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A New Wavelet-based Galerkin Method of Weighted Residual Function for The Numerical Solution of One-dimensional Differential Equations

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Abstract In this paper, we developed a new wavelet-based Galerkin method of weighted residual function. In order to achieve this, we considered the wavelet transform as it relates to orthogonal polynomials, developed new wavelets using the Mamadu-Njoseh Polynomials, and formulated a base function with the newly developed wavelets. We considered the method of implementing solutions with the newly developed wavelet-based Galerkin method of weighted residual function, and applied it in obtaining approximate solutions of some one-dimensional differential equations having the Dirichlet boundary conditions. The results obtained from the newly developed method were compared with the results obtained from the exact solution and that from the classical Finite Difference Method (FDM) in literature. It was observed that the newly developed wavelet-based Galerkin method of weighted residual function demonstrated a high efficiency in providing approximate solutions to differential equations. The study revealed that the newly developed wavelet-based Galerkin method of weighted residual function converges at a good pace to the exact solution, and iterated the accuracy and effectiveness of its solutions. We used the MAPLE 18 software in carrying out all computations in this work.

Keywords Wavelets, New Wavelet-Based Galerkin Method, Mamadu-Njoseh Polynomials, Residual Equations, Weight Functions, Orthogonality and Orthonormality

1 Introduction

Differential equations are very significant in a wide variety of real life situations today. For example, in physics, differential equations can be applied in modelling the movement of particles in a fluid or the trajectory of a projectile; in biology, differential equations can be used to model the growth and development of populations or the spread of diseases and infections, and lots more. According to Iweobodo *et al* [1], many problems involving chemical reactions, wave propagation, heat flow, stock market prediction, etc; are modeled with differential equations. In a nutshell, the capacity to model complex situations with differential equations makes them useful instruments for scientists and engineers, and also, one can get a better prediction and understanding of the future behavior of certain systems and how they can be manipulated in order to achieve expected results through solving differential equations, which can be helpful in designing new technologies or predicting the outcomes of experiments. However, solving some differential equations analytically is usually a difficult task. In [2], it was stated that this difficulty is due to the fact that some linear higher order differential equations in more than one variable are not easily reducible to a simple canonical form. Therefore, there is a need for numerical methods. Also in [3], it was iterated that the impossibility to obtain the exact solution of some differential equations has necessitated either discretization of differential equations leading to numerical solutions, or their qualitative study which is concerned with deduction of important properties of the solutions without actually solving them.
Acoding to Dalquist and Bjorck [4], the main purpose of numerical analysis and scientific computing is to develop efficient and accurate methods to compute approximations to quantities that are difficult or impossible to obtain by analytic means. Many numerical methods for different forms of differential equations already exist in the literature as listed in [1]. Some of these methods include Euler's method, Runge-Kutta method, higher order Taylor's method, linear shooting method, multistep method, finite difference method (FDM), finite element method, spectral method, adaptive and non-adaptive algorithms, the Adomian decomposition method, the variation iteration method, the multigrid

method, the homotopy perturbation method among others. Some others include orthogonal polynomial method, wavelet method, Galerkin method etc. Bsharat [5] went further to separate the methods used for ordinary differential equations (ODE) from the ones used for partial differential equations.

The wavelet method is among the recent methods in the literature, and its importance in both theoretical and applied sciences has really gained ground. It is considered a very significant tool in the theory of approximation of solutions to differential equations. In applied mathematics, the most widely used and one of the best known numerical methods of solutions to differential equations is the Galerkin method, and this is propelled by its efficiency and simplicity. A modification of the Galerkin method is the wavelet-Galerkin method, and in [6], it was pointed out that an approach to studying differential equations is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element methods.

A lot of methods for differential equations already exist in literature. The wavelet-based Galerkin method is amongst the recently existing methods, and due to the orthogonality properties of wavelets, it produces great approximations to equations. Many scholars have done a lot of works on wavelet-based Galerkin methods, some of the works include Daubechies Wavelet-based Galerkin Method of Solving PDEs [1], Wavelet-Galerkin Solutions for One dimensional PDEs [7], A second-generation wavelet-based finite element method for the solution of PDEs [8], Wavelet Transform and Wavelet-based Numerical Methods [9], and Numerical Solution of PDEs using Languerre Wavelets collocation method [6]. Some existing wavelets which have been formulated from orthogonal polynomials include Hermite, Languerre, and Chevbychev. The Mamadu-Njoseh polynomial was developed in 2016 [10].

In this work, we proposed a new wavelet-based Galerkin method for the numerical solutions of differential equations using the Mamadu-Njoseh polynomials, then with the new method, we seek the numerical solution of one-dimensional differential equations having the Dirichlet boundary conditions. It is based on expanding the solution with the newly developed wavelet with unknown coefficients. The properties of the newly developed wavelets are used together with the Galerkin method in evaluating the unknown coefficients, then a numerical solution of the one-dimensional partial differential equation under consideration is obtained. In order to achieve this, we shall define a new wavelet with the Mamadu-Njoseh polynomials, form a base function with the newly defined wavelet, solve some one-dimensional differential equations with the newly developed method, compare the results obtained with those existing in the literature, and test for the convergence of the new method.

2 Materials and Methods

2.1 Orthogonal Polynomials

An orthogonal polynomial is a class of polynomials $\varphi_n(x)$ defined over a domain $[a, b]$, which satisfies the orthogonal relation

$$\int_a^b w(x)\varphi_i(x)\varphi_j(x)dx = h_i\delta_{ij} \tag{1}$$

where $w(x)$ is the weight function, and δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

2.2 Mamadu-Njoseh Polynomials

The Mamadu-Njoseh polynomials are polynomials which are constructed within the interval $[-1, 1]$ with respect to the weight function $x^2 + 1$. Their realization was based on these three properties [10]:

(1.) $\varphi_n(x) = \sum_{i=0}^{n} C_i^{(n)} x^i$

(2.) $< \varphi_m(x), \varphi_n(x) >= 0, m \neq n$

(3.) $\varphi_n(x) = 1$

where $\varphi_i, i = 0, 1, 2, \cdots$ are orthogonal polynomials.
Therefore, the first seven Mamadu-Njoseh polynomials are given as

$$\begin{aligned}
\varphi_0(x) &= 1\\
\varphi_1(x) &= x\\
\varphi_2(x) &= \frac{1}{3}(5x^2 - 2)\\
\varphi_3(x) &= \frac{1}{5}(14x^3 - 9x)\\
\varphi_4(x) &= \frac{1}{648}(333 - 289x^2 + 3213x^4)\\
\varphi_5(x) &= \frac{1}{136}(325x - 1410x^3 + 1221x^5)\\
\varphi_6(x) &= \frac{1}{1064}(-460 + 8685x^2 - 24750x^4 + 17589x^6)
\end{aligned} \tag{2}$$

2.3 Wavelet Transform

Wavelets are made up of a family of functions which are formulated from the dilation and translation of a single function known as the mother wavelet. If the dilation and translation parameters a and b, respectively, vary continuously, we have its mathematical representation as

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right), \forall a, b \in \Re, a \neq 0 \tag{3}$$

If the parameters are discrete values, that is, considering $a = a_0^{-k}$, and $b = nb_0a_0^{-k}$, $a_0 > 1, b_0 > 0$, then the family of discrete wavelets is given as

$$\psi_{k,n}(x) = |a|^{-\frac{1}{2}}\psi\left(a_0^k - nb_0\right), \forall a, b \in \Re, a \neq 0. \tag{4}$$

And $\psi_{k,n}(x)$ forms a wavelet basis for $L_2(\Re)$. To be particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis.

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k). \tag{5}$$

According to Saeed and Rehmann [11], the set $\psi_{j,k}(x)$ forms an orthogonal basis of $L_2(\Re)$, which implies that

$$< \psi_{j,k}(x), \psi_{l,m}(x) >= \delta_{jl}\delta_{km}. \tag{6}$$

2.4 Proposed Wavelet

Using the Mamadu-Njoseh polynomials described in section (2.2) above, we can define an orthonormal wavelet of the form

$$\gamma_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}}\left(\overline{MN}\right)_m (2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leqslant x \leqslant \frac{n}{2^{k-1}} \\ 0, & \text{Otherwise} \end{cases} \tag{7}$$

where

$$(\overline{MN})_m = \sqrt{\frac{2}{\pi}} MN_m, \tag{8}$$

$m = 0, 1, \cdots, M-1$, $n = 1, 2, \cdots, 2^{k-1}$, k is any positive integer, and MN_m are the Mamadu-Njoseh polynomials of degree m with respect to the weight function x^2+1 on the interval $[-1, 1]$ and satisfy equation (2) above. For $n = 1$ and $k = 1$, with the first set of Mamadu-Njoseh polynomials listed in section (2.2) above, we can obtain the new wavelets as

$\gamma_{1,0} = 2^{\frac{1}{2}}\sqrt{\frac{2}{\pi}} \approx \frac{2}{\sqrt{\pi}}$

$\gamma_{1,1} = \frac{2}{\sqrt{\pi}}(2x-1)$

$\gamma_{1,2} = \dfrac{2}{\sqrt{\pi}}\dfrac{1}{3}\left[5(2x-1)^2 - 2\right] = \dfrac{2}{\sqrt{\pi}}\dfrac{1}{3}\left[5(4x^2-4x+1)-2\right]$

$= \dfrac{2}{\sqrt{\pi}}\dfrac{1}{3}\left[20x^2 - 20x + 3\right]$

$\gamma_{1,3} = \frac{2}{\sqrt{\pi}}\frac{1}{5}\left[14(2x-1)^3 - 9(2x-1)\right]$

$\gamma_{1,4} = \frac{2}{\sqrt{\pi}}\frac{1}{648}\left[3213(2x-1)^4 - 289(2x-1)^2 + 333\right]$

2.5 Justification of the new wavelet

Here we want to check if the proposed method satisfies the admissibility, Orthonormality, orthogonality and the regularity condition for wavelets.

1. Admissibility Condition.

$$\Rightarrow \int \frac{|\psi(\omega)|^2}{|\omega|} d\omega < +\infty$$

$\psi(\omega)$ is the Fourier transform of the wavelet. This condition states that the Fourier Transform of a wavelet vanishes at the zero frequency. $|\psi(\omega)|^2_{\omega=0} = 0$.

We know that

$$\psi(\omega) = \int_R \psi(t).\exp^{-j\omega t} dt$$

is the Fourier transform of a function in time domain $\psi(t)$. Also, recall that FT is related to the finiteness and boundedness of a function, therefore the integral of the function must converge.

For this reason, we conclude that $\psi(0) = 0$ ie the FT of the wavelet at the origin is zero,

$$\Rightarrow \psi(0) = \int_{-\infty}^{\infty} \psi(t) dt = 0$$

therefore, we need to check that the integral of the new wavelet at the origin converges.

Considering equation (7) for $n = 1$ and $m = 1, 2, \cdots$, we can see that at $x = 0$

$$\int_0^1 \gamma_{n,m}(x) dx \to 0$$

Hence, the new wavelet satisfies the admissibility condition.

2. Regularity

This condition requires that wavelets should be locally smooth and concentrated in both the time and frequency domains. Therefore, the first vanishing moments must be equal to zero. But it has been shown in the admissibility condition that at $x = 0$

$$\int_0^1 \gamma_{n,m}(x) dx \to 0.$$

Hence, the new wavelet satisfies the regularity condition. 3. Orthogonality

This condition states that for discrete wavelets, the inner product of the wavelet and its translate is equal to zero. ie

$$< \psi(t), \psi(t-r) >= \int_{-\infty}^{\infty} \psi(t), \psi(t-r) dt = 0.$$

Checking with the new wavelet, the Mamadu-Njoseh polynomial where the new wavelet was developed from is an orthogonal polynomial [10], this suffices the orthogonality of the new wavelet.

For clarity sake, evaluating the new wavelet with formulated residual equation

$$\int_0^1 \gamma_{1,m}(x) R(x) dx = 0, m = 0, 1, 2, \cdots$$

as used ahead shows this, and the introduction of the weight function $x - x^2$ into $R(x)$ settles this more.

4. Normalization

This is the process of multiplying with a constant in order to ensure that the probabilty of obtaining a result is 1.

This is why the coefficients $c_{n,m}$ of the new wavelets were introduced. Therefore the coefficients have tackled the normalization condition.

5. Orthonormality

A combination of the last two conditions brings about the orthonormality condition.

2.6 Method of Implementation with the Newly Proposed Method

Given a differential equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u = f(x) \tag{9}$$

with boundaries $u(0) = a$ and $u(1) = b$.

where α, β are constants or functions of u or x, and f(x) is a continuous function.

We shall formulate a residual equation $R(x)$ by rewriting equation (9) above as

$$R(x) = \frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u - f(x) \tag{10}$$

Let $u(x)$ be the solution of (9), defined over [0,1), then $u(x)$ can be expanded as a new wavelet satisfying the boundary conditions

$u(0) = a$, and $u(1) = b$, and having unknown parameters as stated below

$$u(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\gamma_{n,m} \quad (11)$$

where $\gamma_{n,m}$ are Mamadu-Njoseh polynomials and $c_{n,m}$ are the unknown coefficients.
Therefore, we shall differentiate equation (11) twice with respect to x to obtain the term $\frac{\partial^2 u}{\partial x^2}$, and once to obtain the term $\frac{\partial u}{\partial x}$, then we substitute all terms into equation (10).
For us to get the coefficients $c_{n,m}$, we shall use weight functions as base elements and apply the orthogonality condition by integrating on the boundary values together with the residual function and equating it to 0 as considered in [12], this will amount to

$$\int_0^1 \gamma_{1,m}(x)R(x)dx = 0, m = 0, 1, 2, \cdots \quad (12)$$

which is a system of linear equations, we then solve for the unknown parameters which will be substituted into the solution function to achieve the desired numerical solutions.

2.7 Convergence of the proposed method

In [13], it was clearly stated that one striking factor that attracts problem solvers to use a specific algorithm is its convergence behavior. Hence, we consider the convergence behavior of the new method in the theorem below.
Theorem 1.0 [6]
Let $f(x) \in L^2(\Re)$ defined on the interval $[0, 1)$ be a continuous function, if $f(x)$ is bounded, that is $f(x) \leq M$, $M > 0$, then expanding $f(x)$ with the newly proposed wavelet will produce a uniform convergence wavelet converging to $f(x)$.
Proof
Let $f(x)$ be a continuous function defined on $[0, 1)$, expanding $f(x)$ with the newly proposed wavelet as in equation (11), will produce a coefficient $c_{n,m}$, which is defined in [15] as

$$\begin{aligned} c_{n,m} &= \int_0^1 f(x)\gamma_{n,m}(x)dx \\ &= \int_I f(x)\frac{2^{\frac{K+1}{2}}}{\sqrt{\pi}}(MN)_m(2^k x - 2n + 1)dx \end{aligned}$$

$$I = \frac{n-1}{2^{k-1}} \leqslant x \leqslant \frac{n}{2^{k-1}}$$

let $t = 2^k x - 2n + 1$ and $f(x) = u$. It becomes

$$\begin{aligned} &\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\int_{-1}^1 u\left(\frac{t-1+2n}{2^k}\right)MN(t)2^{-k}dx \\ &= \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}\int_{-1}^1 u\left(\frac{t-1+2n}{2^k}\right)MN(t)dx \end{aligned}$$

Using GMVT integrals and for some $g \in (-1, 1)$, we have

$$\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}u\left(\frac{g-1+2n}{2^k}\right)\int_{-1}^1 MN(t)dx$$

Let $h = \int_{-1}^1 MN(t)$, therefore, it becomes

$$\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}u\left(\frac{g-1+2n}{2^k}\right)h$$

$$\Rightarrow |C_{n,m}| = |\frac{2^{-k+1}}{\sqrt{\pi}}||u\left(\frac{g-1+2n}{2^k}\right)|h$$

But u is bounded, it implies that $\sum_n^\infty \sum_m^\infty C_{n,m}$ converges absolutely. Hence, the new wavelet has a uniform convergence.

3 Numerical Illustration

3.1 Test problem 1

Consider the differential equation

$$\frac{\partial^2 u}{\partial u^2} + u = -x, 0 \leqslant x \leqslant 1. \quad (13)$$

having the boundary conditions $u(0) = 0, u(1) = 0$.
The exact solution of the problem is $u(x) = \frac{\sin(x)}{\sin(1)} - x$.
To obtain the numerical solution of this using the proposed wavelet-based method, we obtain the residual equation as

$$R(x) = \frac{\partial^2 u}{\partial u^2} + u + x. \quad (14)$$

We then choose the weight function of the proposed wavelet base as $w(x) = x - x^2$ in order to satisfy the given boundary conditions
$\psi(x) = w(x).\gamma(x)$,
therefore,
$\psi_{1,0}(x) = \gamma_{1,0}(x)(x - x^2) = \frac{2}{\sqrt{\pi}}(x - x^2)$
$\psi_{1,1}(x) = \gamma_{1,1}(x)(x - x^2) = \frac{2}{\sqrt{\pi}}(2x - 1)(x - x^2)$
$\psi_{1,2}(x) = \gamma_{1,2}(x)(x - x^2) = \frac{2}{\sqrt{\pi}}(\frac{20}{3}x^2 - \frac{20}{3}x + 1)(x - x^2)$.
Let

$$u(x) = c_{1,0}\psi_{1,0} + c_{1,1}\psi_{1,1} + c_{1,2}\psi_{1,2} \quad (15)$$

be the assumed solution of the problem for $k = 1$ and $m = 0, 1, 2$, therefore

$$\begin{aligned} u(x) &= c_{1,0}\frac{2}{\sqrt{\pi}}(x - x^2) + c_{1,1}\frac{2}{\sqrt{\pi}}(2x - 1)(x - x^2) + \\ &c_{1,2}\frac{2}{\sqrt{\pi}}(\frac{20}{3}x^2 - \frac{20}{3}x + 1)(x - x^2) \end{aligned} \quad (16)$$

We differentiate with respect to x to get
$\frac{\partial u}{\partial x} = c_{1,0}\frac{2}{\sqrt{\pi}}(-2x) + c_{1,1}\frac{2}{\sqrt{\pi}}(-6x^2 + 6x - 1) + c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(-80x^3 + 120x^2 - 46x)$
$\frac{\partial^2 u}{\partial x^2} = c_{1,0}\frac{2}{\sqrt{\pi}}. - 2 + c_{1,1}\frac{2}{\sqrt{\pi}}(-12x + 6) + c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(-240x^2 + 240x - 46)$.
We therefore substitute into the original equation to obtain

$$\begin{aligned} &(c_{1,0}\frac{2}{\sqrt{\pi}}. - 2 + (c_{1,1}\frac{2}{\sqrt{\pi}}(-12x + 6) + c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(-240x^2 \\ &+ 240x - 46) + (c_{1,0}\frac{2}{\sqrt{\pi}}(x - x^2) + c_{1,1}\frac{2}{\sqrt{\pi}}(2x - 1)(x - x^2) \\ &+ c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(20x^2 - 20x + 1)(x - x^2)) = -x, \end{aligned} \quad (17)$$

equation(14) now becomes

$$R(x) = (c_{1,0}\frac{2}{\sqrt{\pi}}. - 2 + c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6)+ c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(-240x^2+240x-46)) + \left(c_{1,0}\frac{2}{\sqrt{\pi}}(x-x^2) + c_{1,1}\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)\right) + \left(c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(20x^2-20x+3)(x-x^2)\right) + x. \quad (18)$$

Therefore, using the weighted Galerkin method and applying the orthogonality property of the new wavelet, we have $\int_0^1 \gamma_{k,m}(x)R(x)dx = 0$; where $k = 1$, and $m = 3$ (ie $m = 0, 1, 2$) and $\pi = 3.143$, therefore we have
this implies that

$$\int_0^1 \frac{2}{\sqrt{\pi}}(c_{1,0}\frac{2}{\sqrt{\pi}}. - 2 + c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6)+ c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(-240x^2+240x-46))dx = 0$$
$$\int_0^1 \frac{2}{\sqrt{\pi}}(2x-1)(c_{1,0}\frac{2}{\sqrt{\pi}}. - 2+ c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6) + c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(-240x^2+240x-46))dx = 0$$
$$\int_0^1 \frac{2}{\sqrt{\pi}}\frac{1}{3}(20x^2-20x+3)(c_{1,0}\frac{2}{\sqrt{\pi}}. - 2+ c_{1,1}\frac{2}{\sqrt{\pi}}(-12x+6) + c_{1,2}\frac{2}{\sqrt{\pi}}\frac{1}{3}(-240x^2+240x-46))dx = 0 \quad (19)$$

Solving equation(19) above using Maple 18 software with the given boundary condition and in the domain of x, we have $c_{1,0} = 0.2452286140$, $c_{1,1} = 0.07512079715$, $c_{1,2} = -0.00310042198$.
Therefore, substituting into the assumed solution, we have

$$u(x) = 0.2451737048\frac{2}{\sqrt{\pi}}(x-x^2)+ 0.07510397674\frac{2}{\sqrt{\pi}}(2x-1)(x-x^2)- 0.00309972775\frac{2}{\sqrt{\pi}}(\frac{20}{3}x^2 - \frac{20}{3}x+1)(x-x^2) \quad (20)$$

Table 1. Numerical results for Test Problem 1

x	Exact Solution	FDM Solution	NM Solution	Error FDM	Error NM
0.01	0.0186420	0.018660	0.0186708	1.80e-5	2.88e-5
0.02	0.0360977	0.036132	0.0361655	3.40e-5	6.78e-5
0.03	0.0511948	0.051243	0.0512714	4.80e-5	7.66e-5
0.04	0.0627829	0.062842	0.0628316	5.90e-5	4.87e-5
0.05	0.0697470	0.069812	0.0697452	6.50e-5	1.84e-6
0.06	0.0710184	0.071084	0.0709672	6.60e-5	5.12e-5
0.07	0.0655851	0.065646	0.0655087	6.10e-5	7.64e-5
0.08	0.0525025	0.052550	0.0524367	4.80e-5	6.58e-5
0.09	0.0309019	0.030903	0.0308742	2.80e-5	2.77e-5

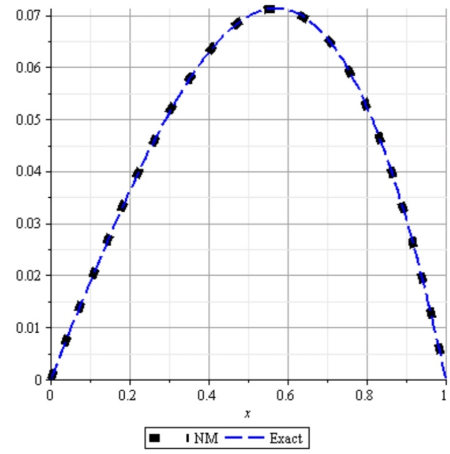

Figure 1. Graphical Comparison of the New Method and the Exact solution for test 1

3.2 Test problem 2

Solve the equation

$$\frac{\partial^2 u}{\partial x^2} - (\pi)^2 u = -2(\pi)^2 \sin(\pi x), 0 \leqslant x \leqslant 1 \quad (21)$$

with boundaries $u(0) = 0$ and $u(1) = 0$.
The exact solution is $u(x) = \sin(\pi x)$.
Solving with the new wavelet-based Galerkin method as in Problem 1 above, we have
$c_1, 0 = 3.235350586$, $c_1, 1 = 0$, $c_1, 2 = -0.4518762140$, and substituting into the assumed solution produced

$$u(x) = 3.235350586\frac{2}{\sqrt{\pi}}(x-x^2)- 0.4518762140\frac{2}{\sqrt{\pi}}(\frac{20}{3}x^2 - \frac{20}{3}x+1)(x-x^2) \quad (22)$$

Table 2. Numerical results for Test Problem 2

x	Exact Solution	FDM Solution	NM Solution	Error FDM	Error NM
0.1	0.309016	0.310289	0.310207	1.27e-3	1.19e-3
0.2	0.588772	0.590204	0.589551	1.43e-3	7.79e-4
0.3	0.809016	0.812347	0.809478	3.33e-3	4.62e-4
0.4	0.951056	0.954971	0.949592	3.92e-3	1.46e-3
0.5	1.000000	1.004126	0.997656	4.13e-3	2.34e-3
0.6	0.951056	0.954971	0.949592	3.92e-3	1.46e-3
0.7	0.809016	0.812347	0.809478	3.33e-3	4.62e-4
0.8	0.587785	0.590204	0.589551	2.42e-3	1.77e-3
0.9	0.309016	0.310289	0.310207	1.27e-3	1.02e-3

3.3 Test Problem 3

Solve the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = -\left(e^{(x-1)}+1\right), 0 \leqslant x \leqslant 1 \quad (23)$$

with boundaries $u(0) = 0$ and $u(1) = 0$.
The exact solution is $u(x) = x\left(-e^{(x-1)}\right)$.

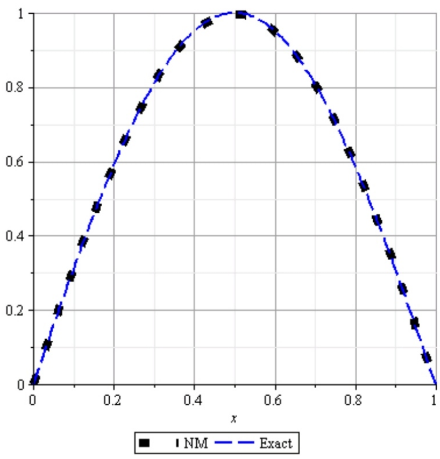

Figure 2. Graphical Comparison of the New Method and the Exact solution for test 2

Solving with the new method produces the solution as

$$u(x) = 0.7075709392\frac{2}{\sqrt{\pi}}(x - x^2)+ \\ 0.1629829344\frac{2}{\sqrt{\pi}}(2x - 1)(x - x^2)+ \\ 0.01564365354\frac{2}{\sqrt{\pi}}(\frac{20}{3}x^2 - \frac{20}{3}x + 1)(x - x^2) \quad (24)$$

Table 3. Numerical results for Test Problem 3

x	Exact Solution	FDM Solution	NM Solution	Error FDM	Error NM
0.1	0.059343	0.061948	0.059251	2.61e-3	9.20e-5
0.2	0.110134	0.115151	0.109902	5.02e-3	3.32e-4
0.3	0.151024	0.158162	0.150735	7.14e-3	2.89e-4
0.4	0.180475	0.189323	0.180249	8.85e-3	2.26e-4
0.5	0.196735	0.206737	0.196660	1.00e-2	7.50e-5
0.6	0.197808	0.208235	0.197904	1.04e-2	9.60e-5
0.7	0.181427	0.191342	0.181631	9.92e-3	2.04e-4
0.8	0.145015	0.153228	0.145212	8.21e-3	7.00e-4
0.9	0.085646	0.090672	0.085733	5.03e-3	4.18e-5

4 Discussion of Results

Having explored the solution of some one-dimensional PDEs using the newly developed method, experimenting with test problem 1 as shown in Table 1, we can observe that the maximum error obtained from the new method is 10^5, while the maximum error obtained from the FDM is 10^5. Also, with test problem 2 as shown in Table 2, the maximum error obtained from the new method is 10^4, while that of the FDM is having 10^3. Finally, with test problem 3 as shown in Table 3, we can see that the maximum error from the new method is 10^5, while that from the FDM is 10^3. From the above, it means that the new method converges better to the exact solution than the FDM in literature.
Moreover, the graphs obtained from the solution of the three test problems show that the new method is in agreement with the exact solution.

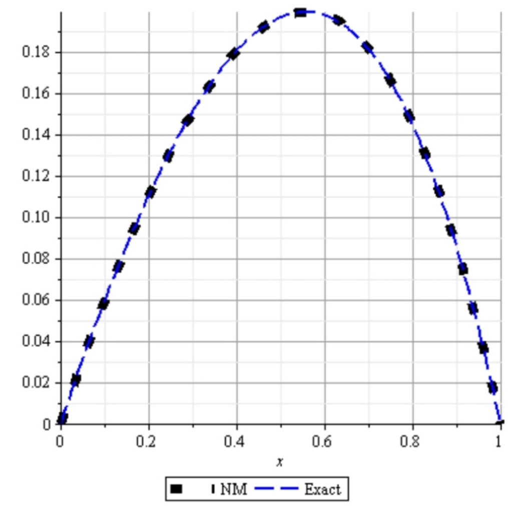

Figure 3. Graphical Comparison of the New Method and the Exact solution for test 3

5 Conclusions

In this work, we have developed a new wavelet-based Galerkin method using the Mamadu-Njoseh polynomials, and we have applied the newly developed method in solving some one-dimensional PDEs with the Dirichlet boundaries. We also carried out a comparison of the new method with the exact solution and the classical FDM in literature, the results as observed in Tables 1 - 3 have shown that the new method is very efficient and converges to the exact solution.

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Not Applicable

Conflict of Interest

The authors declare that they have no competing interests.

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