}_{0^+}x_0(\mathrm{t})\right) \oplus \sum_{i=1}^{\infty}\mathcal{J}^{\gamma}_{0^+}\left(\mathfrak{D}^{\gamma}_{0^+}x_i(\mathrm{t})\right)$$
$$= \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$

By lemma 3, now we have

$$\left(x_0(\mathrm{t})\ominus\frac{\mathcal{J}^{1-\gamma}_{0^+}x_0(0)}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}\right)\oplus\sum_{i=1}^{\infty}\left(x_i(\mathrm{t})\ominus\frac{\mathcal{J}^{1-\gamma}_{0^+}x_i(0)}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}\right)$$
$$= \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$

Suppose $\mathcal{J}^{1-\gamma}_{0^+}x_i(0) = \tilde{0}$, we have

$$x_0(\mathrm{t}) = \frac{x_0}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}$$
$$x_i(\mathrm{t}) = \tilde{0} \oplus \mathcal{J}^{\alpha}_{0^+} L_{x_{i-1}}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+} A_{i-1}(\mathrm{t}), i = 1,2,...$$

(ii) Consider $x_i(\mathrm{t})$ is (2) - differentiable, similar to the proof of (i), we obtain

$$\mathcal{J}^{\gamma}_{0^+}\left(\mathfrak{D}^{\gamma}_{0^+}x_0(\mathrm{t})\right) \oplus \sum_{i=1}^{\infty}\mathcal{J}^{\gamma}_{0^+}\left(\mathfrak{D}^{\gamma}_{0^+}x_i(\mathrm{t})\right)$$
$$= \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$
$$\left((-1)\frac{J^{1-\gamma}_{0^+}x_0(0)}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1} \ominus (-1)x_0(\mathrm{t})\right)\oplus$$
$$\sum_{i=1}^{\infty}\left((-1)\frac{J^{1-\gamma}_{0^+}x_i(0)}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1} \ominus (-1)x_i(\mathrm{t})\right)$$
$$= \mathcal{J}^{\alpha}_{0}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$

Thus,

$$x_0(\mathrm{t}) = \frac{x_0}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}$$
$$x_i(\mathrm{t}) = \tilde{0} \ominus (-1)\mathcal{J}^{\alpha}_{0^+} L_{x_{i-1}}(\mathrm{t}) \ominus (-1)\mathcal{J}^{\alpha}_{0^+} A_{i-1}(\mathrm{t}), i = 1,2,...$$

(iii) Assume that i is even, $x_i(\mathrm{t})$ is (1) - differentiable and j is odd, $x_j(\mathrm{t})$ is (2) - differentiable. Then

$$\mathcal{J}^{\gamma}_{0^+}\left(\mathfrak{D}^{\gamma}_{0^+}x_0(\mathrm{t})\right)\oplus\mathcal{J}^{\gamma}_{0^+}\left(\mathfrak{D}^{\gamma}_{0^+}x_1(\mathrm{t})\right)\oplus\mathcal{J}^{\gamma}_{0^+}\left(\mathfrak{D}^{\gamma}_{0^+}x_2(\mathrm{t})\right)\oplus...$$
$$= \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$
$$\left(x_0(\mathrm{t}) \ominus \frac{\mathcal{J}^{1-\gamma}_{0^+}x_0(0)}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}\right)$$
$$\oplus\left((-1)\frac{\mathcal{J}^{1-\gamma}_{0^+}x_1(0)}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1} \ominus (-1)x_1(\mathrm{t})\right)$$
$$\oplus\left(x_2(\mathrm{t}) \ominus \frac{\mathcal{J}^{1-\gamma}_{0^+}x_2(0)}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}\right) \oplus ...$$
$$= \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}L_{x_i}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+}\sum_{i=0}^{\infty}A_i(\mathrm{t})$$

Therefore

$$x_0(\mathrm{t}) = \frac{x_0}{\Gamma(\gamma)}\mathrm{t}^{\gamma-1}$$
$$x_i(\mathrm{t}) = \tilde{0} \oplus \mathcal{J}^{\alpha}_{0^+} L_{x_{i-1}}(\mathrm{t}) \oplus \mathcal{J}^{\alpha}_{0^+} A_{i-1}(\mathrm{t}),\ i\ is\ even$$
$$x_j(\mathrm{t}) = \tilde{0} \ominus (-1)\mathcal{J}^{\alpha}_{0^+} L_{x_{j-1}}(\mathrm{t}) \ominus (-1)J^{\alpha}_{0^+} A_{j-1}(\mathrm{t}),\ j\ is\ odd$$

Hence proved.

Lemma 5 *[41] Adomian's polynomials are bounded. i.e.,* $D\big(A_n(\mathrm{t}), \tilde{0}\big) \le \mathcal{M}$ *where* $n = 0,1,...,\infty$*, and* D *is Hausdorff metric.*

Remark 3 *[41] If the function* $\big(\mathrm{t}-\tau\big)^{\alpha-1}$ *in fuzzy Riemann-Liouville integral is continuous and bounded, then* $\int_0^{\mathrm{t}}\big|\mathrm{t}-\tau\big|^{\alpha-1}d\tau \le \mathcal{N}$.

Theorem 4.2 *Suppose that Eqn.(8) satisfies the following conditions*

(i) For linear terms, $D\Big(L_{x_n}(\mathrm{t}), L_{x_{n-1}}(\mathrm{t})\Big) \leq \mathcal{P}$ *and* $1 < \mathcal{P} < \mathcal{N}^{-1},\ n \geq 1$

(ii) For nonlinear terms, $D\Big(A_n(\mathrm{t}), A_{n-1}(\mathrm{t})\Big) \leq \mathcal{M}$ *and* $1 < \mathcal{M} < \mathcal{N}^{-1},\ n \geq 1$

Then the successive iterations in theorem 4.1 are uniformly convergent to $x(\mathrm{t})$.

Proof. We prove this theorem for case (i). Other cases' proof is similar to case (i) which is omitted here. As the result of our hypothesis, for case (i) we have

$$\begin{aligned}
D\Big(x_{n+1}(\mathrm{t}), x_n(\mathrm{t})\Big) &= D\Big[\mathcal{J}_{0^+}^{\alpha}\Big(L_{x_n}(\mathrm{t}) \oplus A_n(\mathrm{t})\Big), \\
&\qquad \mathcal{J}_{0^+}^{\alpha}\Big(L_{x_{n-1}}(\mathrm{t}) \oplus A_{n-1}(\mathrm{t})\Big)\Big] \\
&= D\Big[\frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} (\mathrm{t}-\tau)^{\alpha-1}\Big(L_{x_n}(\tau) \oplus A_n(\tau)\Big)d\tau, \\
&\qquad \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} (\mathrm{t}-\tau)^{\alpha-1}\Big(L_{x_{n-1}}(\tau) \oplus A_{n-1}(\tau)\Big)d\tau\Big] \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} D\Big[(\mathrm{t}-\tau)^{\alpha-1}\Big(L_{x_n}(\tau) \oplus A_n(\tau)\Big), \\
&\qquad (\mathrm{t}-\tau)^{\alpha-1}\Big(L_{x_{n-1}}(\tau) \oplus A_{n-1}(\tau)\Big)\Big]d\tau \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} \big|\mathrm{t}-\tau\big|^{\alpha-1} D\Big[\Big(L_{x_n}(\tau) \oplus A_n(\tau)\Big), \\
&\qquad \Big(L_{x_{n-1}}(\tau) \oplus A_{n-1}(\tau)\Big)\Big]d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \sup_{0\leq \mathrm{t}\leq 1} D\Big[\Big(L_{x_n}(\mathrm{t}) \oplus A_n(\mathrm{t})\Big), \\
&\qquad \Big(L_{x_{n-1}}(\mathrm{t}) \oplus A_{n-1}(\mathrm{t})\Big)\Big]\int_0^{\mathrm{t}} \big|\mathrm{t}-\tau\big|^{\alpha-1}d\tau \\
&\leq \frac{\mathcal{N}}{\Gamma(\alpha)} \sup_{0\leq \mathrm{t}\leq 1} D\Big(x_n(\mathrm{t}), x_{n-1}(\mathrm{t})\Big) \qquad (10)
\end{aligned}$$

Thus we have

$$\sup_{0\leq \mathrm{t}\leq 1} D\Big(x_{n+1}(\mathrm{t}), x_n(\mathrm{t})\Big) \leq \frac{\mathcal{N}}{\Gamma(\alpha)} \sup_{0\leq \mathrm{t}\leq 1} D\Big(x_n(\mathrm{t}), x_{n-1}(\mathrm{t})\Big) \qquad (11)$$

In this manner we get

$$\begin{aligned}
\sup_{0\leq \mathrm{t}\leq 1} D\left(x_{n+1}(\mathrm{t}),\ x_n(\mathrm{t})\right) &\leq \frac{\mathcal{N}}{\Gamma(\alpha)} \sup_{0\leq \mathrm{t}\leq 1} D\left(x_n(\mathrm{t}),\ x_{n-1}(\mathrm{t})\right) \\
&\leq \frac{\mathcal{N}^2}{\Gamma(\alpha)^2} \sup_{0\leq \mathrm{t}\leq 1} D\left(x_{n-1}(\mathrm{t}),\ x_{n-2}(\mathrm{t})\right) \\
&\leq \ldots \leq \frac{\mathcal{N}^{n-1}}{\Gamma(\alpha)^{n-1}} \sup_{0\leq \mathrm{t}\leq 1} D\left(x_2(\mathrm{t}),\ x_1(\mathrm{t})\right) \qquad (12)
\end{aligned}$$

For $n = 1$, we have

$$\begin{aligned}
D(x_2(\mathrm{t}), x_1(\mathrm{t})) &= D\Big[\mathcal{J}_{0^+}^{\alpha}\Big(L_{x_1}(\mathrm{t}) \oplus A_1(\mathrm{t})\Big), \\
&\qquad \mathcal{J}_{0^+}^{\alpha}\Big(L_{x_0}(\mathrm{t}) \oplus A_0(\mathrm{t})\Big)\Big] \\
&= D\Big[\frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} (\mathrm{t}-\tau)^{\alpha-1}\Big(L_{x_1}(\tau) \oplus A_1(\tau)\Big)d\tau, \\
&\qquad \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} (\mathrm{t}-\tau)^{\alpha-1}\Big(L_{x_0}(\tau) \oplus A_0(\tau)\Big)d\tau\Big] \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} D\Big[(\mathrm{t}-\tau)^{\alpha-1}\Big(L_{x_1}(\tau) \oplus A_1(\tau)\Big), \\
&\qquad (\mathrm{t}-\tau)^{\alpha-1}\Big(L_{x_0}(\tau) \oplus A_0(\tau)\Big)\Big]d\tau \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} \big|\mathrm{t}-\tau\big|^{\alpha-1}\Big(D\Big[L_{x_1}(\tau), \\
&\qquad L_{x_0}(\tau)\Big] \oplus D\Big[A_1(\tau), A_0(\tau)\Big]\Big)d\tau
\end{aligned}$$

Then

$$\sup_{0\leq \mathrm{t}\leq 1} D\Big(x_2(\mathrm{t}), x_1(\mathrm{t})\Big) \leq \frac{(\mathcal{P}+\mathcal{M})\mathcal{N}}{\Gamma(\alpha)} \qquad (13)$$

Substitute (13) to (12), we have

$$\begin{aligned}
\sup_{0\leq \mathrm{t}\leq 1} D\Big(x_{n+1}(\mathrm{t}), x_n(\mathrm{t})\Big) &\leq \frac{\mathcal{N}^{n-1}}{\Gamma(\alpha)^{n-1}}\frac{(\mathcal{P}+\mathcal{M})\mathcal{N}}{\Gamma(\alpha)} \\
&\leq \frac{\mathcal{N}^n}{\Gamma(\alpha)^n}(\mathcal{P}+\mathcal{M})
\end{aligned}$$

Since $\mathcal{P} \leq \mathcal{P}^n$ and $\mathcal{M} \leq \mathcal{M}^n$, we can write

$$\begin{aligned}
\lim_{n\to\infty}\sup_{0\leq \mathrm{t}\leq 1} D\Big(x_{n+1}(\mathrm{t}),\ x_n(\mathrm{t})\Big) &\leq \lim_{n\to\infty} \frac{\mathcal{N}^n}{\Gamma(\alpha)^n}\left(\mathcal{P}^n + \mathcal{M}^n\right) \\
&= \lim_{n\to\infty} \frac{(\mathcal{N}\mathcal{P})^n + (\mathcal{N}\mathcal{M})^n}{\Gamma(\alpha)^n} \\
&= 0
\end{aligned}$$

As a result, we have the series $\{x_n(\mathrm{t})\}$ converges.

5 Numerical Examples

5.1 Example

Consider the following fuzzy Hilfer fractional differential equation

$$\mathfrak{D}_{0^+}^{\alpha,\beta} x(\mathrm{t}) = \lambda \odot x(\mathrm{t}), 0 < \alpha < 1, 0 \leq \beta \leq 1 \qquad (14)$$

initial condition

$$\mathcal{J}_{0^+}^{1-\gamma} x(0) = \big[0.5 + 0.5r, 1.5 - 0.5r\big] \qquad (15)$$

where $[x(\mathrm{t})]^r = \big[\underline{x_r}(\mathrm{t}), \overline{x_r}(\mathrm{t})\big]$ and $\odot$ represents fuzzy multiplication,

Case 1. Assume $\lambda = 1$ then using (1) - differentiable, we have

$$\begin{cases}\mathfrak{D}_{0^+}^{\alpha,\beta}\underline{x_r}(\mathrm{t}) = \underline{x_r}(\mathrm{t})\\ \mathfrak{D}_{0^+}^{\alpha,\beta}\overline{x_r}(\mathrm{t}) = \overline{x_r}(\mathrm{t})\\ \underline{x_r}(0) = 0.5 + 0.5r\\ \overline{x_r}(0) = 1.5 - 0.5r\end{cases} \tag{16}$$

Operating $\mathcal{J}_{0^+}^{\alpha}$ on both sides of Eqn.(16), we have

$$\begin{cases}\underline{x_r}(\mathrm{t}) = (0.5+0.5r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + J_{0^+}^{\alpha}\underline{x_r}(\mathrm{t})\\ \overline{x_r}(\mathrm{t}) = (1.5-0.5r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + J_{0^+}^{\alpha}\overline{x_r}(\mathrm{t})\end{cases} \tag{17}$$

In the view of the ADM, the solutions $\underline{x_r}(\mathrm{t})$ and $\overline{x_r}(\mathrm{t})$ are decomposed into infinite series as follows

$$\underline{x_r}(\mathrm{t}) = \sum_{n=0}^{\infty}\underline{x_{nr}}(\mathrm{t}) \; and \; \overline{x_r}(\mathrm{t}) = \sum_{n=0}^{\infty}\overline{x_{nr}}(\mathrm{t}) \tag{18}$$

Substituting the decomposition series (18) into both sides of (17) yields

$$\begin{cases}\sum_{n=0}^{\infty}\underline{x_{nr}}(\mathrm{t}) = (0.5+0.5r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + \mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}\underline{x_{nr}}(\mathrm{t})\\ \sum_{n=0}^{\infty}\overline{x_{nr}}(\mathrm{t}) = (1.5-0.5r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + \mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}\overline{x_{nr}}(\mathrm{t})\end{cases}$$

Hence the solutions $\underline{x_r}(\mathrm{t})$ and $\overline{x_r}(\mathrm{t})$ are obtained as follows

$$\begin{aligned}\underline{x_r}(\mathrm{t}) &= (0.5+0.5r)\mathrm{t}^{\gamma-1}\sum_{n=0}^{\infty}\frac{(\mathrm{t}^{\alpha})^n}{\Gamma(n\alpha+\gamma)}\\ &= (0.5+0.5r)\mathrm{t}^{\gamma-1}E_{\alpha,\gamma}(\mathrm{t}^{\alpha})\\ \overline{x_r}(\mathrm{t}) &= (1.5-0.5r)\mathrm{t}^{\gamma-1}\sum_{n=0}^{\infty}\frac{(\mathrm{t}^{\alpha})^n}{\Gamma(n\alpha+\gamma)}\\ &= (1.5-0.5r)\mathrm{t}^{\gamma-1}E_{\alpha,\gamma}(\mathrm{t}^{\alpha})\end{aligned}$$

If $\beta = 0$, then

$$\begin{cases}\underline{x_r}(\mathrm{t}) = (0.5+0.5r)\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(\mathrm{t}^{\alpha})\\ \overline{x_r}(\mathrm{t}) = (1.5-0.5r)\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(\mathrm{t}^{\alpha})\end{cases} \tag{19}$$

is the solution of following FFDE involving the Riemann-Liouville derivative.

$$\begin{cases}{}^{RL}\mathfrak{D}_{0^+}^{\alpha}x(\mathrm{t}) = \lambda\odot x(\mathrm{t}), 0<\alpha<1\\ \mathcal{J}_{0^+}^{1-\alpha}x(0) = \left[0.5+0.5r, 1.5-0.5r\right]\end{cases} \tag{20}$$

Salahshour et al.[31] considered the following FFDE with Riemann-Liouville derivative.

$$\begin{cases}{}^{RL}\mathfrak{D}_{0^+}^{\alpha}x(\mathrm{t}) = \lambda\odot x(\mathrm{t}), 0<\alpha<1\\ \mathcal{J}_{0^+}^{1-\alpha}x(0) = x_0\end{cases} \tag{21}$$

The exact solution of Eqn.(21) obtained by Laplace transform in [31] is as follows.

$$\begin{cases}\underline{x_r}(\mathrm{t}) = \underline{x_0}\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(\lambda\mathrm{t}^{\alpha})\\ \overline{x_r}(\mathrm{t}) = \overline{x_0}\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(\lambda\mathrm{t}^{\alpha})\end{cases} \tag{22}$$

By comparing equations (20) and (21), we have $x_0 = \left[0.5+0.5r, 1.5-0.5r\right]$. It shows that the result (19) is matched with the result (22) of the Riemann-Liouville FFDE in [31], where $\lambda = 1$.

If $\beta = 1$, then

$$\begin{cases}\underline{x_r}(\mathrm{t}) = (0.5+0.5r)E_{\alpha,1}(\mathrm{t}^{\alpha})\\ \overline{x_r}(\mathrm{t}) = (1.5-0.5r)E_{\alpha,1}(\mathrm{t}^{\alpha})\end{cases} \tag{23}$$

The result (23) coincides with the exact solution of the following Caputo FFDE in [38].

$$\begin{cases}{}^{C}\mathfrak{D}_{0^+}^{\alpha}x(\mathrm{t}) = \lambda\odot x(\mathrm{t}), 0<\alpha<1\\ x(0) = \left[0.5+0.5r, 1.5-0.5r\right]\end{cases}$$

Case 2. Assume $\lambda = -1$ then using (2) - differentiable, we can be written Eqn. (14) as follows

$$\begin{cases}\mathfrak{D}_{0^+}^{\alpha,\beta}\underline{x_r}(\mathrm{t}) = -\underline{x_r}(\mathrm{t})\\ \mathfrak{D}_{0^+}^{\alpha,\beta}\overline{x_r}(\mathrm{t}) = -\overline{x_r}(\mathrm{t})\\ \underline{x_r}(0) = 0.5 + 0.5r\\ \overline{x_r}(0) = 1.5 - 0.5r\end{cases} \tag{24}$$

Operating $\mathcal{J}_{0^+}^{\alpha}$ on both sides of Eqn.(24), we have

$$\begin{cases}\underline{x_r}(\mathrm{t}) = (0.5+0.5r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} - \mathcal{J}_{0^+}^{\alpha}\underline{x_r}(\mathrm{t})\\ \overline{x_r}(\mathrm{t}) = (1.5-0.5r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} - \mathcal{J}_{0^+}^{\alpha}\overline{x_r}(\mathrm{t})\end{cases} \tag{25}$$

In the view of the ADM, the solutions $\underline{x_r}(\mathrm{t})$ and $\overline{x_r}(\mathrm{t})$ are decomposed into infinite series as follows

$$\underline{x_r}(\mathrm{t}) = \sum_{n=0}^{\infty}\underline{x_{nr}}(\mathrm{t}) \;\; and \;\; \overline{x_r}(\mathrm{t}) = \sum_{n=0}^{\infty}\overline{x_{nr}}(\mathrm{t}) \tag{26}$$

Substituting the decomposition series (26) into both sides of (25) yields

$$\begin{cases}\sum_{n=0}^{\infty}\underline{x_{nr}}(\mathrm{t}) = (0.5+0.5r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} - \mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}\underline{x_{nr}}(\mathrm{t})\\ \sum_{n=0}^{\infty}\overline{x_{nr}}(\mathrm{t}) = (1.5-0.5r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} - \mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}\overline{x_{nr}}(\mathrm{t})\end{cases}$$

Hence the solutions $\underline{x_r}(\mathrm{t})$ and $\overline{x_r}(\mathrm{t})$ are obtained as follows

$$\begin{aligned}\underline{x_r}(\mathrm{t}) &= (0.5+0.5r)\mathrm{t}^{\gamma-1}\sum_{n=0}^{\infty}\frac{(-\mathrm{t}^{\alpha})^n}{\Gamma(n\alpha+\gamma)}\\ &= (0.5+0.5r)\mathrm{t}^{\gamma-1}E_{\alpha,\gamma}(-\mathrm{t}^{\alpha})\\ \overline{x_r}(\mathrm{t}) &= (1.5-0.5r)\mathrm{t}^{\gamma-1}\sum_{n=0}^{\infty}\frac{(-\mathrm{t}^{\alpha})^n}{\Gamma(n\alpha+\gamma)}\\ &= (1.5-0.5r)\mathrm{t}^{\gamma-1}E_{\alpha,\gamma}(-\mathrm{t}^{\alpha})\end{aligned}$$

If $\beta = 0$, then

$$\begin{cases}\underline{x_r}(\mathrm{t}) = (0.5+0.5r)\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(-\mathrm{t}^{\alpha})\\ \overline{x_r}(\mathrm{t}) = (1.5-0.5r)\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(-\mathrm{t}^{\alpha})\end{cases} \tag{27}$$

is the solution of following FFDE involving the Riemann-Liouville derivative.

$$\begin{cases}{}^{RL}\mathfrak{D}_{0^+}^{\alpha}x(\mathrm{t}) = \lambda\odot x(\mathrm{t}), 0<\alpha<1\\ \mathcal{J}_{0^+}^{1-\alpha}x(0) = \left[0.5+0.5r, 1.5-0.5r\right]\end{cases} \tag{28}$$

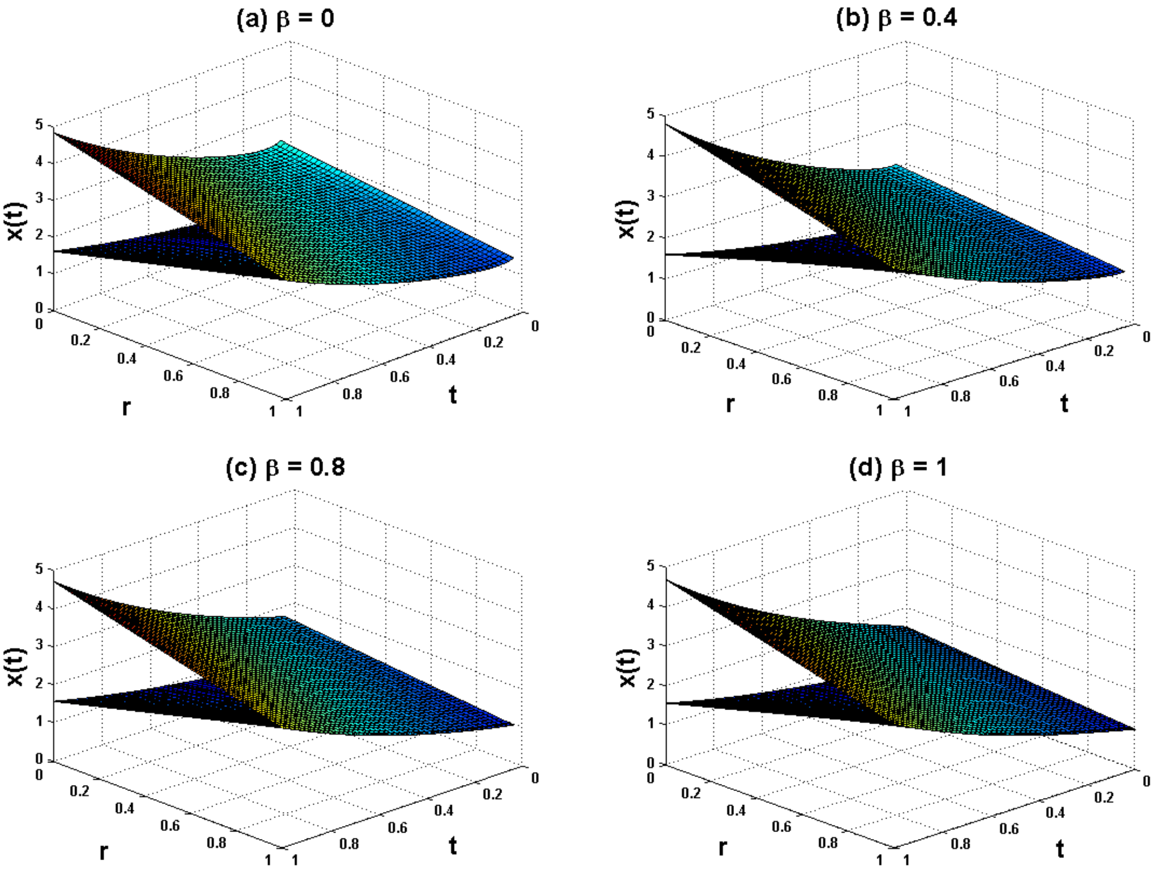

Figure 1. Approximation solutions for Example 5.1 with $\alpha = 0.85$ and $\beta = 0, 0.4, 0.8, 1$ for case 1

Table 1. Numerical results for Example 5.1 with $\alpha = 0.85$ for case 1

r	$\beta = 0$		$\beta = 0.4$		$\beta = 0.8$		$\beta = 1$	
	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$
0	1.6144	4.8431	1.5956	4.7869	1.5743	4.7230	1.5627	4.6882
0.1	1.7758	4.6817	1.7552	4.6273	1.7318	4.5655	1.7190	4.5320
0.2	1.9373	4.5203	1.9147	4.4677	1.8892	4.4081	1.8753	4.3757
0.3	2.0987	4.3588	2.0743	4.3082	2.0466	4.2507	2.0316	4.2194
0.4	2.2601	4.1974	2.2339	4.1486	2.2040	4.0932	2.1878	4.0631
0.5	2.4216	4.0360	2.3934	3.9891	2.3615	3.9358	2.3441	3.9069
0.6	2.5830	3.8745	2.5530	3.8295	2.5189	3.7784	2.5004	3.7506
0.7	2.7445	3.7131	2.7126	3.6699	2.6763	3.6209	2.6567	3.5943
0.8	2.9059	3.5516	2.8721	3.5104	2.8338	3.4635	2.8129	3.4380
0.9	3.0673	3.3902	3.0317	3.3508	2.9912	3.3061	2.9692	3.2818
1	3.2288	3.2288	3.1912	3.1912	3.1486	3.1486	3.1255	3.1255

Salahshour et al. [31] considered the following FFDE with Riemann-Liouville derivative.

$$\begin{cases} {}^{RL}\mathfrak{D}_{0^+}^{\alpha} x(\mathrm{t}) = \lambda \odot x(\mathrm{t}), 0 < \alpha < 1 \\ \mathcal{J}_{0^+}^{1-\alpha} x(0) = x_0 \end{cases} \tag{29}$$

The exact solution of Eqn.(29) obtained by Laplace transform in [31] is as follows.

$$\begin{cases} \underline{x_r}(\mathrm{t}) = \underline{x_0}\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(\lambda \mathrm{t}^{\alpha}) \\ \overline{x_r}(\mathrm{t}) = \overline{x_0}\mathrm{t}^{\alpha-1}E_{\alpha,\alpha}(\lambda \mathrm{t}^{\alpha}) \end{cases} \tag{30}$$

By comparing equations (28) and (29), we have $x_0 = \left[0.5 + 0.5r, 1.5 - 0.5r\right]$. It shows that the result (27) is matched with the result (30) of the Riemann-Liouville FFDE in [31], where $\lambda = -1$.

If $\beta = 1$, then

$$\begin{cases} \underline{x_r}(\mathrm{t}) = (0.5 + 0.5r)E_{\alpha,1}(-\mathrm{t}^{\alpha}) \\ \overline{x_r}(\mathrm{t}) = (1.5 - 0.5r)E_{\alpha,1}(-\mathrm{t}^{\alpha}) \end{cases} \tag{31}$$

The result (31) coincides with the exact solution of the following Caputo FFDE in [38].

$$\begin{cases} {}^{C}\mathfrak{D}_{0^+}^{\alpha} x(\mathrm{t}) = \lambda \odot x(\mathrm{t}), 0 < \alpha < 1 \\ x(0) = \left[0.5 + 0.5r, 1.5 - 0.5r\right] \end{cases}$$

Figures 1 and 2 show the approximation solutions of Example 5.1 at $\alpha = 0.85$ and $\alpha = 0.75$ for different values of β for case 1 and case 2 respectively. Approximation solutions of $\underline{x_r}$ and $\overline{x_r}$ for various values of r and β at $\mathrm{t} = 1$ are shown numerically in Tables 1 and 2. It can be seen that the solution of Example 5.1 coincides with the exact solution of FFDE with Riemann-Liouville derivative in [31] when $\beta = 0$ and also it coincides with the exact solution of FFDE with Caputo derivative in [38] when $\beta = 1$.

5.2 Example

Consider the following fuzzy Hilfer fractional differential equation

$$\mathfrak{D}_{0^+}^{\alpha,\beta} x = 0.5x(1 - x), \ \ 0 < \alpha < 1, \ \ 0 \le \beta \le 1 \tag{32}$$

with initial condition

$$\mathcal{J}_{0^+}^{1-\gamma} x(0) = [0.4 + 0.2r, 0.9 - 0.3r] \tag{33}$$

where $[x(\mathrm{t})]^r = \left[\underline{x_r}(\mathrm{t}), \overline{x_r}(\mathrm{t})\right]$

Operating $\mathcal{J}_{0^+}^{\alpha}$ on both sides of Eqn.(32), we have

$$\begin{cases} \underline{x_r}(\mathrm{t}) = (0.4 + 0.2r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + 0.5\mathcal{J}_{0^+}^{\alpha}\underline{x_r}(\mathrm{t}) - 0.5\mathcal{J}_{0^+}^{\alpha}\underline{x_r^2}(\mathrm{t}) \\ \overline{x_r}(\mathrm{t}) = (0.9 - 0.3r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + 0.5\mathcal{J}_{0^+}^{\alpha}\overline{x_r}(\mathrm{t}) - 0.5\mathcal{J}_{0^+}^{\alpha}\overline{x_r^2}(\mathrm{t}) \end{cases} \tag{34}$$

Using the Adomian decomposition method, we have

$$\begin{cases} \sum_{n=0}^{\infty}\underline{x_{nr}}(\mathrm{t}) & = (0.4 + 0.2r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + 0.5\mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}\underline{x_{nr}}(\mathrm{t}) \\ & -0.5\mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}\underline{A_n} \\ \sum_{n=0}^{\infty}\overline{x_{nr}}(\mathrm{t}) & = (0.9 - 0.3r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + 0.5\mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}\overline{x_{nr}}(\mathrm{t}) \\ & -0.5\mathcal{J}_{0^+}^{\alpha}\sum_{n=0}^{\infty}\overline{A_n} \end{cases}$$

where

$$\underline{A_0} = \underline{x_0^2} \quad and \quad \overline{A_0} = \overline{x_0^2}$$

$$\underline{A_1} = 2\underline{x_0}\,\underline{x_1} \qquad \overline{A_1} = 2\overline{x_0}\,\overline{x_1}$$

$$\underline{A_2} = 2\underline{x_0}\,\underline{x_2} + \underline{x_1^2} \qquad \overline{A_2} = 2\overline{x_0}\,\overline{x_2} + \overline{x_1^2}$$

and so on.

Then $\underline{x_r}(\mathrm{t})$ and $\overline{x_r}(\mathrm{t})$ can be approximated as

$$\begin{cases} \underline{x_r}(\mathrm{t}) & = (0.4 + 0.2r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + 0.5(0.4 + 0.2r) \\ & \left[\frac{\mathrm{t}^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} - (0.4 + 0.2r)\frac{\Gamma(2\gamma-1)\mathrm{t}^{\alpha+2\gamma-2}}{\Gamma(\gamma)^2\Gamma(\alpha+2\gamma-1)}\right] + ... \\ \overline{x_r}(\mathrm{t}) & = (0.9 - 0.3r)\frac{\mathrm{t}^{\gamma-1}}{\Gamma(\gamma)} + 0.5(0.9 - 0.3r) \\ & \left[\frac{\mathrm{t}^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} - (0.9 - 0.3r)\frac{\Gamma(2\gamma-1)\mathrm{t}^{\alpha+2\gamma-2}}{\Gamma(\gamma)^2\Gamma(\alpha+2\gamma-1)}\right] + ... \end{cases} \tag{35}$$

If $\beta = 1$, then

$$\begin{cases} \underline{x_r}(\mathrm{t}) & = (0.4 + 0.2r) + 0.5(0.4 + 0.2r)(0.6 - 0.2r)\frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)} + \\ & 0.5^2(0.4 + 0.2r)(0.6 - 0.2r)(0.2 - 0.4r)\frac{\mathrm{t}^{2\alpha}}{\Gamma(2\alpha+1)} + ... \\ \overline{x_r}(\mathrm{t}) & = (0.9 - 0.3r) + 0.5(0.9 - 0.3r)(0.1 + 0.3r)\frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)} + \\ & 0.5^2(0.9 - 0.3r)(0.1 + 0.3r)(-0.8 + 0.6r)\frac{\mathrm{t}^{2\alpha}}{\Gamma(2\alpha+1)} + ... \end{cases} \tag{36}$$

The result (36) is the solution of the Caputo FFDE.

Figure 3 shows the approximate solutions of Example 5.2 for $\alpha = 0.5$ and $\alpha = 0.25$ at $\mathrm{t} = 1$ when $\beta = 1$. It can be seen that the numerical result of Example 5.2 is almost matched with the solution of Caputo FFDE in [32] when $\beta = 1$. Furthermore, Figure 4 depicts the approximate solutions of Example 5.2 for various values of β when $\alpha = 0.85$. For various values of r at $\mathrm{t} = 1$, approximation solutions of $\underline{x_r}$ and $\overline{x_r}$ are shown numerically in Table 3.

6 Conclusions

In this work, Hilfer fractional differential equations in the fuzzy sense are considered. In addition, a numerical technique based on ADM is suggested to solve fuzzy Hilfer fractional differential equations. The examples presented in this study demonstrate the possibility of obtaining numerical solutions for fuzzy Hilfer fractional differential equations in various scenarios: if β tends to 0, then the obtained results are matched with the solutions of Riemann-Liouville FFDE and if β tends to 1, the obtained results are also matched with the solutions of Caputo FFDE. As a result, instead of studying various problems with Riemann-Liouville and Caputo fractional derivatives, it is more convenient to investigate fuzzy fractional differential equations using this new Hilfer fractional derivative. Most FFDEs are difficult to develop analytical solutions for in general. Therefore, in this paper, ADM is used to solve this new fuzzy Hilfer fractional differential equation. The convergence analysis of the ADM is proved by the theorems. The two numerical examples presented in this study show that the

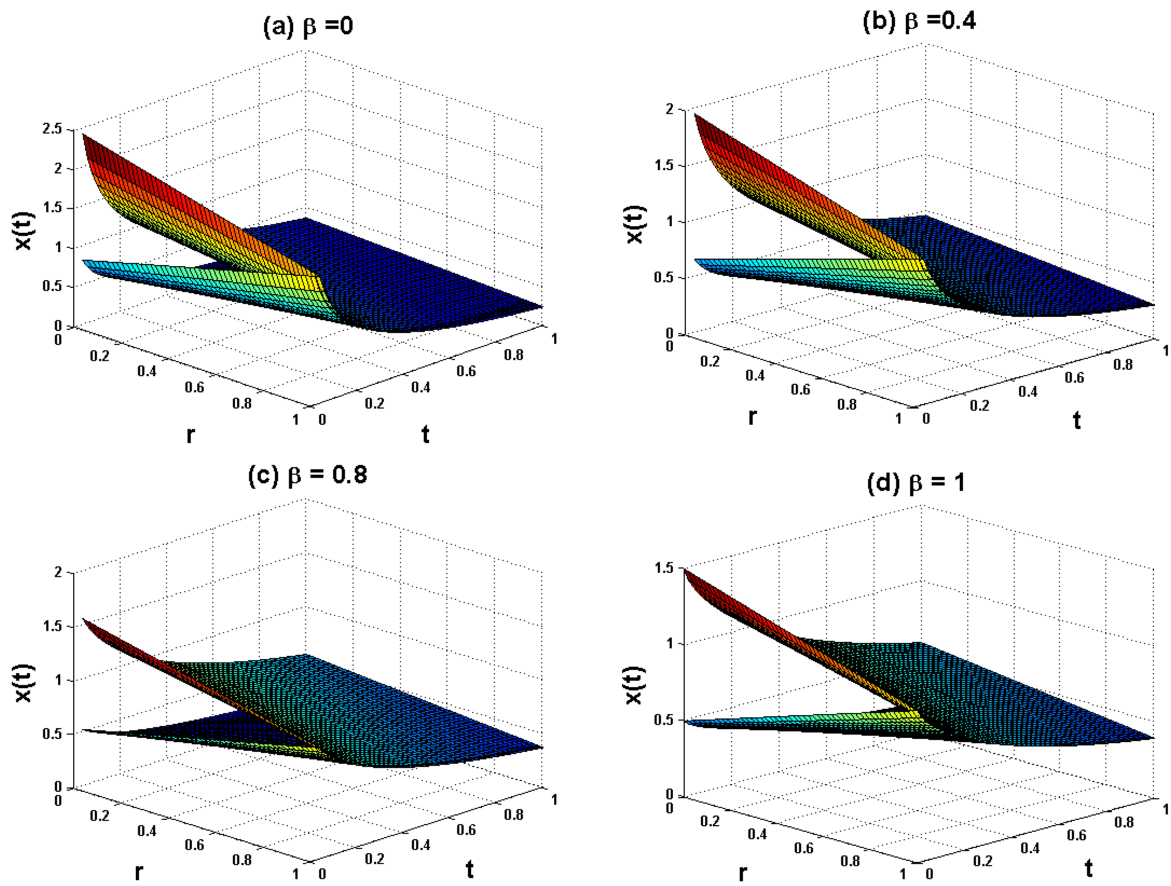

Figure 2. Approximation solutions for Example 5.1 with $\alpha = 0.75$ and $\beta = 0, 0.4, 0.8, 1$ for case 2

Table 2. Numerical results for Example 5.1 with $\alpha = 0.75$ for case 2

r	$\beta = 0$		$\beta = 0.4$		$\beta = 0.8$		$\beta = 1$	
	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$
0	0.1161	0.3484	0.1502	0.4507	0.1819	0.5456	0.1966	0.5897
0.1	0.1277	0.3367	0.1653	0.4357	0.2001	0.5274	0.2162	0.5700
0.2	0.1393	0.3251	0.1803	0.4207	0.2182	0.5092	0.2359	0.5504
0.3	0.1510	0.3135	0.1953	0.4056	0.2364	0.4910	0.2555	0.5307
0.4	0.1626	0.3019	0.2103	0.3906	0.2546	0.4729	0.2752	0.5110
0.5	0.1742	0.2903	0.2253	0.3756	0.2728	0.4547	0.2948	0.4914
0.6	0.1858	0.2787	0.2404	0.3606	0.2910	0.4365	0.3145	0.4717
0.7	0.1974	0.2671	0.2554	0.3455	0.3092	0.4183	0.3341	0.4521
0.8	0.2090	0.2555	0.2704	0.3305	0.3274	0.4001	0.3538	0.4324
0.9	0.2206	0.2438	0.2854	0.3155	0.3455	0.3819	0.3735	0.4128
1	0.2322	0.2322	0.3005	0.3005	0.3637	0.3637	0.3931	0.3931

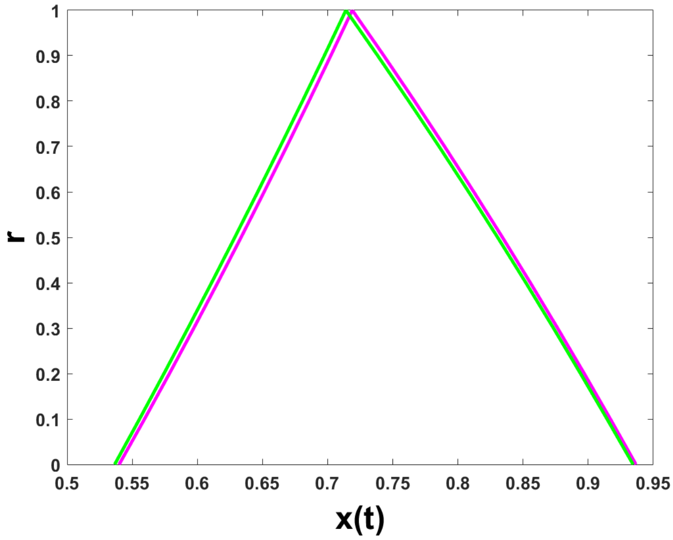

Figure 3. Approximate solution for $\alpha = 0.5$ (Pink line) and $\alpha = 0.25$ (green line) at $t = 1$ with $\beta = 1$ for Example 5.2

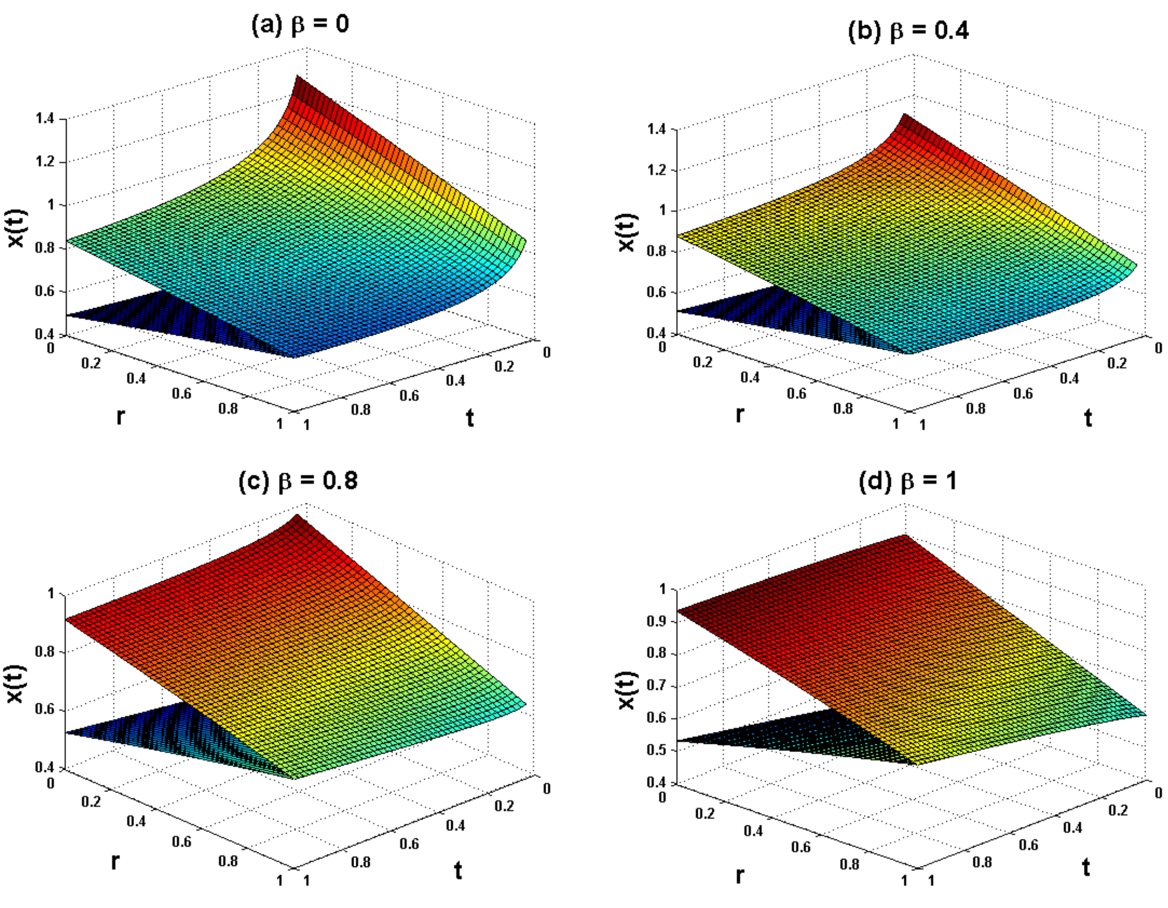

Figure 4. Approximate solution for Example 5.2 for $\alpha = 0.85$ with $\beta = 0, 0.4, 0.8, 1$

Table 3. Numerical results for Example 5.2 with $\alpha = 0.85$

r	$\beta = 0$		$\beta = 0.4$		$\beta = 0.8$		$\beta = 1$	
	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$	$\underline{x_r}$	$\overline{x_r}$
0	0.4906	0.8366	0.5107	0.8791	0.5274	0.9180	0.5347	0.9359
0.1	0.5083	0.8185	0.5296	0.8606	0.5474	0.8988	0.5551	0.9163
0.2	0.5255	0.8005	0.5480	0.8420	0.5669	0.8793	0.5751	0.8963
0.3	0.5422	0.7824	0.5659	0.8231	0.5859	0.8594	0.5946	0.8759
0.4	0.5585	0.7642	0.5834	0.8040	0.6044	0.8392	0.6136	0.8552
0.5	0.5744	0.7458	0.6004	0.7845	0.6225	0.8186	0.6322	0.8340
0.6	0.5898	0.7271	0.6171	0.7646	0.6402	0.7975	0.6504	0.8122
0.7	0.6049	0.7080	0.6333	0.7443	0.6575	0.7758	0.6681	0.7899
0.8	0.6196	0.6886	0.6492	0.7234	0.6744	0.7536	0.6855	0.7670
0.9	0.6340	0.6686	0.6646	0.7019	0.6909	0.7307	0.7025	0.7434
1	0.6480	0.6480	0.6798	0.6798	0.7071	0.7071	0.7191	0.7191

ADM for solving fuzzy Hilfer fractional differential equations is very effective, helpful, and simple to apply. Also, we have shown graphically that the approximate solution is indistinguishable from the exact solution. Our future work will be focused on investigating the solutions of fuzzy Hilfer partial fractional differential equations using ADM.

REFERENCES

[1] A. A. Kilbas, H. M. Srivastava, & J. J. Trujillo, "Theory and applications of fractional differential equations" Vol. 204. Elsevier (2006).

[2] V. Lakshmikantham, S. Leela, & J. V. Devi, "Theory of Fractional Dynamic Systems", Cambridge Scientific Publishers, 2009.

[3] K.S. Miller, & B. Ross, "An introduction to the fractional calculus and differential equations", John Wiley, New York, 1993.

[4] I. Podlubny, "Fractional Differential Equations", Academic Press, San Diego, 1999.

[5] R. Hilfer, "Application of Fractional Calculus in Physics", World Scientific, Singapore 35.12 (2000) 87-130.

[6] C. P. Chitalkar-Dhaigude, S. P. Bhairat, & D. B. Dhaigude, "Solution of fractional differential equations involving Hilfer fractional derivative: method of successive approximations", Bull. Marathwada Math. Soc, 18.2 (2017) 1-12.

[7] R. Kamocki, & C. Obczynski, "On fractional Cauchy-type problems containing Hilfer's derivative", Electronic Journal of Qualitative Theory of Differential Equations, 2016.50 (2016) 1-12.

[8] K. M. Furati, & M. D. Kassim, "Existence and uniqueness for a problem involving Hilfer fractional derivative", Computers & Mathematics with Applications,64.6 (2012) 1616-1626.

[9] K. M. Furati, M. D. Kassim, & N. E.Tatar, "Non-existence of global solutions for a differential equation involving Hilfer fractional derivative" Electron. J. Differ. Equ. 235 (2013) 1–10.

[10] H. Gu, & J. J. Trujillo, "Existence of mild solution for evolution equation with Hilfer fractional derivative", Applied Mathematics and Computation, 257 (2015) 344-354.

[11] R. Hilfer, Y. Luchko, & Z. Tomovski, "Operational method for the solution of fractional differential equations with generalized Riemann–Lioville fractional derivative", Fract. Calc. Appl. Anal. 12.3 (2009) 299–318.

[12] M. A. Almalahi, M. S. Abdo, & S. K. Panchal, "Positive solution of Hilfer fractional differential equations with integral boundary conditions", preprint, arXiv:1910.07887v1 [math.GM], (2019) 1-10.

[13] S. S. Haider, M. ur Rehman, & T. Abdeljawad, "On Hilfer fractional Difference operator", Advances in Difference Equations, 2020.1(2020)1-20.

[14] J. Wang, & Y. Zhang, "Nonlocal initial value problems for differential equations with Hilfer fractional derivative", Applied Mathematics and Computation, 266 (2015) 850-859.

[15] R. P. Agarwal, V. Lakshmikantham, & J. J. Nieto, "On the concept of solution for fractional differential equations with uncertainty", Nonlinear Analysis: Theory, Methods & Applications, 72.6 (2010) 2859-2862.

[16] T. Allahviranloo, S. Salahshour, & S. Abbasbandy, "Explicit solutions of fractional differential equations with uncertainty", Soft Computing 16.2 (2012) 297–302.

[17] M. Mazandarani, & A. V. Kamyad, "Modified fractional Euler method for solving fuzzy fractional initial value problem" Communications in Nonlinear Science and Numerical Simulation, 18.1(2013)12-21.

[18] V. Lupulescu, "Fractional calculus for interval-valued functions", Fuzzy Sets and Systems 265 (2015) 63–85.

[19] T. Allahviranloo, A. Armand, & Z. Gouyandeh, "Fuzzy fractional differential equations under generalized fuzzy Caputo derivative", Journal of Intelligent & Fuzzy Systems, 26.3 (2014) 1481-1490.

[20] O. S. Fard, & M. Salehi, "A survey on fuzzy fractional variational problems", Journal of Computational and Applied Mathematics, 271 (2014) 71-82.

[21] R. Dai, & M. Chen, “The structure stability of periodic solutions for first-order uncertain dynamical systems”, Fuzzy Sets and Systems, 400 (2020) 134-146.

[22] N.V. Hoa, H. Vu, & T.M. Duc, “Fuzzy fractional differential equations under Caputo–Katugampola fractional derivative approach”, Fuzzy Sets and Systems, 375 (2019) 70–99.

[23] N.V. Hoa, & Ho Vu, “A survey on the initial value problems of fuzzy implicit fractional differential equations”, Fuzzy Sets and Systems, 400 (2020) 90-133.

[24] A. Ahmadian, S. Salahshour, & C.S. Chan, “Fractional differential systems: a fuzzy solution based on operational matrix of shifted Chebyshev polynomials and its applications”, IEEE Transactions on Fuzzy Systems, 25.1 (2016) 218–236.

[25] A. Ahmadian, F. Ismail, S. Salahshour, D. Baleanu, & F. Ghaemi, “Uncertain viscoelastic models with fractional order: a new spectral tau method to study the numerical simulations of the solution”, Communications in Nonlinear Science and Numerical Simulation 53 (2017) 44-64.

[26] C. Vinothkumar, J.J. Nieto, A. Deiveegan, & P. Prakash, ”Invariant solutions of hyperbolic fuzzy fractional differential equations”, Modern Physics Letters B 34.1 (2020) 2050015.

[27] T. Allahviranloo, & B. Ghanbari, “On the fuzzy fractional differential equation with interval Atangana–Baleanu fractional derivative approach”, Chaos, Solitons Fractals 130 (2020) 109397.

[28] Z. Yang, & W. Ren, “Existence and stability results for quaternion fuzzy fractional differential equations in the sense of Hilfer”, Journal of Intelligent & Fuzzy Systems, 34.1 (2018) 167-175.

[29] E. Khodadadi, E. Celik, “The variational iteration method for fuzzy fractional differential equations with uncertainty”, Fixed Point Theory and Applications 2013.1 (2013) 1-7.

[30] A. Ahmadian, M. Suleiman, S. Salahshour, & D. Baleanu, “A Jacobi operational matrix for solving a fuzzy linear fractional differential equation”, Advances in Difference Equations 2013.1 (2013) 1-29.

[31] S. Salahshour, T. Allahviranloo, & S. Abbasbandy, “Solving fuzzy fractional differential equations by fuzzy Laplace transforms”, Communications in Nonlinear Science and Numerical Simulation, 17.3 (2012)1372-1381.

[32] A. Rivaz, O.S. Fard, & T.A. Bidgoli, “Solving fuzzy fractional differential equations by a generalized differential transform method”, SeMA Journal, 73.2 (2016) 149-170.

[33] S. Narayanamoorthy, & K. Thangapandi, “Modified Fractional Runge–Kutta Method for Solving Fuzzy Differential Equations of Fractional Order”, National Academy Science Letters, (2020) 1-5.

[34] N. Azhar, & S. Iqbal, ”Solution of fuzzy fractional order differential equations by fractional Mellin transform method”, Journal of Computational and Applied Mathematics, 400 (2022) 113727.

[35] O. Kaleva, “Fuzzy differential equations”, Fuzzy Sets and Systems, 24.3 (1987) 301–317.

[36] B. Bede, & L. Stefanini, “Generalized differentiability of fuzzy-valued Functions”, Fuzzy Sets and System, 230 (2013) 119–141.

[37] J. Duan, J. An, & M. Xu, “Solution of system of fractional differential equations by Adomian decomposition method”, Applied Mathematics-A Journal of Chinese Universities, 22.1(2007) 7-12.

[38] A. Ahmadian, M. Suleiman, & S. Salahshour, “An operational matrix based on legendre polynomials for solving fuzzy fractional-order differential equations”, In: Abstract and Applied Analysis, vol. 2013, Hindawi; 2013.

[39] E. A. A. Ziada, “Solution of Nonlinear Fractional Differential Equations Using Adomain Decomposition Method”, Journal of Systems Science and Applied Mathematics.Vol, 6.4 (2021) 111-119.

[40] B. R. Sontakke, & R. Pandit, “Convergence analysis and approximate solution of fractional differential equations”, Malaya Journal of Matematik (MJM), 7.2 (2019) 338-344.

[41] S. Askari, T. Allahviranloo & S. Abbasbandy, “Solving fuzzy fractional differential equations by adomian decomposition method used in optimal control theory”, International Transaction Journal of Engineering, Management,& Applied Science & Technologies 10.12 (2019) 1–10.

[42] M.D. Johansyah,A.K. Supriatna, E. Rusyaman, & J. Saputra, ”Solving Differential Equations of Fractional Order Using Combined Adomian Decomposition Method with Kamal Integral Transformation”. Mathematics and Statistics, 10.1 (2022) 187-194. DOI: 10.13189/ms.2022.100117.

[43] R. I.Nuruddeen, ”Laplace-based method for the linearized dynamical models in the presence of Hilfer fractional operator”, Partial Differential Equations in Applied Mathematics, 5 (2022) 100248.