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Σ-uniserial Modules and Their Properties

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Abstract The close association of abelian group theory and the theory of modules have been extensively studied in the literatures. In fact, the theory of abelian groups is one of the principal motives of new research in module theory. As it is well-known, module theory can only be processed by generalizing the theory of abelian groups that provide novel viewpoints of various structures for torsion abelian groups. The theory of torsion abelian groups is significant as it generates the natural problems in $QTAG$-module theory. The notion of $QTAG$ (torsion abelian group like) module is one of the most important tools in module theory. Its importance lies behind the fact that this module can be applied in order to generalized torsion abelian group accurately. Significant work on $QTAG$-module was produced by many authors, concentrating on establishing when torsion abelian groups are actually $QTAG$-modules. There are two rather natural problems which arise in connection with the Σ-uniserial modules. Namely: The $QTAG$-module M is Σ-uniserial if and only if all N-high submodules of M are Σ-uniserial, for some basic submodules N of M, and M is not a Σ-uniserial module if and only if it contains a proper $(\omega + 1)$-projective submodule. The current work explores these two problems for $QTAG$-modules. Some related concepts and problems are also considered. Our global aim here is to review the relationship between the aspects of group theory in the form of torsion abelian groups and theory of modules in the form of $QTAG$-modules.

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1 Introduction and Fundamentals

The field of algebra which goes under the name of module theory is concerned with all problems directly or indirectly related to abelian groups. The concept of TAG (torsion abelian group like) module was introduced and well studied by Singh [1]. The notion of TAG-modules in classical module theory is, of course, an important generalization of torsion abelian groups. This notion has many more pleasing properties which have been the focus of attention of many authors since 1976 (see, for instance, [2, 3]).

Over an arbitrary (associative, unitary) ring R, a module M is called a TAG-module if it satisfies the following two conditions relating to uniserial modules.

"(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any nonzero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided the composition length $d(U_1/N) \leq d(U_2/\phi(N))$ holds."

In 1987 Singh [4] showed that the second condition, with minimal additional hypotheses, can be deduced from the first and named modules satisfying at least the first condition $QTAG$-modules. Since then, many notions of this concept have been defined and some of their interesting properties and characterizations are investigated via these types of notions (see, for instance, [5, 6] and the references cited therein). It is worthwhile noticing that many of the developments in this direction are analogous to the earlier development of torsion abelian groups. About the exploration of uniserial modules, and their closely related problems, we refer the interested reader to [7, 8, 9, 10].

In this study, we present a systematic and successful development of torsion abelain groups for $QTAG$-modules and more specially, the problems about Σ-uniserial modules between the basic submodules of such $QTAG$-modules and the structure of the $QTAG$-modules such as N-high submodules, Σ-modules and $(\omega+1)$-projective modules. For this purpose, we organize our work as follows: in Section 1, i.e. here, we have presented the fundamental as well as the preliminary necessary for applicable purposes. In Section 2, we foremost add two crucial technical claims which are needed to obtain a simplified but more convenient for us, major necessary and sufficient condition when a $QTAG$-module can be a Σ-uniserial. After that, in Section 3, we relate the Σ-uniserial module to some other interesting classes of $QTAG$-modules. In Section 4, we quote some open problems that remain unanswered.

We begin with some terminology: "Throughout the text, we assume that all rings into consideration are associative with unity $(1 \neq 0)$ and modules are unital $QTAG$-modules. By the term 'uniserial module', we will mean a module M over a ring R, whose submodules are totally ordered by inclusion, i.e., for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. Likewise, we shall say M is uniform if intersection of any two of its non-zero submodules is non-zero. An element $u \in M$ is called uniform if uR is a non-zero uniform (hence uniserial) module. Concerning decomposition series, we suppose that all decomposition series are unique. For any module M, the symbol $d(M)$ will denote its decomposition length. In addition, if u is an uniform element of M (i.e., $u \in M$), then $e(u)$ is called the exponent of u, and $e(u) = d(uR)$. As usual, for such a module M, we set the height of u in M as $H_M(u) = \sup\{d(vR/uR) : v \in M,\ u \in vR \text{ and } v \text{ uniform}\}$."

Next, we review the following concepts: "For every non-negative integer t, $H_t(M) = \{u \in M \mid H_M(u) \geq t\}$ denotes the t-th copies of M which can be viewed as a submodule of M consisting of all elements of height at least t. The topology of M which admits as a base neighbourhoods of zero is known as h-topology. This topology has the submodules $H_t(M)$ with $t = 0, 1, \ldots, \infty$. In this way, a submodule N of M is called the closure in M if $\overline{N} = \cap_{t=0}^{\infty}(N + H_t(M))$ and N is closed with respect to this topology provided that $\overline{N} = N$. For a module M, the letter M^1 will always denote in the sequel the submodule of M, containing elements of infinite height. Moreover, we denote by $Soc(M)$, the socle of M, i.e., the sum of all simple submodules of M. For any $t \geq 0$, $Soc^t(M)$ is defined inductively as follows: $Soc^0(M) = 0$ and $Soc^{t+1}(M)/Soc^t(M) = Soc(M/Soc^t(M))$."

We add some fundamental definitions as well from [6, 11]. "The module M is named h-divisible if $M = M^1 = \cap_{t=0}^{\infty} H_t(M)$, or equivalently, if $H_1(M) = M$. With this in hand, we say that the module M is h-reduced if it does not contain any h-divisible submodule. The module M is termed separable if $M^1 = 0$. Moreover, M is defined to be bounded if $\exists\, t \geq 0$ such that $H_M(u) \leq t$ for some $u \in M$. A submodule N of a module M is said to be an h-pure in M if for every non-negative integer t the equality $N \cap H_t(M) = H_t(N)$ holds. In particular, if $t = 1$, N is called h-neat in M. A submodule $N \subseteq M$ is a basic submodule of M, if N is h-pure in M, $N = \oplus N_i$, where each N_i is the direct sum of uniserial modules of length i and M/N is h-divisible. The cardinality of the minimal generating set of M is denoted by the symbol $g(M)$ that plays a significant role in our further investigation. By analogy, for all ordinals σ, one can define $f_M(\sigma)$, the σ^{th}-Ulm invariant of M as follows: $f_M(\sigma) = g\big(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M))\big)$."

We also make use of concepts related to high submodules that can be found, for instance, in [12] and [13], respectively, and that we briefly review: "We say that a submodule N of M is high, if it is maximal with respect to $\cap M^1 = 0$. Likewise, a submodule N_1 of M is N_2-high, if $N_1 \cap N_2 = 0$ and N_1 is maximal with respect to this intersection."

It is well to note that various results for TAG-modules are also valid for $QTAG$-modules [14]. Our present work is motivated by many significant results from the papers [15, 16, 17]. We follow notations and terminology which can be found in the classical books [18] and [19].

2 The Main Result

One of the most significance concepts in the theory of modules is to study when a $QTAG$-module M is isomorphic to a direct sum of uniserial; known as Σ-uniserial modules [5]. Notice that Σ-uniserial modules are separable (i.e., ω-bounded). It is apparent to see that these modules are necessarily ω-bounded, that is they have zero first Ulm submodule.

This section deals with the problem concerning properties of Σ-uniserial modules. A natural problem arises, that of characterizing the Σ-uniserial modules. It is evident that this problem follows from others (see, for instance, [20, 6]) but as it often happens the proof precedes in the different perspective. Here we wish to consider the situation where the arbitrary submodules are replaced by basic submodules and thus to explore a characterization of Σ-uniserial modules, which is of some importance.

We begin with the following preliminary technical claim.

Lemma 2.1. *Suppose M is an h-reduced $QTAG$-module with a basic submodule N of M such that $\aleph_0 \leq g(M/N) < g(N)$. If T is any submodule of N, then $M = S \oplus T$, and the following holds:*

$$g(S \cap N) \leq g(S/S \cap N),$$

where $S \cap N$ is a basic submodule of S.

Proof. Let us assume that L is the submodule of M induced by a set of representatives for the cosets of M/N. Evidently, $M = L + N$, then, because of the cardinality of M/N, we get that $g(L) = g(M/N)$. If now M is h-reduced, we deduce that $g(L \cap N) \geq \aleph_0$. Therefore, there is a summand K of N with $L \cap N \subset K$, and thus $g(K) = g(L \cap N) \leq g(L)$. Suppose that T is a complementary summand of K in N. Then S has the form $S = L + K$. By hypothesis, we may write $M = S \oplus T$ and $N = (S \cap N) \oplus K$. Since $S \cap N$ is an h-pure submodule of S, it is elementary to see that $S \cap N$ is a Σ-uniserial, which yields that $S/(S \cap N) \simeq (S + N)/N = M/N$ is h-divisible. Certainly, $S \cap N$ is a basic submodule of S. Indeed, since $K = S \cap N$, one may see that $g(S/S \cap N) = g(M/N) \geq g(K) = g(S \cap N)$, and thus the assertion sustained. □

The last assertion can be somewhat refined like the following.

Lemma 2.2. *Suppose N and L are submodules of the $QTAG$-module M, and $g(M/N) \geq g(N)$. If N is a basic submodule of M, then L is also a basic submodule of M with $N \cap L = 0$.*

Proof. Write $N = \oplus_{\gamma\in\Gamma}(x_\gamma R)$ and $M/N = (\oplus_{\gamma\in\Gamma}T_\gamma/N) \oplus K/N$, where $d((T_\gamma/N)R) \simeq \infty$, for every $\gamma \in \Gamma$. In fact, assume that $y_\gamma \in T_\gamma$ such that $e(y'_\gamma) = e(y'_\gamma + N) = e(x_\gamma)$ where $d(y_\gamma R/y'_\gamma R) = 1$. Then, because of the h-pureness of N and the well-known classical fact that $d((T_\gamma/N)R) \simeq \infty$, we now let $z_\gamma = x_\gamma + y'_\gamma$ with $d(y_\gamma R/y'_\gamma R) = 1$, and that $L = (\{z_\gamma\}_{\gamma\in\Gamma}R)$. Of course we claim that the submodule L of M must be a basic submodule and $N \cap L = 0$. It is to verify that $L = \oplus_{\gamma\in\Gamma}(z_\gamma R)$ and $N \cap L = 0$. Besides, it is simple to check that M/L is h-divisible, since $M = N + H_1(M)$ and $N + H_1(M) = L + H_1(M)$.

Turning, therefore, to the claim, we need to show that L is h-pure in M. To that goal, let us consider elements of the form $\Sigma u_\gamma z_\gamma$ in L, therefore,

$$H_M(\Sigma u_\gamma x_\gamma) = H_N(\Sigma u_\gamma x_\gamma) = \min\{H_M(u_\gamma x_\gamma)\}.$$

Similarly, $H_M(\Sigma u'_\gamma y_\gamma) \geq \min\{H_M(u'_\gamma y_\gamma)\}$ where $d(u_\gamma R/u'_\gamma R) = 1$. But $H_M(u_\gamma x_\gamma) < H_M(u'_\gamma y_\gamma)$ where $d(u_\gamma R/u'_\gamma R) = 1$. Therefore, $H_M(\Sigma u_\gamma x_\gamma) < H_M(\Sigma u_\gamma y'_\gamma)$ such that $d(y_\gamma R/y'_\gamma R) = 1$, and that

$$H_M(\Sigma u_\gamma z_\gamma) = H_M(\Sigma u_\gamma x_\gamma + \Sigma u_\gamma y'_\gamma) = H_M(u_\gamma x_\gamma),$$

where $d(y_\gamma R/y'_\gamma R) = 1$. Suppose $w \in M$, and let $w' = \Sigma u_\gamma z_\gamma$ such that for some $t > 0$, $d(wR/w'R) = t$. Then, $\Sigma u_\gamma x_\gamma = p'$ where $p = \Sigma v_\gamma x_\gamma$ and $d(pR/p'R) = t$. Observe that $(v'_\gamma - u_\gamma)x_\gamma = 0$ with $d(v_\gamma R/v'_\gamma R) = t$, and hence it is readily seen that $(v'_\gamma - u_\gamma)y'_\gamma = 0$ where $d(v_\gamma R/v'_\gamma R) = t$ and $d(y_\gamma R/y'_\gamma R) = 1$. Setting $q = \Sigma v_\gamma z_\gamma$, $r = \Sigma v_\gamma x_\gamma$ and $s = \Sigma v_\gamma(z_\gamma - y'_\gamma)$ where $d(y_\gamma R/y'_\gamma R) = 1$, we obtain that

$$\begin{aligned} q' &= q' - \Sigma(v'_\gamma - u_\gamma)y'_\gamma, \\ &= s' + \Sigma u_\gamma y'_\gamma, \\ &= r' + \Sigma u_\gamma y'_\gamma, \\ &= \Sigma u_\gamma(x_\gamma + y'_\gamma), \\ &= \Sigma u_\gamma z_\gamma, \end{aligned}$$

where

$$d(v_\gamma R/v'_\gamma R) = d(qR/q'R) = d(rR/r'R) = d(sR/s'R) = t$$

and $d(x_\gamma R/x'_\gamma R) = d(y_\gamma R/y'_\gamma R) = 1$. The proof of the lemma is finished. □

The main thesis is now the following.

Theorem 2.1. *If M is a $QTAG$-module with $H_\omega(M) = 0$, then for any basic submodule N of M, all N-high submodules of M are Σ-uniserial if and only if M is Σ-uniserial.*

Proof. In accordance with Lemma 2.1, one may see that $g(M/N) \geq g(N)$. Now, we may break the argument into two cases. Consider firstly the case $g(M/N) < \aleph_0$. In this case, when $M^1 = 0$, it is plainly seen that M is Σ-uniserial. For the second case if $\aleph_0 \leq g(M/N) < g(N)$, again by virtue of Lemma 2.1, it is easily observed that $M = S \oplus T$, where T is Σ-uniserial. Thus, it can be checked that for all S, $(N \cap S)$-high submodules (which will clearly be Σ-uniserial) will also be N-high in M. We next employ Lemma 2.2, to get that, for each basic submodule N, there is a submodule L (which is actually basic) of M and $N \cap L = 0$. Assume now that a submodule T which is N-high in M and containing L, it is, in fact, an h-pure submodule of M, which, because of the h-neatness of N in M. It is well known that this submodule class properly contains the classes of h-neat submodules and h-pure submodules (in particular, basic submodules). Now, construct, using h-purity, a sequence of submodules T_m in the following manner:

$$N = \oplus_{m=1}^{\infty} N_m,\ H_m(N_m) = 0,\ T_0 = T,$$

such that $T_k \oplus N_{k+1} \subset T_{k+1}$ and

$$Soc(T_{k+1}) = Soc(T_k) \oplus Soc(N_{k+1}) = Soc(T_k \oplus N_{k+1}),$$

where T_{k+1} is h-neat in M. Bearing in mind this construction, it is apparent that $T_m \cap H_m(M) = H_m(T_m)$ for all $m \in Z^+$, since $T_1 \cap H_1(M) = H_1(T_1)$ and contains the basic submodule L of M. Furthermore, since $\cup T_m \cap H_m(M) = H_m(\cup T_m)$ and $Soc(N) \oplus Soc(T) = Soc(M) \subseteq \cup T_m$, we conclude that $M = \cup T_m$.

What remains to show is that T_k is a Σ-uniserial module for every $k \in Z^+$. Let $U_k = \oplus_{m=1}^{k} N_m$, $k \geq 1$, then $U_k \subset T_k$. Since U_k is bounded and $U_k \cap H_k(M) = H_k(U_k)$, it is obvious that $U_k \cap H_k(T_k) = H_k(U_k)$. Therefore, there exists a submodule V_k of T_k such that $T_k = U_k \oplus V_k$. In fact,

$$\begin{aligned} Soc(M) &= Soc(\oplus_{m=k+1}^{\infty} N_m) \oplus Soc(T_k), \\ &= Soc(\oplus_{m=k+1}^{\infty} N_m) \oplus Soc(U_k) \oplus Soc(V_k), \\ &= Soc(N) \oplus Soc(V_k). \end{aligned}$$

Thereby, because by h-pureness of V_k in M, it follows at once that V_k is an N-high submodule of M. Thus, both V_k and T_k are Σ-uniserial. If now M is the union of an ascending sequence of h-pure submodules; say W_k such that $g(W_k) = \aleph_k$ for $0 \leq k < \omega$. The claim then follows directly from Theorem 1 of [21].

The converse implication is obvious. □

3 Some Related Results

As the motivation for the development of $QTAG$-modules, we shall discuss here certain related concepts of $QTAG$-modules by observing that our results really only depend upon the behavior of Σ-uniserial modules.

We start with

(*a*) Σ-modules. "A $QTAG$-module M is called a Σ-module (see [12]) if its high submodules are the direct sum of uniserial modules. It is well known that if M is a Σ-module, then all its high submodules are Σ-uniserial, and that the separable Σ-modules are precisely the Σ-uniserial."

We pause now for a quick observation about Σ-uniserial modules.

Theorem 3.1. *If M is a Σ-module with $g(M^1) \leq \aleph_0$, then M/M^1 is Σ-uniserial.*

Proof. Assume that $N = U + K$ is a high submodule of M, where U is a closed Σ-uniserial and $K \subseteq M$. Then $M/M^1 \cong V + K$, where V is a closed Σ-uniserial with $g(M^1) \leq \aleph_0$. Observing that $V/U \cong L/M^1$, where L is minimal h-divisible containing M^1. However, L/M^1 is obviously checked to be countably generated with $H_\omega(V) = 0$, and consequently V is Σ-uniserial, as wanted. □

A question arises quite usually, inspired by [22], whether all submodules of Σ-modules are again Σ-modules. We conjecture that the question has a negative answer in general, but nevertheless we shall inspect in the sequel its validity for a cardinality of the minimal generating set.

Theorem 3.2. *If M is a Σ-module with a submodule N such that $g(M^1) \leq \aleph_0$, then N is a Σ-module precisely when M is closed.*

Proof. We claim that N is a Σ-module. Then it suffices to show that M/M^1 is Σ-uniserial. To that claim, write $N/(N \cap M^1) \cong (N + M^1)/M^1$, it readily implies that $N/(N \cap M^1)$ is Σ-uniserial. But it is easily checked that $N/(N \cap M^1)$ is isomorphic to $(N/N^1)/(N \cap M^1/N^1)$. In fact, since $N \cap M^1/N^1$ is countably generated and $H_\omega(N/N^1) = 0$, we detect that M/M^1 is Σ-uniserial, and this gives the desired claim. □

We come now to a significant reformulation of Theorem 2.1.

Theorem 3.3. *Let N denote a basic submodule of the $QTAG$-module M. Then $M \supset N$, and all N-high submodules of M are Σ-uniserial if and only if M is a Σ-module with a finite number of elements of infinite height.*

Proof. By what we have used in Theorem 2.1, $M^1 = 0$ and $g(M/N) < \aleph_0$, where M must necessarily be h-reduced. Consequently, we get that $g(M^1) < \aleph_0$. Suppose by assumption that there is a M^1-high submodule K of M containing N such that $g(K/N) < \aleph_0$ and $K^1 = 0$. Therefore, the submodule K being Σ-uniserial implies, with the aid of [12], that M is a Σ-module.

Conversely, suppose M is a Σ-module whose submodule M^1 is finitely generated and that N is a M^1-high submodule of M. Then N is basic submodule of M. Indeed, it is pretty easy to see that all N-high submodules are finitely generated. Therefore, all N-high submodules are Σ-uniserial. □

(*b*) (*ω* + 1)-projective modules. "A $QTAG$-module M is $(\omega+1)$-projective (see [23]) precisely when there is a submodule $N \subset H^1(M)$ with the property that M/N is a direct sum of uniserial modules. Clearly, a $QTAG$-module is ω-projective if and only if it is Σ-uniserial. It follows easily that the class of $(\omega+1)$-projective module is closed under arbitrary submodules."

The next theorem gives an interesting consequence of Theorem 2.1.

Theorem 3.4. *A $QTAG$-module M is not a Σ-uniserial module if and only if it contains a proper $(\omega + 1)$-projective submodule, provided $M^1 = 0$.*

Proof. Let X and Y be submodules of M, then there is an h-pure resolution $0 \to X \to Y \to M \to 0$ of M such that $X/Soc(X) \subseteq Y/Soc(X)$. Observing that $X/Soc(X) \simeq H_1(X)$ is Σ-uniserial, we may extend it in a basic submodule $N/Soc(X)$ of $Y/Soc(X)$. In fact, $Y/X \simeq M$ where it is easily checked that Y/X is Σ-uniserial [6], and $Y/Soc(X)$ is not a

Σ-uniserial module. Further, since $H_\omega(Y/Soc(X)) = 0$, there is a $N/Soc(X)$-high submodule $K/Soc(X)$ of $Y/Soc(X)$ which is not a Σ-uniserial module. Indeed, K being Σ-uniserial forces that $K/Soc(X)$ is $(\omega+1)$-projective. Now, consider the canonical homomorphism

$$\psi : Y/Soc(X) \to (Y/Soc(X))/(Y/Soc(X))$$

with $(K/Soc(X)) \cap (X/Soc(X)) = 0$. It is easy to check that ψ is an isomorphism on $K/Soc(X)$. Consequently, M contains isomorphically in $K/Soc(X)$, and we are done. □

The following result argues about the class of modules for which any member of the class has the condition that all of its high submodules are isomorphic but not necessarily Σ-uniserial.

Theorem 3.5. *Suppose M is a $QTAG$-module such that $M/H_\omega(M)$ is $(\omega+1)$-projective. Then any two high submodules of M are isomorphic, provided the $(\omega+n)$-th Ulm invariant of M is zero for some $n \geq 0$.*

Proof. Let H_1 and H_2 be high submodules of M, and if we let $\phi : M \to M/H_\omega(M)$ be a natural map. Then $Soc(M) = Soc(H_1) \oplus Soc(H_\omega(M)) = Soc(H_2) \oplus Soc(H_\omega(M))$, since $f_M(\omega+n) = 0$ for some $n \geq 0$. Thus, $Soc(\phi(H_1)) = Soc(\phi(H_2))$. Next, since $\phi(H_1)$ and $\phi(H_2)$ are submodules of a $(\omega+1)$-projective module $M/H_\omega(M)$, it readily follows that $\phi(H_1)$ and $\phi(H_2)$ are $(\omega+1)$-projective. Moreover, if there is an isomorphism from $Soc(H_1)$ into $Soc(H_2)$ that preserves heights in H_1 and H_2, then both $(\omega+1)$-projective submodules H_1 and H_2 of M are isomorphic. Since $\phi(H_1)$ and $\phi(H_2)$ are h-pure submodules of $M/H_\omega(M)$, the identity map from $Soc(\phi(H_1))$ into $Soc(\phi(H_2))$ preserves heights such that $Soc(\phi(H_1)) = Soc(\phi(H_2))$. This shows that $\phi(H_1) \simeq \phi(H_2)$. But ϕ restricted to H_1 and ϕ restricted to H_2 are isomorphisms. Consequently, it is plainly seen that $H_1 \simeq H_2$, and we are finished. □

4 Open Problems

We shall pose in this section some questions that remain unanswered yet.

Problem 4.1. *Describe those subsocles N of a $QTAG$-module M such that all N-high submodules of M are Σ-uniserial?*

Problem 4.2. *Does it follow that Σ-modules of countable length are $(\omega+1)$-projective if and only if they are Σ-uniserial?*

Problem 4.3. *Does it follow that the Theorem 3.2 remains true without the restriction $g(M^1) \leq \aleph_0$?*

Problem 4.4. *Suppose in the Theorem 3.5, ω is replaced by any limit ordinal α, and high submodule by α-high submodule. Is this new statement true?*

5 Conclusion

The Σ-uniserial $QTAG$-module is specifically discussed in this study including an application of N-high submodules of $QTAG$-modules. We have also provided some notions such as basic submodules, Σ-modules and $(\omega$+1)-projective modules for solving the problems and their developments. We have examined how these different algebraic structures are related. Finally, we close the work by formulating some problems that seems to be interesting.

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