introduced a method to test the difference between Multiplicative and Non-multiplicative seasonal models under the assumption that these models are based on Gaussian noises. This test utilizes the covariance structure of the model, but it is complex, and its properties have not been studied and therefore has a limited use in practice. Suhartono [5] conducted an empirical comparison of Multiplicative and Non-multiplicative models by evaluating how closely the estimated interaction parameter with the estimated product of corresponding non-seasonal and seasonal parameters, without formal testing.

When examining the previous research that employed MINLP to analyze time series using Box and Jenkins methodology, we notice a lack of literature on this topic. MINLP is considered a special case of mathematical programming. It is a class of optimization problems that involve both continuous and integer variables, where the objective function and/or constraints are nonlinear. MINLP is a generalization of mixed-integer linear programming (MILP) and can be used to model many real-world problems see Sahinidis [6]. Emet [7] presented a method that utilizes Mixed Integer Nonlinear Programming (MINLP) techniques to identify the structure of a dynamic system by minimizing Akaike's Information Criterion (AIC). It is demonstrated by using Autoregressive Moving Average (ARMA) time series to identify their structure and parameters, and the Extended Cutting Plane (ECP) method is used to solve the problems. Uilhoorn [8] compared two MINLP algorithms employed in ARMA model structure and parameter estimation. Gangi et al. [9] proposed a mixed integer nonlinear optimization approach to identifying the parameters of autoregressive (AR) models based on time series data. The approach combines model selection and parameter estimation as a single optimization problem. Bichescu & Polak [10] employed a mathematical programming approach to simultaneously selecting suitable lagged terms and estimating their coefficients in ARIMA-like model (this model includes free variation, and seasonality is not treated in the manner of a seasonal ARIMA (SARIMA) model). The objective of this approach is to minimize normed errors while considering specific constraints. To model the white noise component, a wavelet-based estimator is utilized, which helps in formulating integer linear constraints.

The purpose of this paper is to introduce a new method for identifying and estimating SMA models by using MINLP. The goal of utilizing the MINLP technique is to determine the appropriate SMA model, whether it is Multiplicative or Non-multiplicative, for the given time series data. Our approach is considered an extension for Emet [7].

The structure of the paper is organized as follows: Section 2 introduces the proposed method, Sections 3 and 4 present simulation study and real-world applications, respectively, and finally, Section 5 concludes.

2. The Proposed Method

2.1. SMA Model

A time series is considered to generate from SMA model of non-seasonal order q and seasonal order Q if:

$$y_t = \sum_{i=1}^{q} \phi_i \epsilon_{t-i} + \sum_{j=1}^{Q} \Phi_j \epsilon_{t-js} + \sum_{i=1}^{q} \sum_{j=1}^{Q} \alpha_{ij} \epsilon_{t-i-js} + \epsilon_t, \mathrm{t} = 1, \dots, n \qquad (2.1)$$

where ϵ_t is a sequence of independent variables with zero mean and variance σ^2. The non-seasonal and seasonal moving average orders are represented by q and Q, respectively. The non-seasonal and seasonal moving average coefficients are $\phi = (\phi_1, \phi_2, \dots, \phi_q)^T$ and $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_q)^T$, respectively. The interaction coefficients are denoted by α_{ij}, and s represents the number of seasons in a year.

Let $\alpha_{ij} = \phi_i \Phi_j + \Upsilon_{ij}, (i = 1,2, \dots, q; j = 1,2, \dots, Q)$.

Model (2.1) can be rewritten as follows:

$$y_t = \sum_{i=1}^{q} \phi_i \epsilon_{t-i} + \sum_{j=1}^{Q} \Phi_j \epsilon_{t-js} + \sum_{i=1}^{q} \sum_{j=1}^{Q} \phi_i \Phi_j \epsilon_{t-i-js} + \sum_{i=1}^{q} \sum_{j=1}^{Q} \Upsilon_{ij} \epsilon_{t-i-js} + \epsilon_t \qquad (2.2)$$

If all $\Upsilon_{ij} (i = 1, \dots, q, j = 1, \dots, Q)$ are insignificant, model (2.2) can be simplified into a Multiplicative model, which can be represented as follows:

$$y_t = \sum_{i=1}^{q} \phi_i \epsilon_{t-i} + \sum_{j=1}^{Q} \Phi_j \epsilon_{t-js} + \sum_{i=1}^{q} \sum_{j=1}^{Q} \phi_i \Phi_j \epsilon_{t-i-js} + \epsilon_t \qquad (2.3)$$

It is important to point out that the model will not be Multiplicative if at least one of Υ_{ij} is significant ($\Upsilon_{ij} \neq 0$).

The objective of our study is to estimate the model parameters $\phi_i, \Phi_j, \Upsilon_{ij}$ and identify the number of parameters that distinguish between Multiplicative and Non-multiplicative models. To achieve this, we can utilize an information criterion such as Akaike's Information Criterion (AIC) Akaike [11]. AIC can be defined in the following manner:

$$AIC = -2 \ Log_e(L) + 2k \qquad (2.4)$$

Where $Log_e(L)$ is the natural logarithm of the likelihood function of the model, and k is the number of parameters in the model.

The Akaike Information Criterion (AIC) is designed to balance the goodness of fit of a model with its complexity, penalizing overly complex models. It considers the log-likelihood of the model and the number of parameters used. The model with the lowest AIC value is considered the best fitting model among the candidate models being compared, as it reflects a better trade-off between model fit and complexity.

Referring to model (2.2) and assuming that the random errors $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ are independent and normally distributed with mean zero and variance σ^2, we can write the log-likelihood function for that model (2.2) as follows:

$$Log_e(L) = -\frac{n}{2} Log_e(2\pi) - \frac{n}{2} Log_e(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} \epsilon_t^2 \quad (2.5)$$

Where: $\epsilon_t = y_t - \sum_{i=1}^{q} \phi_i \epsilon_{t-i} - \sum_{j=1}^{Q} \Phi_j \epsilon_{t-js} - \sum_{i=1}^{q} \sum_{j=1}^{Q} \phi_i \Phi_j \epsilon_{t-i-js} - \sum_{i=1}^{q} \sum_{j=1}^{Q} \Upsilon_{ij} \epsilon_{t-i-js}$

Taking the first derivative of the log-likelihood function (2.5) and setting it equal to zero, the maximum likelihood estimator for the parameter σ^2 becomes:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{t=1}^{n} \epsilon_t^2 \quad (2.6)$$

By substituting (2.6) into (2.5), the log-likelihood function can be expressed as follows:

$$Log_e(L) = -\frac{n}{2} \log 2\pi - \frac{n}{2} \log\left(\frac{1}{n} \sum_{t=1}^{n} \epsilon_t^2\right) - \frac{n}{2} \quad (2.7)$$

By substituting (2.7) into (2.4), we can define AIC for the SMA model (2.2) as follows:

$$AIC = (n \ \log 2\pi + n \ \log\left(\frac{1}{n} \sum_{t=1}^{n} \epsilon_t^2\right) + n) + 2(q + Q + q \times Q + 1) \quad (2.8)$$

Where:

$K = q + Q + qQ + 1$ is the number of parameters $\phi_i, \Phi_j, \Upsilon_{ij}, and\ \sigma^2$ in SMA model respectively, n is the number of observations and $\sum_{t=1}^{n} \epsilon_t^2$ is the sum of squared errors.

Note that the difference between Multiplicative and Non-multiplicative SMA model lies only in the Υ_{ij} terms. Therefore, we can eliminate all the fixed terms in equation (2.8), resulting in the objective function being simplified to the following:

$$\min(n \log\left(\frac{1}{n} \sum_{t=1}^{n} \epsilon_t^2\right) + 2(\ q \times Q) \ ) \quad (2.9)$$

It appears that if the sum of squared $\sum_{t=1}^{n} \epsilon_t^2$ is equal in two models, then the model containing the smallest $q \times Q$ is better. Conversely, if $q \times Q$ is fixed in advance, minimizing $\sum_{t=1}^{n} \epsilon_t^2$ leads to the best model.

2.2. The MINLP Model

Mixed Integer Non-Linear Programming (MINLP) Lee & Leyffer [12] refers to a class of optimization problems that involve both integer and continuous decision variables, as well as non-linear objective functions or constraints.

In MINLP, the objective is to find the optimal values for the decision variables that minimize or maximize the objective function, subject to a set of constraints. The decision variables can take on both integer and continuous values, allowing for a more flexible representation of real-world problems.

Solving MINLP problems can be challenging because the presence of non-linearities and discrete variables makes the problem more complex than linear programming or non-linear programming alone. Various techniques and algorithms have been developed to tackle MINLP, including branch and bound methods, outer approximation, and mixed integer linear programming (MILP) relaxations.

To formulate the proposed MINLP model, we begin by introducing binary decision variables $\varphi_{ij}, i = 1, \dots, q, j = 1, \dots, Q$, where each variable is equal to either zero or one, and it is associated with $\Upsilon_{ij}, i = 1, \dots, q, j = 1, \dots, Q$.

The proposed MINLP model can be formulated as follows:

Find the values $\phi_i, \Phi_j, \Upsilon_{ij}$ and ϵ_t which

$$\text{Minimize} \ \sum_{t=1}^{n} \epsilon_t^2 \quad (2.10)$$

Subject to

$$\epsilon_t = y_t - (\sum_{i=1}^{q} \phi_i \epsilon_{t-i} + \sum_{j=1}^{Q} \Phi_j \epsilon_{t-js} + \sum_{i=1}^{q} \sum_{j=1}^{Q} \phi_i \Phi_j \epsilon_{t-i-js} + \sum_{i=1}^{q} \sum_{j=1}^{Q} \Upsilon_{ij} \epsilon_{t-i-js} \ ) \ \ (t = 1,2, \dots, n) \quad (2.11)$$

$$\phi_i \ and \ \Phi_j satisfy \ \ invertibility \ constraints. \quad (2.12)$$

$$-M\varphi_{ij} \leq \ \Upsilon_{ij} \ \leq M \ \varphi_{ij} \quad (2.13)$$

$$\sum_{i}^{q} \sum_{j}^{Q} \varphi_{ij} = \text{r}, \quad (2.14)$$

$$\varphi_{ij} \ \in \{0,1\}, i = 1, \dots, q, j = 1, \dots, Q. \quad (2.15)$$

Where:

n: Sample size.

ϕ_i: Unrestricted in sign decision variables that represent the coefficients of non-seasonal moving averages, $i = 1, \dots, q$.

Φ_j: Unrestricted in sign decision variables that represent the coefficients of seasonal moving averages, $j = 1, \dots, Q$.

Υ_{ij}: Unrestricted in sign decision variables that represent the coefficients that distinguish between the Multiplicative and Non-multiplicative models, $i = 1, \dots, q, j = 1, \dots, Q$.

M: a sufficiently large positive constant.

φ_{ij}: Binary decision variables, where each variable is equal to either zero or one, $i = 1, \dots, q, j = 1, \dots, Q$. Note that if $\varphi_{ij} = 0$, the ij[th] candidate variable is eliminated from the model, because its coefficient Υ_{ij} has to be 0 from Constraint (2.13); if $\varphi_{ij} = 1$, Constraint (2.13) is invalidated.

ϵ_t: Decision variables that represent the errors.

r: The number of coefficients that distinguish between Multiplicative and Non-multiplicative models selected in the SMA equation, determined sequentially as $\text{r} = 0,1, \dots, qQ$.

It is possible to summarize the process of determining the value of r with the following algorithm:

Step 1: Set r = 0, which means the model is Multiplicative. Solve the model and obtain the sum of squares of errors, $\sum_{t=1}^{n}\epsilon_t^2$.

Step 2: Calculate $z_0 = n\log\left(\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2\right)$ as defined in Equation (2.9).

Step 3: Set $r = r + 1$, solve the model and obtain Z_r, where $Z_r = n\log\left(\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2\right) + 2r$.

Step 4: If z_0 is less than or equals to Z_r, or if $r = qQ$, stop and the model becomes Multiplicative SMA model. Otherwise, go back to Step 3.

Note that for a SMA model to be invertible, it should satisfy certain conditions. These conditions depend on the parameters of the model. In general, for a $SMA(q,Q)_s$ model, the invertibility conditions are as follows:

The roots of the MA polynomial $\phi(B) = 0$ should lie outside the unit circle.

The roots of the seasonal MA polynomial $\Phi(B^s) = 0$ should also lie outside the unit circle [13].

3. Simulation Study

3.1. Simulation Design

To evaluate the proposed MINLP model, we generated 1000 time series datasets from 12 SMA models. These models had different orders of q and Q, along with seasonal periods of s=4 and 12. The first six models represent Multiplicative models, namely $SMA(1,1)_4$, $SMA(1,1)_{12}$, $SMA(2,1)_4$, $SMA(2,1)_{12}$, $SMA(1,2)_4$ and $SMA(1,2)_{12}$, respectively. The last six models represent Non-multiplicative models, namely $SMA(1,1)_4$, $SMA(1,1)_{12}$, $SMA(2,1)_4$, $SMA(2,1)_{12}$, $SMA(1,2)_4$ and $SMA(1,2)_{12}$, respectively.

Table 1 presents the specifics of the simulation design, which encompass true parameter values and seasonal periods.

Table 1. Simulation design

Model	ϕ_1	ϕ_2	Φ_1	Φ_2	Υ_{11}	Υ_{21}	Υ_{12}	Υ_{22}	s
I	-0.5	-	-0.3	-	-	-	-	-	4
II	-0.3	-	-0.4	-	-	-	-	-	12
III	0.4	0.5	0.6	-	-	-	-	-	4
IV	-0.6	0.3	0.5	-	-	-	-	-	12
V	0.4	-	0.5	0.4	-	-	-	-	4
VI	0.6	-	-0.3	-0.4	-	-	-	-	12
VII	0.7	-	0.5	-	0.3	-	-	-	4
VIII	-0.8	-	-0.4	-	0.1	-	-	-	12
IX	0.5	-0.4	0.5	-	0.2	0.3	-	-	4
X	0.4	0.4	0.4	-	0.1	0.2	-	-	12
XI	0.5	-	-0.5	0.4	0.3	0.3	-	-	4
XII	-0.4	-	-0.4	-0.3	0.3	0.4	-	-	12

In our simulation study, we employ various sample sizes including 100, 200, 300, and 500.

3.2. Simulation Steps

To evaluate the proposed model, the simulation study consists of the following steps:

1) Generating 1000 time series datasets from a specific SMA model.
2) Applying the proposed MINLP approach to each generated dataset to determine the appropriate SMA model (Multiplicative or Non-multiplicative).
3) Computing the efficiency criterion.

The efficiency criterion (EC): the percentage of time series datasets n^* in which the MINLP approach succeeded in identifying the appropriate SMA model (Multiplicative or Non-multiplicative). It is defined as: Albassam et al. [14]

$$EC = \frac{n^*}{Total\ number\ of\ time\ series\ datasets} * 100$$

Simulation study was conducted using MATLAB R2010a and GAMS 40.4.0 software. MATLAB R2010a was used to generate the data, and GAMS 40.4.0 solver DICOPT was used for the MINLP approach.

3.3. Simulation Results

In the following, the results of the simulation study will be presented, along with an explanation of all the results that have been obtained.

Tables 2 to 7 display the percentage of correct identification (The efficiency criterion) achieved by the MINLP method for the following Multiplicative SMA models $SMA(1,1)_4$, $SMA(1,1)_{12}$, $SMA(2,1)_4$, $SMA(2,1)_{12}$, $SMA(1,2)_4$ and $SMA(1,2)_{12}$.

Table 2. The percentage of correct identification by MINLP for model I

	$y_t = -0.5\epsilon_{t-1} - 0.3\epsilon_{t-4} + 0.15\ \ \epsilon_{t-5} + \epsilon_t$			
n	100	200	300	500
Multiplicative	81.3	80.8	83.3	83

Table 2 displays the percentage of correct identification by MINLP for model I, based on 1000 time series datasets generated by Multiplicative $SMA(1,1)_4$. The table shows the efficiency of the MINLP method in accurately identifying the appropriate model across different sample sizes: 100, 200, 300, and 500. The results indicate that the proposed MINLP model successfully identified the true model, which is the Multiplicative SMA, in over 80% of cases for all sample sizes.

In tables 3 to 7, the proposed MINLP model was applied on 1000 time series datasets generated by Multiplicative $SMA(1,1)_{12}$, $SMA(2,1)_4$, $SMA(2,1)_{12}$, $SMA(1,2)_4$ and $SMA(1,2)_{12}$ models, and the percentage of correct model identification was computed. Consistent with the previous findings, the results demonstrate that the MINLP model achieved an accuracy of over 80% for various sample sizes across all these different models.

Table 3. The percentage of correct identification by MINLP for model II

	$y_t = -0.3\epsilon_{t-1} - 0.4\epsilon_{t-12} + 0.12\ \ \epsilon_{t-13} + \epsilon_t$			
n	100	200	300	500
Multiplicative	81.2	84	84.5	82.9

Table 4. The percentage of correct identification by MINLP for model III

	$y_t = 0.4\epsilon_{t-1} + 0.5\epsilon_{t-2} + 0.6\epsilon_{t-4} + 0.24\ \ \epsilon_{t-5} + 0.30\epsilon_{t-6} + \epsilon_t$			
n	100	200	300	500
Multiplicative	82.6	82.9	84	83.2

Table 5. The percentage of correct identification by MINLP for model IV

	$y_t = -0.6\epsilon_{t-1} + 0.3\epsilon_{t-2} + 0.5\epsilon_{t-12} - 0.30\ \ \epsilon_{t-13} + 0.15\ \ \epsilon_{t-14} + \epsilon_t$			
n	100	200	300	500
Multiplicative	81.5	85.1	85.4	84.8

Table 6. The percentage of correct identification by MINLP for model V

	$y_t = 0.6\epsilon_{t-1} + 0.5\epsilon_{t-4} + 0.30\ \ \epsilon_{t-5} + 0.4\epsilon_{t-8} + 0.24\ \ \epsilon_{t-9} + \epsilon_t$			
n	100	200	300	500
Multiplicative	81.9	81.4	82	82.7

Table 7. The percentage of correct identification by MINLP for model VI

	$y_t = 0.6\epsilon_{t-1} - 0.3\epsilon_{t-12} - 0.18\ \ \epsilon_{t-13} - 0.4\epsilon_{t-24} - 0.24\ \ \epsilon_{t-25} + \epsilon_t$			
n	100	200	300	500
Multiplicative	82.6	83.8	84.4	83.9

Tables 8 to 13 display the percentage of correct identification achieved by the MINLP method for various Non-multiplicative SMA models: $SMA(1,1)_4$, $SMA(1,1)_{12}$,$SMA(2,1)_4$,$SMA(2,1)_{12}$,$SMA(1,2)_4$ and $SMA(1,2)_{12}$ with different values of $\Upsilon_{ij}, i = 1, \dots, q, j = 1, \dots, Q$. Some values of Υ_{ij} are close to zero, while others are far from zero.

Table 8 presents the results of Model VII obtained from 1000 time series datasets generated using Non-multiplicative $SMA(1,1)_4$ with $\Upsilon_{11} = 0.3$. Our findings indicate that the proposed MINLP model successfully identified the true model, which is the Non-multiplicative SMA, with 97.4% when sample size was 100. Furthermore, the percentage increased to 100% as we increased the sample sizes to 200, 300, and 500.

Table 9 presents the results of Model VIII obtained from 1000 time series datasets generated using Non-multiplicative $SMA(1,1)_{12}$ with $\Upsilon_{11} = 0.1$. Our findings indicate that even though the value of Υ_{11} is close to zero, the proposed MINLP model successfully identified the true model, which is the Non-multiplicative SMA, with 53.8% when a sample size was 100, with 80.3% when a sample size was 200, 90.3% when a sample size was 300 and 98.2% when a sample size was 500.

Table 10 displays the results of Model IX obtained from 1000 time series datasets generated using a Non-multiplicative $SMA(2,1)_4$ with $\Upsilon_{11} = 0.2$ and $\Upsilon_{21} = 0.3$. Our findings indicate that the proposed MINLP model successfully identified the Non-multiplicative SMA model, as the model is considered Non-multiplicative SMA if at least one of $\Upsilon_{ij}, i = 1,2\ and\ j = 1$ is not equal to zero. The percentage of identification was 76.5% for a sample size of 100, 96.1% for a sample size of 200, 99.6% for a sample size of 300, and 100% for a sample size of 500.

Moreover, the proposed MINLP model successfully identified the true model (which is Non-multiplicative $SMA(2,1)_4$ with both Υ_{11} and Υ_{21} not equal to zero) with a percentage of 64.2% for a sample size of 100, 89.1% for a sample size of 200, 96.9% for a sample size of 300, and 100% for a sample size of 500.

Table 11 presents the results of Model X obtained from 1000 time series datasets generated using Non-multiplicative $SMA(2,1)_{12}$ with $\Upsilon_{11} = 0.1$ and $\Upsilon_{21} = 0.2$. Our findings indicate that despite the values of Υ_{11} and Υ_{21} being close to zero, the proposed MINLP model successfully identified the Non-multiplicative SMA model. The identification percentage was 62.8% for a sample size of 100, 88.5% for a sample size of 200, 97.3% for a sample size of 300, and 99.8% for a sample size of 500.

Moreover, the proposed MINLP model successfully identified the true model (which is Non-multiplicative $SMA(2,1)_{12}$ with both Υ_{11} and Υ_{21} not equal to zero) for sample sizes of 200,300 and 500. The identification percentages were 55.3% for a sample size of 200, 69.5% for a sample size of 300, and 86.8% for a sample size of 500.

Table 12 displays the results of Model XI obtained from 1000 time series datasets generated using Non-multiplicative $SMA(1,2)_4$ with $\Upsilon_{11} = 0.3$ and $\Upsilon_{12} = 0.3$. Our findings indicate that the proposed MINLP model successfully identified the Non-multiplicative SMA model. The identification percentage was 96% for a sample size of 100 and reached 100% for sample sizes 200, 300 and 500.

Furthermore, the proposed MINLP model successfully identified the true model (which is Non-multiplicative $SMA(1,2)_4$ with both Υ_{11} and Υ_{12} not equal to zero) with a percentage of 64.8% for a sample size of 100, 93.7% for a sample size of 200, 99% for a sample size of 300, and 100% for a sample size of 500.

Table 13 presents the results of Model XII obtained from 1000 time series datasets generated using Non-multiplicative $SMA(1,2)_{12}$ with $\Upsilon_{11} = 0.3$ and $\Upsilon_{12} = 0.4$. Our findings indicate that the proposed MINLP model successfully identified the Non-multiplicative SMA model. The identification percentage was 94.4% for a sample size of 100, 99.9% for a sample size of 200, and 100% for sample sizes of 300 and 500.

Furthermore, the proposed MINLP model also successfully identified the true model, which is the Non-multiplicative $SMA(1,2)_{12}$ with both Υ_{11} and Υ_{12} not equal to zero. The identification percentages were 63.2% for a sample size of 100, 94.3% for a sample size of 200, 98.9% for a sample size of 300, and 100% for a sample size of 500.

Overall, the simulation study demonstrated that the proposed MINLP method exhibits potential for effectively identifying various Multiplicative and Non-multiplicative SMA models.

Table 8. The percentage of correct identification by MINLP for model VII

	$y_t = 0.7\epsilon_{t-1} + 0.5\epsilon_{t-4} + (0.35 + 0.3)\ \ \epsilon_{t-5} + \epsilon_t$			
n	100	200	300	500
Non-multiplicative	97.4	100	100	100

Table 9. The percentage of correct identification by MINLP for model VIII

	$y_t = -0.8\epsilon_{t-1} - 0.4\epsilon_{t-12} + (0.32 + 0.1)\epsilon_{t-13} + \epsilon_t$			
n	100	200	300	500
Non-multiplicative	53.8	80.3	90.3	98.2

Table 10. The percentage of correct identification by MINLP for model IX

	$y_t = 0.5\epsilon_{t-1} - 0.4\epsilon_{t-2} + 0.5\epsilon_{t-4} + (0.25 + 0.2)\epsilon_{t-5} + (-0.20 + 0.3)\epsilon_{t-6} + \epsilon_t$			
n	100	200	300	500
Non-multiplicative	76.5	96.1	99.6	100
True model	64.2	89.1	96.9	100

Table 11. The percentage of correct identification by MINLP for model X

	$y_t = 0.4\epsilon_{t-1} + 0.4\epsilon_{t-2} + 0.4\epsilon_{t-12} + (0.16 + 0.1)\ \ \epsilon_{t-13} + (0.16 + 0.2)\epsilon_{t-14} + \epsilon_t$			
n	100	200	300	500
Non-multiplicative	62.8	88.5	97.3	99.8
True model	38.2	55.3	69.5	86.8

Table 12. The percentage of correct identification by MINLP for model XI

	$y_t = 0.5\epsilon_{t-1} - 0.5\epsilon_{t-4} + (-0.25 + 0.3)\epsilon_{t-5} + 0.4\epsilon_{t-8} + (0.20 + 0.3)\ \ \epsilon_{t-9} + \epsilon_t$			
n	100	200	300	500
Non-multiplicative	96	100	100	100
True model	64.8	93.7	99	100

Table 13. The percentage of correct identification by MINLP for model XII

	$y_t = -0.4\epsilon_{t-1} - 0.4\epsilon_{t-12} + (0.16 + 0.3)\epsilon_{t-13} - 0.3\epsilon_{t-24} + (0.12 + 0.4)\ \ \epsilon_{t-25} + \epsilon_t$			
n	100	200	300	500
Non-multiplicative	94.4	99.9	100	100
True model	63.2	94.3	98.9	100

4. Real-World Applications

In the following, we explain how the proposed MINLP model can be used to identify and estimate SMA models in the application of two real datasets: Carbon Dioxide Levels data Cryer and Chan [15] and College Enrollment data Pankratz [2].

Carbon Dioxide Levels Data

Carbon dioxide (CO_2) levels are monitored at several locations around the world to verify atmospheric changes. One of these locations is Alert, Northwest Territories, Canada, near the Arctic Circle.

The Carbon Dioxide Levels data set consists of 132 monthly values from January 1994 to December 2004. As shown in Figures 1 and 2, the CO_2 series exhibits an increasing trend and a seasonal pattern. Therefore, we used the first differences and seasonal differences to obtain a stationary difference series, as shown in Figures 3 and 4.

Based on Figure 4, the SMA model has been proposed. To determine whether the model is Multiplicative or Non-multiplicative, we used the proposed MINLP model.

Table 14 shows the results of identifying and estimating SMA model for the Carbon Dioxide Levels data series using the proposed MINLP.

From Table 14, we find that r = 0, which means that the proposed model for the carbon dioxide levels time series data is Multiplicative SMA model.

Table 14. Identifying and Estimating SMA model for CO_2

Estimated model	r
Multiplicative SMA $\phi = -0.551$ $\Phi = -0.720$	0

College Enrollment Data

Typically, enrollment in colleges is higher during the fall season than in the spring because new students usually begin their studies in the fall. However, some students withdraw between semesters due to academic or social issues. As a result, we anticipate that new student enrollment will follow a seasonal pattern with s = 2. The data from Pankratz [2] supports this expectation, as they have recorded new college student enrollment since the fall of 1954. Visual analysis confirms that each year's second half (the fall semester) consistently has a higher enrollment than the following spring semester. For more information, refer to figure 5. Figures 5 and 6 show that the college enrollment series exhibits a seasonal pattern, so we used seasonal differences to obtain a stationary difference series. See figure 7 for further details.

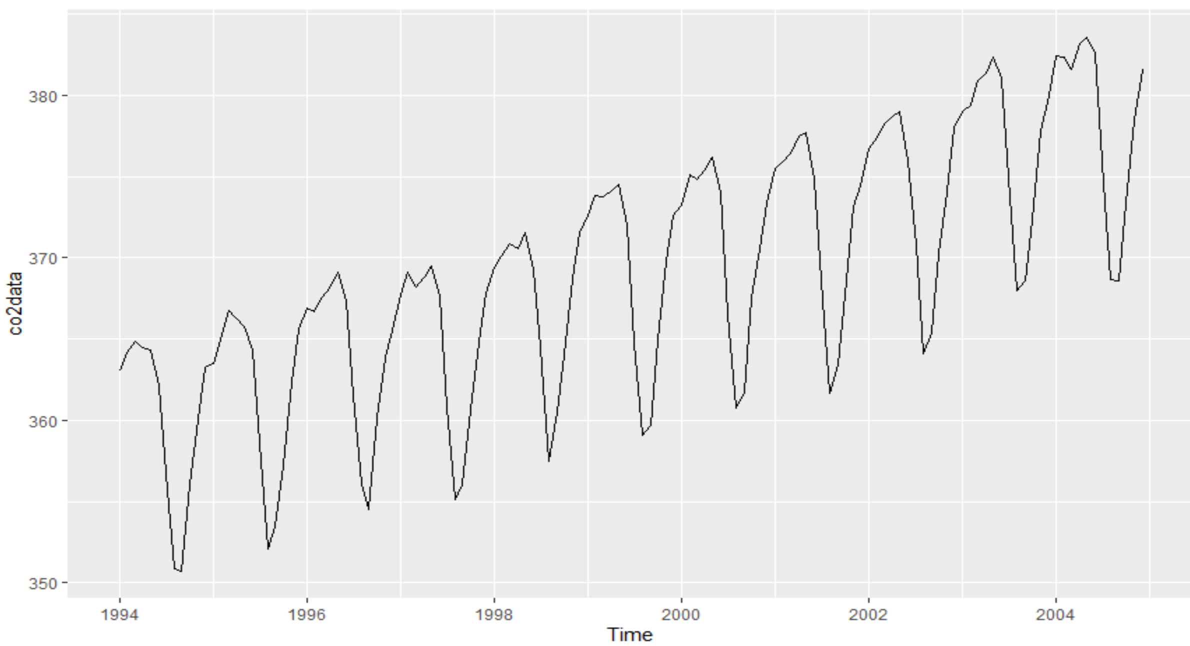

Figure 1. Monthly CO_2 levels, January 1994- December 2004

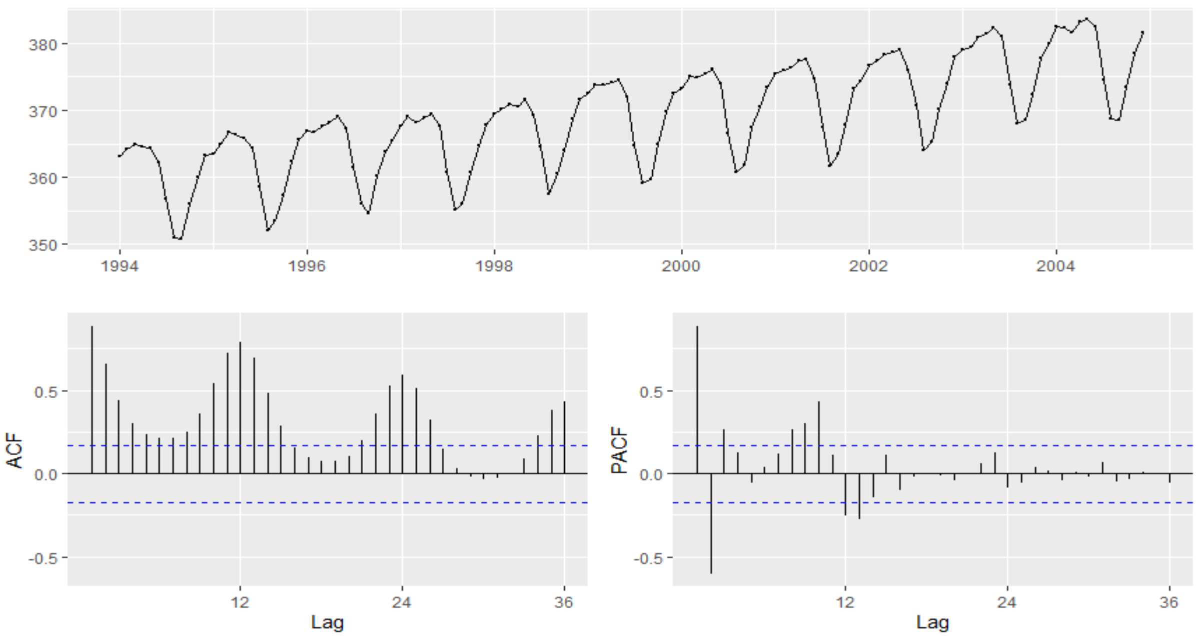

Figure 2. ACF and PACF for CO_2 levels

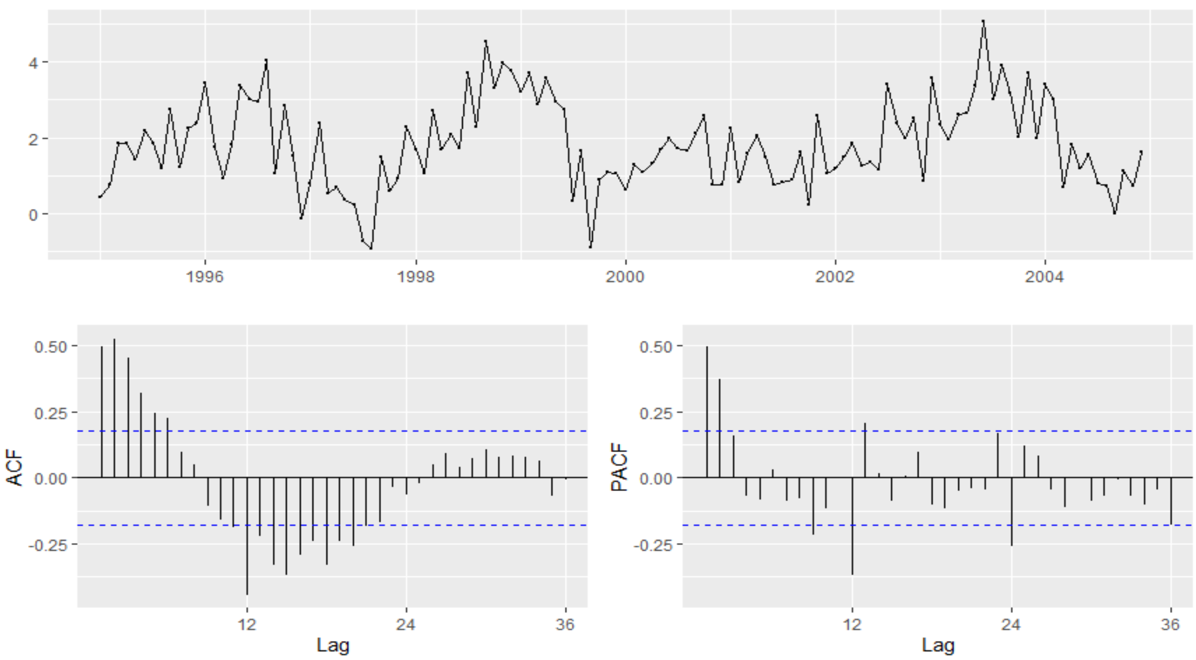

Figure 3. ACF and PACF for CO_2 levels after taking seasonal differences

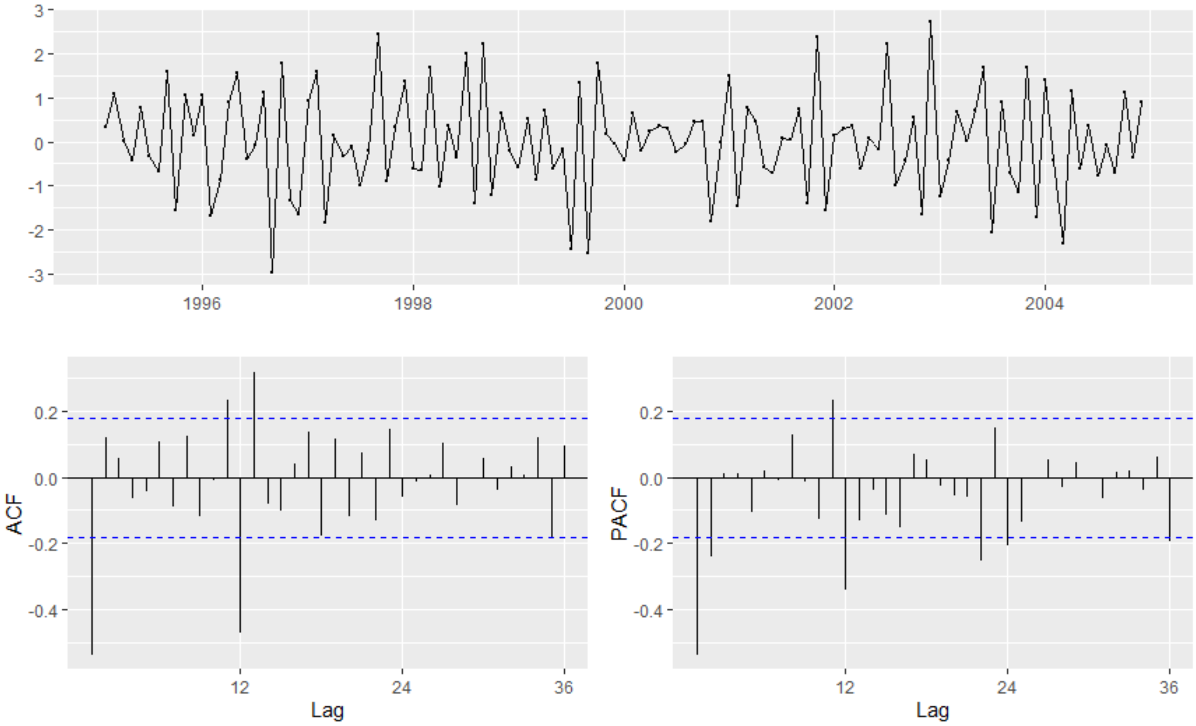

Figure 4. ACF and PACF for CO_2 levels after taking first differences for seasonal differences

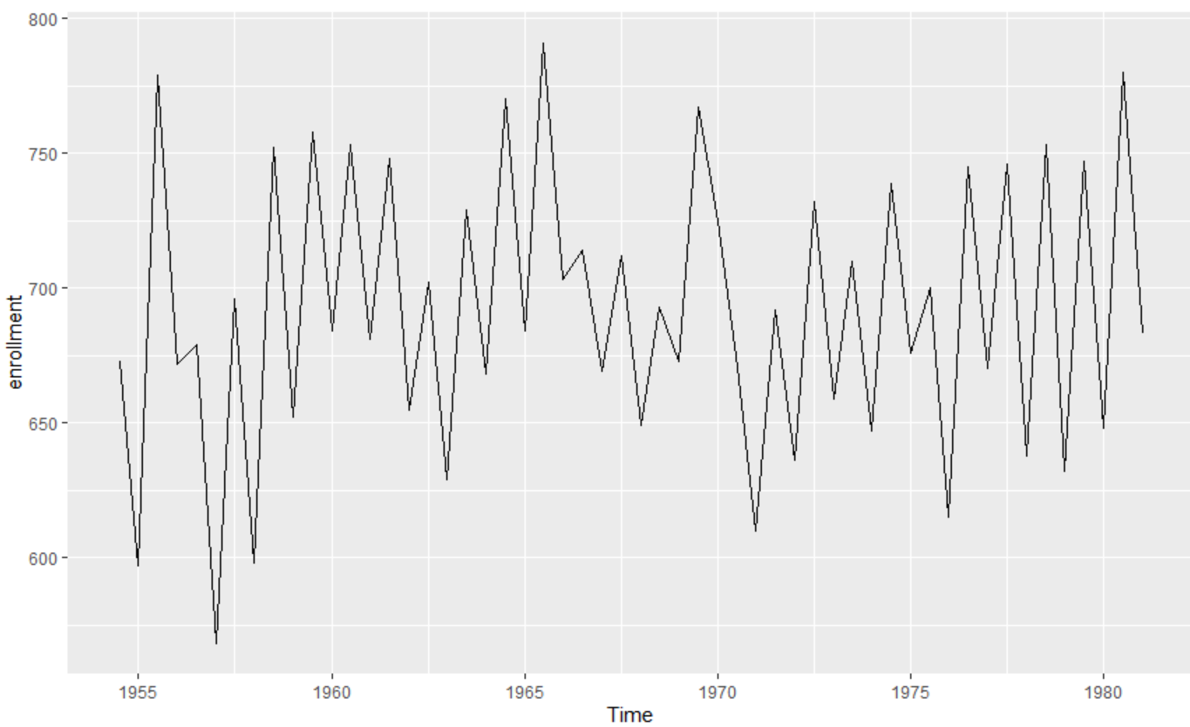

Figure 5. College- enrollment data

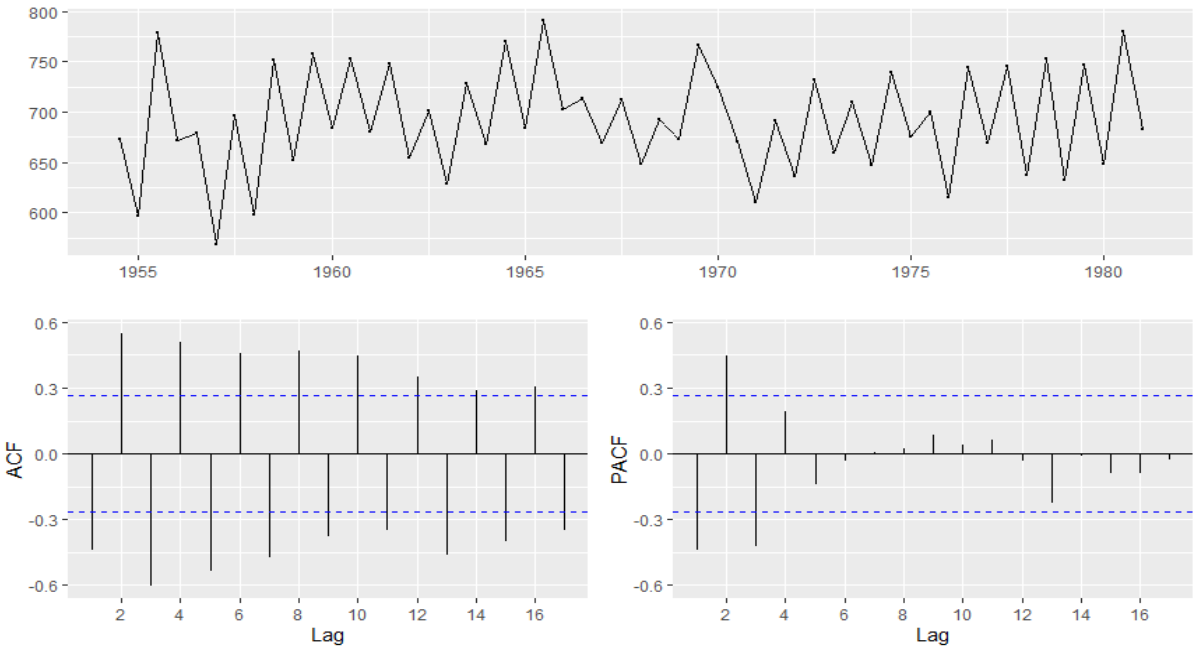

Figure 6. ACF and PACF for College- enrollment data

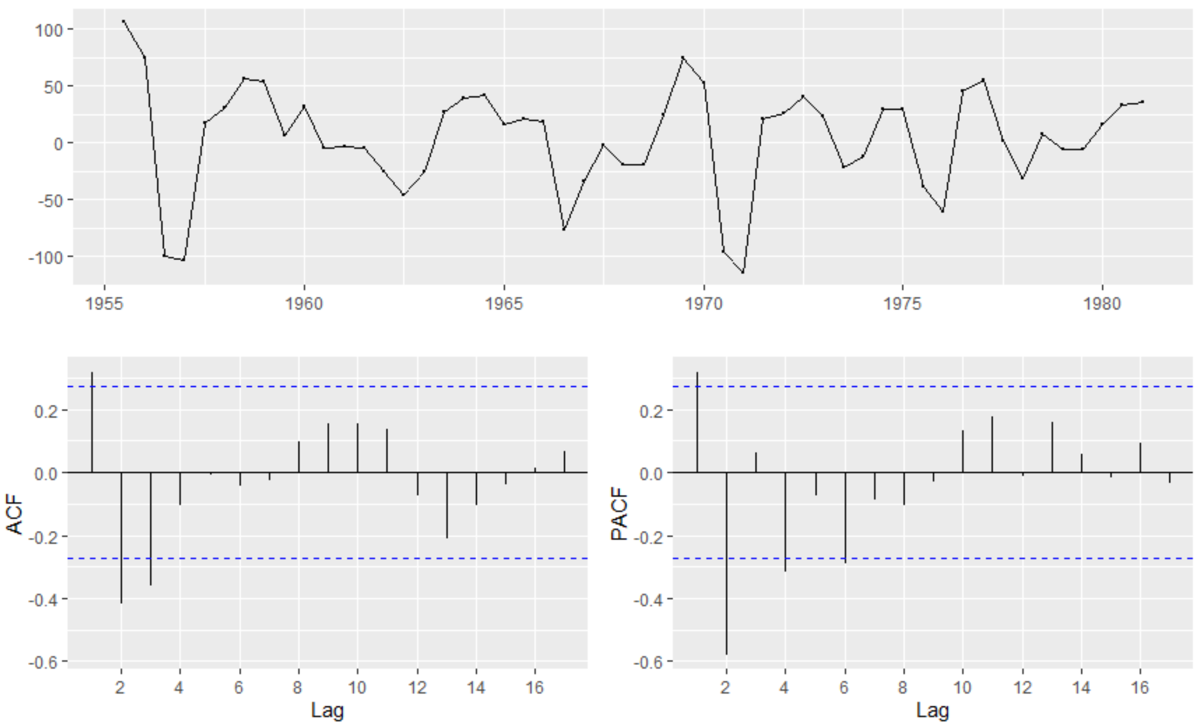

Figure 7. ACF and PACF for College- enrollment data after taking seasonal differences

From figure 7, the SMA model has been proposed. To determine whether the model is Multiplicative or Non-multiplicative, we used the proposed MINLP model.

Table 15 presents the results of identifying and estimating SMA model for the college enrollment time series using the proposed MINLP.

Table 15. Identifying and Estimating SMA model for the college enrollment

Estimated model	r
Non-Multiplicative SMA $\phi = 0.499$ $\Phi = -0.616$ $\Upsilon = -0.112$	1

From table 15, we find that r = 1, indicating that the proposed model for the college enrollment time series is Non-multiplicative SMA model. This is consistent with the findings of Pankratz [2].

5. Conclusions

A new approach is introduced for identifying and estimating SMA models using a mixed-integer nonlinear programming model. The proposed method aims to identify the appropriate moving average model (Multiplicative or Non-multiplicative model). The effectiveness of the suggested method is demonstrated by applying it to real-world applications and conducting a simulation study. The results indicate that the proposed approach is useful for identifying and estimating SMA models and can provide valuable insights into time series data analysis. Overall, this study provides a valuable contribution to the field of time series analysis and provides a practical tool for researchers and practitioners who work with time series data.

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