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Some Properties of Cyclic and Dihedral Homology for Schemes

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Abstract A scheme is a type of mathematical construction that extends the concept of algebraic variety in a number of ways, including accounting for multiplicities and being defined over any commutative ring. In this article, we study some properties of the cyclic and dihedral homology theory in schemes. We study the long exact sequence of cyclic homology of scheme and prove some results. So, we introduce and study Morita-equivalence in cyclic homology of schemes and proof the main relation between trace map and inclusion map. Our goal is to explain product structures on cyclic homology groups $\mathbb{HC}_*(\mathcal{X}/\mathcal{S})$. Especially, we show $\mathbb{HC}_*(\mathcal{X}/\mathcal{S}) = \otimes_{\ n\in\mathbb{Z}} \mathbb{HC}_n(\mathcal{X}/\mathcal{S})$ of algebra. We give the relations between dihedral homology $(\mathbb{HD}(\mathcal{U}))$ and cyclic homology $(\mathbb{HC}(\mathcal{U}))$ of schemes, therefore: $\mathbb{HC}_n(\mathcal{U}) = {}^{-}\mathbb{HD}_{n+1}(\mathcal{U}) \oplus {}^{+}\mathbb{HD}_n(\mathcal{U})$. We explain the trace map and inclusion map of cyclic homology for scheme algebra which takes form: $inc: \mathbb{HC}_n(\mathcal{U},\mathcal{V}) \to \mathbb{HC}_n(M_m(\mathcal{U}), M_m(\mathcal{V}))$ and $tr: \mathbb{HC}_n(M_m(\mathcal{U}), M_m(\mathcal{V})) \to \mathbb{HC}_n(\mathcal{U},\mathcal{V})$. For the shuffle map $sh: \mathcal{U}\otimes\mathcal{V} \to \mathcal{U}\times\mathcal{V}$, we obtain the long exact sequence of cyclic homology for scheme:

$$\cdots \to \mathbb{HC}_n(\mathcal{U}\times\mathcal{V}) \xrightarrow{i} \bigoplus_{r+s=n} \mathbb{HC}_r(\mathcal{U})\otimes\mathbb{HC}_s\ B(\mathcal{V}) \xrightarrow{S\otimes -1-\otimes S} \bigoplus_{p+q=n-2} \mathbb{HC}_p(\mathcal{U})\otimes\mathbb{HC}_q(\mathcal{V}) \xrightarrow{\partial} \mathbb{HC}_{n-1}(\mathcal{U}\times\mathcal{V}) \to \cdots.$$

We give the long exact sequence of dihedral homology for scheme:

$$\cdots \to \mathbb{HC}_n(\mathcal{U}\times\mathcal{V}) \xrightarrow{i} \bigoplus_{r+s=n} \mathbb{HC}_r(\mathcal{U})\otimes\mathbb{HC}_s\ B(\mathcal{V}) \xrightarrow{S\oplus -1-\otimes S} \bigoplus_{p+q=n-2} \mathbb{HC}_p(\mathcal{U})\otimes\mathbb{HC}_q(\mathcal{V}) \xrightarrow{\partial} \mathbb{HC}_{n-1}(\mathcal{U}\times\mathcal{V}) \to \cdots .$$

For any three $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$ algebra, we write the next long exact sequence as a commutative diagram:

$$\cdots \to \mathbb{HC}_n(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HC}_n(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HC}_n(\mathcal{W}_*) \xrightarrow{\delta} \mathbb{HC}_{n-1}(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HC}_{n-1}(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HC}_{n-1}(\mathcal{W}_*) \to \cdots.$$

For all $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$ schemes, we give the long exact sequence of dihedral homology as:

$$\cdots \to \mathbb{HD}_n(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HD}_n(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HD}_n(\mathcal{W}_*) \xrightarrow{\delta} \mathbb{HD}_{n-1}(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HD}_{n-1}(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HD}_{n-1}(\mathcal{W}_*) \to \cdots.$$

Keywords Cyclic Homology, Mayer-Vietortis, Morita Invariance, Scheme

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1. Introduction

The topic of algebraic homology was introduced in Riemann and Betty's work on homological numbers in the nineteenth century. Poincar created this science for the first time in 1895. In 1925, Emmy Noether concentrated on the homology groups in space. But until about 1945, homology was still a factor in the study of topology. Between 1940 and 1955, the homology and cohomology of a number of algebraic systems, including the cohomology of associative algebras, groups, and Lie algebras, as well as Tor and Ext for abelian groups, were defined and studied using these topologically inspired methods for computing homology. Sheaves, spectral sequences, and sheaf cohomology were also introduced by Leray. The book by Cartan and Eilenberg was clarified and entirely redirected. The Abelian categories are a result of the quest for a general environment for derived functors, while algebraic $\mathcal{K}$-theory is a result of the search for nontrivial examples of projective modules.

Loday [1] offered the definition pertains to the relationship between cyclic homology of algebra and cyclic homology of Scheme. For any finite-dimensional $\mathcal{A}$ over $\mathcal{K}$ -algebra, Geller and Welbel [2] demonstrated that $\mathcal{HC}_*(\mathrm{A}) \cong \mathbb{HC}_*$. Additionally, Geller and Welbel [2] show that for schemes over $\mathcal{K}$, sheafifying and taking hypercohomology result in a "Hochschild homology theory". A natural "SBI" sequence that generalises the typical SBI sequence for algebras connects Hochschild $(\mathbb{HH}_n(\mathrm{X}))$ and cyclic homology $(\mathbb{HC}_n(X))$ of schemes:

$$\cdots \mathbb{HC}_n(X) \xrightarrow{S} \mathbb{HC}_{n-2}(X) \xrightarrow{B} \mathbb{H}H_{n-1}(X) \xrightarrow{I} \mathbb{HC}_{n-1}(X) \cdots.$$

The literature has developed the theory for Hochschild and cyclic homologies of $\mathcal{A}$ alegebra for great detail (see [1]). By sheafifying the simplicial complex of $\mathcal{X}$ and using hyper-(co)homology, it is possible to define the Hochschild homology of the scheme $\mathcal{X}$ if it is a scheme over $\mathrm{Spec}(\mathcal{K})$ [2]. Cyclic homology of $\mathcal{X}$ is obtained in [1], also [3]. For more about methods of the Hochschild cohomology of schemes, see [4].

Here is a summary of what will happen in the subsequent sections:

In section one and two, we will define the cyclic homology $\mathbb{HC}_n(\mathcal{X})$ of schemes over $\mathcal{K}$ as a collection of $\mathcal{K}$ –modules associated with each scheme $\mathcal{X}$ over $\mathcal{K}$ that satisfies the following conditions:

1- They are contravariant and natural in $\mathcal{X}$,
2- Natural isomorphisms exist for each affine scheme $\mathcal{X} = \mathrm{Spec}\ \mathcal{A}$:

$$\mathbb{HC}_n(\mathcal{X}) \cong \mathcal{HC}_n(\mathcal{A}) \text{ for all n.}$$

We will get Mayer-Vietoris sequence, if $X = \mathcal{U} \cup \mathcal{V}$ is as follows:

$$\cdots \mathbb{HC}_n(\mathcal{X}) \to \mathbb{HC}_n(\mathcal{U}) \oplus \mathbb{HC}_n(\mathcal{V}) \to \mathbb{HC}_n(\mathcal{U} \cup \mathcal{V}) \to \mathbb{HC}_{n-1}(\mathcal{X}) \cdots. \quad (1)$$

In section three, we consider that there is a product structure given schemes $\mathcal{X}$ and $\mathcal{Y}$ over a base scheme:

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \otimes \mathbb{HC}_r(\mathcal{Y}/\mathcal{S}) \to \mathbb{HC}_{q+r}(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}/\mathcal{S}) \quad \forall q,r \in \mathbb{Z}. \quad (2)$$

This leads us to conclude that a product structure exists for a particular scheme X over S:

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \otimes \mathbb{HC}_r(\mathcal{Y}/\mathcal{S}) \to \mathbb{HC}_{q+r}(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}/\mathcal{S}), \quad (3)$$

It makes $\oplus_{r\in\mathbb{Z}}\ \mathbb{HC}_r(\mathcal{X}/\mathcal{S})$ into algebra. Additionally, if we modify a morphism's base with $\mathrm{t}: \mathcal{S}' \to \mathcal{S}$, to it by setting $\mathcal{X}_{\mathcal{S}'} := \mathcal{X} \times_{\mathcal{S}} \mathcal{S}'$, we obtain a product.

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \otimes \mathbb{HC}_r(\mathcal{S}'/\mathcal{S}) \to \mathbb{HC}_{q+r}(\mathcal{X}_{\mathcal{S}'}/\mathcal{S}'), \forall q,r \in \mathbb{Z}. \quad (4)$$

In section four, we will discuss the relationship between dihedral homology $(\mathbb{HD}(\mathcal{U}))$ and cyclic homology $(\mathbb{HC}(\mathcal{U}))$. We will discuss the inverse between trace map and inclusion map of cyclic homology for Scheme algebra. For the shuffle map $\mathrm{sh}: \mathcal{U} \otimes \mathcal{V} \to \mathcal{U} \times \mathcal{V}$, we will obtain the exact long sequence of cyclic homology for Scheme. For any three $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ algebra, we will write the next long exact sequence as a commutative diagram:

$$\cdots \to \mathbb{HC}_n(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HC}_n(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HC}_n(\mathcal{W}_*) \xrightarrow{\delta} \mathbb{HC}_{n-1}(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HC}_{n-1}(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HC}_{n-1}(\mathcal{W}_*) \to \cdots. \quad (5)$$

2. Homology Theory of Schemes

Simplicial homology of $\mathcal{A}$ is equal to the homology of the common Hochschild complex, $C_*^h(\mathcal{A})$, which has the formula $C_n^h(\mathcal{A}) = \mathcal{A}^{\otimes n+1} = \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}$. For the sheafification of the associated complex of presheaves $\mathcal{U} \to C_*^h(O_{\mathcal{X}}(\mathcal{U}))$, let's write C_*^h:

$$C_*^h: \cdots \xrightarrow{b} O_{\mathcal{X}}^{\otimes\ *+1} \xrightarrow{b} \cdots \xrightarrow{b} O_{\mathcal{X}} \otimes_{\mathcal{K}} O_{\mathcal{X}} \xrightarrow{0} O_{\mathcal{X}} \to 0, \quad (6)$$

Hypercohomology of the unbounded cochain complex $C^n = C_{-n}^h$ is used to determine the Hochschild homology of $\mathcal{X}$ over $\mathcal{K}$:

$$\mathbb{HH}_n(\mathcal{X}) = \mathbb{H}^{-n}(\mathcal{X}, C_*^h) = \mathbb{H}^{-n}(\mathcal{X}, C^{-*}). \quad (7)$$

Proposition (1-1): [3]

For any scheme $\mathcal{X}$ over $\mathcal{K}$. A quasi-coherent sheafon $\mathcal{X}$ is then each HH_n. Further, we obtain natural isomorphisms $\mathcal{HH}_n(\mathcal{A}) \xrightarrow{\cong} H^0(\mathcal{U}, HH_n)$ on each affine open $\mathcal{U} = \mathrm{Spec}(\mathcal{A})$ of $\mathcal{X}$.

Now, the cyclic homology $\mathbb{HC}_n(\mathcal{X})$ of a scheme over $\mathcal{K}$ is defined. Remember that a cyclic homology of $\mathcal{A}$ is $\mathcal{HC}_*(\mathcal{A}) = H_*(Tot\ B_{**}(\mathcal{A}))$ where $B_{**}(\mathcal{A})$ is the double complex of Connes'(B, b). We denote the result of sheafifying B_{**} as a double complex of sheaves. Of course, the related (b, B) double complex in a category of sheaves is B_{**}, and the $O_{\mathcal{X}}^{\otimes n+1}$ form is the cyclic object in a category of sheaves.

In order to consider $\mathrm{Tot}\ \mathrm{B}_{**}$, the (product) total chain complex, as a cochain complex, reindexing is used to define $\mathbb{HC}_n(X) = H^{-n}(\mathcal{X}, Tot\ B_{**})$. We will demonstrate that the natural maps $\mathcal{HC}_n(\mathcal{A}) \to \mathbb{HC}_n(\mathcal{X})$ are isomorphisms if $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$ is affine.

Example (1-2):

Let $\mathcal{A}$ be the $\mathcal{K}$-algebra and $\mathcal{X}$ be the projective m-space $P_{\mathcal{K}}^m$. The following equations are

$$\mathbb{HH}_*(\mathbb{P}_{\mathcal{K}}^m) = \begin{cases} A & if\ * = 0 \\ 0 & if\ * \neq 0' \end{cases} \quad (8)$$

and

$$\mathbb{H}\mathcal{C}_*(\mathbb{P}^m_{\mathcal{K}}) = \mathcal{HC}_*(k)\otimes A = \begin{cases} A & if * is \ even \\ 0 & if * \ is \ odd \end{cases} \quad (9)$$

The computation for $\mathbb{H}\mathcal{H}$ is given by the hypercohomology spectral sequence $E_2^{pq} = H^p(X, HH_{-q}) \Longrightarrow \mathbb{H}\mathcal{H}_{-n}(X)$, which degenerates for $\mathcal{X} = \mathbb{P}^m_{\mathcal{K}}$.

Lemma (1-3):

For any scheme $\mathcal{X} = \text{Spec}(\mathcal{A})$, we have

$$\mathbb{H}\mathcal{H}_n\langle r\rangle(\mathcal{X}) = \begin{cases} \mathcal{HH}_n(\mathcal{A}) & if \ \ n < r \\ 0 & if \ \ n \geq r \end{cases}. \quad (10)$$

Proof:

The good truncation of C_*^h at level r should be denoted by ${}_{\tau<r}C_*^h$, and using [5]. This is a quotient complex of sheaves whose homology $\mathbb{H}\mathcal{H}_n\langle r\rangle = H_n({}_{\tau<r}C_*^h)$ equals $H_n(C_*^h) = \mathbb{H}\mathcal{H}_n$ if $n < r$ and equals zero otherwise. Furthermore, with quasi-coherent cohomology, ${}_{\tau<r}\mathrm{C}_*^{\mathrm{h}} = 0$ is a bounded complex if $n > r$, and ${}_{\tau<r}C_*^h = 0$ otherwise. For $\mathrm{H}^{-\mathrm{n}}(\mathrm{X}, {}_{\tau<r}\mathrm{C}_*^{\mathrm{h}})$, we will write $\mathbb{H}\mathcal{H}_n\langle r\rangle(\mathcal{X})$.

The hypercohomology spectral sequence $E_2^{pq} = H^p(\mathcal{X}, H_{-q}({}_{\tau<r}C_*^h)) \Longrightarrow \mathbb{H}\mathcal{H}_{-n}\langle r\rangle(\mathcal{X})$ converges, because ${}_{\tau<r}C_*^h$ is a bounded complex. Due to the quasicoherent cohomology of ${}_{\tau<r}C_*^h$, it degenerates, producing $\mathbb{H}\mathcal{H}_n\langle r\rangle(X) = H^0(X, HH_n\langle r\rangle)$.

Definition (1-4):

The double chain complex ${}_{\tau_q<r}B_{**}(\mathcal{A})$ is the double complex $B_{**}(\mathcal{A})$ that we derive from Connes' (b, B) with $\tau < r$. The bounded triangular region of the plane defined by the equations $0 \leq p \leq q, 0 \leq q \leq r$ is the sole location where this truncated double complex is non-zero. The top row of ${}_{\tau_q<r}\mathrm{B}_{**}(\mathcal{A})$ is as follows:

$$0 \leftarrow b(\mathcal{A}^{\otimes n+1}) \overset{B}{\leftarrow} b(\mathcal{A}^{\otimes n}) \overset{B}{\leftarrow} \cdots \overset{B}{\leftarrow} b(\mathcal{A}^{\otimes 2}) \leftarrow 0. \quad (11)$$

The homology of $Tot \ {}_{\tau_q<r}B_{**}(\mathcal{A})$ is denoted by $\mathcal{HC}_n\langle r\rangle(\mathcal{A})$. Since ${}_{\tau_q<0}B_{**}(\mathcal{A}) = 0$, ${}_{\tau_q<r}\mathrm{B}_{**}(\mathcal{A})$ is $\mathcal{A}$ concentrated in degree zero. For $n \neq 0$, the first nonzero truncation is $\mathcal{HC}_0\langle 1\rangle(\mathcal{A}) = \mathcal{A}$, with $\mathcal{HC}_n\langle 1\rangle(\mathcal{A}) = 0$.

3. The Mayer – Vietoris Sequence of Cyclic Homology for Scheme

In order to construct $\mathbb{H}\mathcal{C}_n\langle r\rangle(\mathcal{X}) = \mathbb{H}^{-n}\left(\mathcal{X}, Tot \ {}_{\tau_q<r}B_{**}(\mathcal{A})\right)$ on $\mathcal{X}$, we can sheafify the truncations of definition (1-4) to create a bounded double complex of sheaves ${}_{\tau_q<r}B_{**}(\mathcal{A})$ on $\mathcal{X}$.

Lemma (2-1):

$$\mathcal{HC}_n\langle r\rangle(\mathcal{A}) = \begin{cases} \mathcal{HC}_n(\mathcal{A}) & if \ \ n < r \\ S(\mathcal{HC}_{2r-n}(\mathcal{A})) & if \ \ r \leq n \leq 2r-2 \\ 0 & if \ \ n > 2r-2 \end{cases}. \quad (12)$$

Proof:

This is clear unless $r \leq n \leq 2r-2$. Typically, a exact sequence of chain complexes is produced from the first column:

$$0 \to {}_{\tau<r}C_*^h(\mathcal{A}) \to Tot_{\tau_q<r}B_{**}(\mathcal{A}) \overset{S}{\to} Tot_{\tau_q<r-1}B_{**}(\mathcal{A})[-2] \to 0. \quad (13)$$

The lengthy exact homology sequence produces $\mathcal{HC}_n\langle r\rangle(\mathcal{A}) \cong \mathcal{HC}_{n-2}(r-1)(\mathcal{A})$ for $n > r$ and $\mathcal{HC}_n\langle r\rangle(\mathcal{A}) \cong \ image \ \mathcal{HC}_r(\mathcal{A}) \overset{S}{\to} \mathcal{HC}_{r-2}(\mathcal{A})$. The outcome is now clear-cut.

Lemma (2-2):

We have the expression

$$\mathcal{HC}_n\langle r\rangle(\mathcal{A}) \geq \mathbb{H}\mathcal{C}_n\langle r\rangle(\mathcal{X}), \quad (14)$$

for every affine scheme $\mathcal{X} = \text{Spec}(\mathcal{A})$ for all n and r.

Proof:

We have a following exact sequence of complexes of sheaves on $\mathcal{X}$ for each n:

$$0 \to {}_{\tau<r}C_*^h \to Tot_{\tau_q<r}B_{**}(\mathcal{A}) \overset{S}{\to} Tot \ {}_{\tau_q<r-1}B_{**}(\mathcal{A})[-2] \to 0. \quad (15)$$

A map of lengthy precise sequences is produced by taking hypercohomology and comparing it to (13).

$$\begin{array}{ccccccccc} \mathcal{HC}_{n-1}\langle r-1\rangle & \to & \mathcal{HH}_n\langle r\rangle(\mathcal{A}) & \to & \mathcal{HC}_n\langle r\rangle(\mathcal{A}) & \to & \mathcal{HC}_{n-2}\langle r-1\rangle(\mathcal{A}) & \to & \mathcal{HH}_{n-1}\langle r\rangle \\ \mathcal{A}\downarrow & & \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\ \mathbb{H}\mathcal{C}_{n-1}\langle r-1\rangle & \to & \mathbb{H}\mathcal{H}_n\langle r\rangle(\mathcal{X}) & \to & \mathbb{H}\mathcal{C}_n\langle r-1\rangle(\mathcal{X}) & \to & \mathbb{H}\mathcal{C}_{n-2}\langle r-1\rangle(\mathcal{X}) & \to & \mathbb{H}\mathcal{H}_{n-1}\langle r\rangle \end{array} \quad (16)$$

We now move forward by induction on r, beginning with the simple case $r = 0$.

Remark (2-3):

The isomorphism $\mathcal{A} \cong H^0(\mathcal{X}, O_{\mathcal{X}})$ is the case $r = 1, n = 0$ of this lemma.

Theorem (2-4):

For any scheme $\mathcal{X} = spec\ (\mathcal{A})$ over k-algebra, we get

$$\mathcal{HC}_n(\mathcal{A}) \cong \mathbb{HC}_n(\mathcal{X}). \quad (17)$$

Proof:

Tot B_{**} is the tower of complexes' inverse limit. We have an accurate sequence for each n in $Tot(_{\tau_q<r}B_{**})$:

$$0 \to \underset{\leftarrow}{lim}^1 \mathbb{HC}_{n+1}\langle r\rangle(\mathcal{X}) \to \mathbb{HC}_{n+1}(\mathcal{X}) \to \underset{\leftarrow}{lim}\ \mathbb{HC}_{n+1}\langle r\rangle(\mathcal{X}) \to 0. \quad (18)$$

As soon as $r > n$ by Lemma (2-1) and Lemma (2-2), the tower $\{\mathbb{HC}_{n+1}\langle r\rangle(\mathcal{X})\}$ stabilises at $\mathcal{HC}_n(\mathcal{A})$.

4. Products on Cyclic Homology

In this study, we define $O_{\mathcal{X}}$ as the structure sheaf of the scheme $\mathcal{X}$ and define $f: \mathcal{X} \to \mathcal{S}$ as a morphism of schemes over a $\mathcal{K}$-algebra. We define the following for any open set $\mathcal{U} \subseteq \mathcal{X}$ and any $n \geq 0$:

$$CC_n(\mathcal{X}/\mathcal{S})(\mathcal{U}) := O_{\mathcal{X}}(\mathcal{U}) \otimes_{\Gamma(\mathcal{S},O_{\mathcal{S}})} \cdots \otimes_{\Gamma(\mathcal{S},O_{\mathcal{S}})} O_{\mathcal{X}}(\mathcal{U}\ ) \quad (19)$$

where $\Gamma(\mathcal{S}, O_{\mathcal{S}})$ stands for the global sections of the $O_{\mathcal{S}}$ of $\mathcal{S}$'s structural sheaf.

Then the objects $CC_n(\mathcal{X}/\mathcal{S})(\mathcal{U})$ carry a well-known cyclic differential $b(\mathcal{U})_n: CC_n(\mathcal{X}/\mathcal{S})(\mathcal{U}) \to CC_{n-1}(\mathcal{X}/\mathcal{S})(\mathcal{U})$ for $n \geq 1$ (see [1]) defining a cyclic complex $(CC_h^*(\mathcal{X}/\mathcal{S})(\mathcal{U}), b(\mathcal{U}))$ given by $CC_n(\mathcal{X}/\mathcal{S})(U) := CC_{-n}(\mathcal{X})(\mathcal{U}), n \leq 0$. Further, we have Connes' operator $B(U)_n: CC_n(X/S)(\mathcal{U}) \to CC_{n+1}(\mathcal{X}/\mathcal{S})(\mathcal{U})$ for $n \geq 0$ (see [1]) and the differentials $b(\mathcal{U})_*$ and $B(\mathcal{U})_*$ together to form a "mixed (bi) complex" $(BC^{**}(\mathcal{X}/\mathcal{S})(\mathcal{U}), B(\mathcal{U}), b(\mathcal{U}))$ with (for $p, q \leq 0$):

$$BC^{p,q}(\mathcal{X}/\mathcal{S})(\mathcal{U}) := CC_{p-q}(\mathcal{X}/\mathcal{S})(\mathcal{U}) \quad (20)$$

$$b(\mathcal{U}): BC^{p,q}(\mathcal{X}/\mathcal{S})(\mathcal{U}) \to BC^{p,q+1}(\mathcal{X}/\mathcal{S})(\mathcal{U}) \quad (21)$$

$$B(\mathcal{U}): BC^{p,q}(\mathcal{X}/\mathcal{S})(\mathcal{U}) \to BC^{p+1,q}(\mathcal{X}/\mathcal{S})(\mathcal{U}) \quad (22)$$

Definition (3-1):

Suppose that $f: \mathcal{X} \to \mathcal{S}$ is a morphism of schemes over a $\mathcal{K}$ -algebra. Let $(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S}), B, b)$) call the sgeafification of the bi-complex of $U \to \left(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S})(\mathcal{U}), B(\mathcal{U}), b(\mathcal{U})\right)$ to a bi-complex of sheaves of abelian groups on $\mathcal{X}$.

Then, for each $q \in \mathbb{Z}$, we refer to the $(-q)$th total of the bi-complex ($Tot \to \left(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S})(\mathcal{U}), B, b\right)$ as the qth cyclic homology $\mathbb{HC}_q(\mathcal{X}/\mathcal{S})$ of $\mathcal{X}$ with respect to $\mathcal{S}$.

Theorem (3-2):

Assume that $\mathcal{S} = \mathrm{Spec}(\mathcal{A})$ and that $\mathcal{A}$ is $\mathcal{K}$-algebra. The form $f: \mathcal{X} \to \mathcal{S}$ is a morphism of schemes. Then, we get natural isomorphisms

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \cong \mathbb{HC}_q(\mathcal{X}_{\mathcal{A}}) \quad (23)$$

In particular, natural isomorphisms

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \cong \mathbb{HC}_q(\mathcal{B}|\mathcal{A}) \quad (24)$$

exist when $\mathcal{X} = \mathrm{Spec}(\mathcal{B})$ is also affine. $\mathbb{HC}_q(\mathcal{B}|\mathcal{A})$ expresses the cyclic homologies of $\mathcal{B}$ when viewed as an $\mathcal{A}$-algebra, respectively, for any $q \in \mathbb{Z}$.

Proof:

$\mathcal{S} = Spec(\mathcal{A})$ is affine if and only if $\Gamma(\mathcal{S}, O_{\mathcal{S}}) = \mathcal{A}$. Then, we obtain $CC_n(\mathcal{X}/\mathcal{S})(\mathcal{U}) = O_{\mathcal{X}}(\mathcal{U}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} O_{\mathcal{X}}(\mathcal{U})$ for any open set $\mathcal{U} \subseteq \mathcal{X}$ and any $n \geq 0$. As a result, the cyclic complex of $\mathcal{X}$ taken into account in [2] and the sheafified complex $Tot(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S})), B, b)$ computing the cyclic homology of $\mathcal{X}$ is the same. As a result, we get natural isomorphisms of the form

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \cong \mathbb{HC}_q(\mathcal{X}_{\mathcal{A}}) \quad (25)$$

Particularly, suppose that $\mathcal{X} = Spec(\mathcal{B})$, i.e., $\mathcal{X}$ corresponds to an affine scheme corresponding to an $\mathcal{A}$-algebra $\mathcal{B}$. It then follows ([2] and [3]) that we have natural isomorphisms

$$\mathbb{HC}_q(\mathcal{X}_{\mathcal{A}}) = \mathbb{H}^{-q}(Tot(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S}))) \cong \mathbb{HC}_q(\mathcal{B}|\mathcal{A}), \quad (26)$$

The result of combining the isomorphisms in (25) and (26) is (24).

Let $\mathcal{A}$ and $\mathcal{B}$ be two k-algebras that are known. A permutation $\sigma \in \mathcal{S}_p$ works on $(a_0, a_1, \ldots, a_p)$ as follows for every $p \geq 0$, and given $(a_0, a_1, \ldots, a_p) \in C_p(A)$ (see [12]):

$$\sigma \cdot (a_0, a_1, \ldots, a_p) := (a_0\ , a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(p)}). \quad (27)$$

The shuffle product is known if $p, q \geq 0$ are integers and $\mathcal{S}_{p,q}$ indicates the set of (p, q) shuffles, that is, of all $\sigma \in \mathcal{S}_{p,q}$, such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$ (see [1]),

$$\mathcal{S}h_{p,q}((a_0, a_1, \ldots, a_p) \otimes (b_0, b_1, \ldots, b_p)) = \sum_{\sigma \in \mathcal{S}_{p,q}} \mathcal{S}gn(\sigma)\sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, \ldots, a_p \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_q) \quad (28)$$

generate a product $\mathcal{S}h_{p,q}$: $\mathbb{HC}_p(\mathcal{A}) \otimes_k \mathbb{HC}_q(\mathcal{B}) \to \mathbb{HC}_{p+q}(\mathcal{A} \otimes \mathcal{B})$. In relation to the fundamental scheme $\mathcal{S}$, our goal is to extend this to the simplicial homology groups of schemes.

Theorem (3-3):

Assume that schemes over k-algebra are morphisms of $f: \mathcal{X} \to \mathcal{S}$ and $g: \mathcal{Y} \to \mathcal{S}$. Then, with regard to $\mathcal{S}$, there is a multiplication on cyclic homology groups:

$$\mathbb{H}\mathcal{C}_q(\mathcal{X}/\mathcal{S}) \otimes \mathbb{H}\mathcal{C}_r(\mathcal{Y}/\mathcal{S}) \rightarrow \mathbb{H}\mathcal{C}_{q+r}(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}/\mathcal{S}), \quad \forall q,r \in \mathbb{Z}. \tag{29}$$

Proof:

We take into account the fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ in addition to the projections $p_{\mathcal{Y}}$: $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \rightarrow \mathcal{Y}$ and $p_{\mathcal{X}}: \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \rightarrow \mathcal{X}$. The next step is to select an affine cover of $\{\mathcal{S}_i\}_{i\in I}$ for $\mathcal{S}$ with the parameters $\mathcal{Y}_i = g^{-1}(\mathcal{S}_i)$ and $\mathcal{X}_i := f^{-1}(\mathcal{S}_i), i \in I$. Let $\{\mathcal{X}_{im}\}_{m\in M_i}$ and. $\{\mathcal{Y}_{in}\}_{n\in N_i}$ be an affine open subset-based basis of $\mathcal{Y}_i$ and. $\mathcal{X}_i$ for each it. The fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ then has a basis made up of open sets with the formula $\{\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}\}_{i\in I, m\in M_i, n\in N_i}$.

Additionally, we take into account the complex of abelian group presheaves on $(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S}), B, b)$ and $(\widetilde{BC}^{**}(\mathcal{Y}/\mathcal{S}), B, b)$. We observe that, for all $q, r \geq 0$, the inverse image presheaves may be characterised as:

$$p_{\mathcal{X}}^{-1}(CC_h^{-q}(\mathcal{X}/\mathcal{S}))(\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}) = O_{\mathcal{X}}(\mathcal{X}_{im})^{\otimes q+1},$$
$$p_{\mathcal{Y}}^{-1}(CC_h^{-q}(\mathcal{Y}/\mathcal{S}))(\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}) = O_{\mathcal{Y}}(\mathcal{Y}_{in})^{\otimes q+1}, \tag{30}$$

for any given $i \in I$, $m \in M_i$ and $n \in N_i$, where $\Gamma\ (\mathcal{S}, O_{\mathcal{S}})$ is the sum of all tensor products. Then, we have shuffle maps for all $q, r \geq 0$.

$$\mathcal{S}\mathrm{h}_{q,r}: O_{\mathcal{X}}(\mathcal{X}_{im})^{\otimes q+1} \otimes_{\Gamma(\mathcal{S}, O_{\mathcal{S}})} O_{\mathcal{Y}}(\mathcal{Y}_{in})^{\otimes r+1} \rightarrow O_{\mathcal{X}}(\mathcal{X}_{im}) \otimes_{\Gamma(\mathcal{S}, O_{\mathcal{S}})} O_{\mathcal{Y}}(\mathcal{Y}_{in})^{\otimes q+r+1} \tag{31}$$

We also take morphisms into account:

$$\left(O_{\mathcal{X}}(\mathcal{X}_{im}) \otimes_{\Gamma(\mathcal{S}, O_{\mathcal{S}})} O_{\mathcal{Y}}(\mathcal{Y}_{in})\right)^{\otimes q+r+1} \rightarrow \left(O_{\mathcal{X}}(\mathcal{X}_{im}) \otimes_{O_{\mathcal{S}}(\mathcal{S}_i)} O_{\mathcal{Y}}(\mathcal{Y}_{in})\right)^{\otimes q+r+1} = O_{\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}}\left(\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}\right)^{\otimes q+r+1} \tag{32}$$

resulting from natural morphism of algebras:

$$\left(O_{\mathcal{X}}(\mathcal{X}_{im}) \otimes_{\Gamma(\mathcal{S}, O_{\mathcal{S}})} O_{\mathcal{Y}}(\mathcal{Y}_{in})\right) \rightarrow \left(O_{\mathcal{X}}(\mathcal{X}_{im}) \otimes_{O_{\mathcal{S}}(\mathcal{S}_i)} O_{\mathcal{Y}}(\mathcal{Y}_{in})\right)$$

Natural morphisms are created by combining the morphisms in (31) and (32) and using:

$$\begin{gathered} p_{\mathcal{X}}^{-1}\left(Tot(\widetilde{BC}^{-q}(\mathcal{X}/\mathcal{S}))\right)(\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}) \otimes p_{\mathcal{Y}}^{-1}\left(Tot(\widetilde{BC}^{-r}(\mathcal{Y}/\mathcal{S}))\right)(\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}) \\ \downarrow \\ p_{\mathcal{X}}^{-1}\left(Tot(\widetilde{BC}^{-q}(\mathcal{X}/\mathcal{S}))\right)(\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}) \otimes_{\Gamma(\mathcal{S}, O_{\mathcal{S}})} p_{\mathcal{Y}}^{-1}\left(Tot(\widetilde{BC}^{-r}(\mathcal{Y}/\mathcal{S}))\right)(\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}) \\ \downarrow \\ \left(Tot(\widetilde{BC}^{-q-r}(\mathcal{X} \times_{\mathcal{S}_i} \mathcal{Y}/\mathcal{S}))\right)(\mathcal{X}_{im} \times_{\mathcal{S}_i} \mathcal{Y}_{in}) \end{gathered} \tag{33}$$

It follows from the morphisms in (33) that we have a morphism for complexes of sheaves of abelian groups on $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$, because the open sets $\{\mathcal{X} \times_{\mathcal{S}_i} \mathcal{Y}/\mathcal{S}\}_{i\in I, m\in M_i, n\in N_i}$ form a basis of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$:

$$p_{\mathcal{X}}^{-1}(Tot(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S}))) \otimes p_{\mathcal{Y}}^{-1}(Tot(\widetilde{BC}^{**}(\mathcal{Y}/\mathcal{S})) \rightarrow Tot(\widetilde{BC}^{**}(\mathcal{X} \times_{\mathcal{S}_i} \mathcal{Y}/\mathcal{S})). \tag{34}$$

The morphism in (34) therefore causes a multiplication on hypercohomology for any $q, r \in \mathbb{Z}$:

$$\mathbb{H}^{-q}\left(p_{\mathcal{X}}^{-1}(Tot(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S}))\right) \otimes \mathbb{H}^{-r}\left(p_{\mathcal{Y}}^{-1}(Tot(\widetilde{BC}^{**}(\mathcal{Y}/\mathcal{S}))\right) \rightarrow \mathbb{H}^{-q-r}(Tot(\widetilde{BC}^{**}(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}/\mathcal{S}))). \tag{35}$$

We have natural morphisms

$$\mathbb{H}^{-q}\left(\mathrm{Tot}(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S}))\right) \rightarrow \mathbb{H}^{-q}\left(p_{\mathcal{X}}^{-1}\left(\mathrm{Tot}(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S}))\right)\right) \tag{36}$$

and

$$\mathbb{H}^{-r}\left(Tot(\widetilde{BC}^{**}(\mathcal{Y}/\mathcal{S}))\right) \rightarrow \mathbb{H}^{-r}\left(p_{\mathcal{Y}}^{-1}\left(Tot(\widetilde{BC}^{**}(\mathcal{Y}/\mathcal{S}))\right)\right) \tag{37}$$

as a result of the universal hypercohomological qualities (see [6]). We have a multiplication when we combine this with (35).

$$\mathbb{H}^{-q}\left(Tot(\widetilde{BC}^{**}(\mathcal{X}/\mathcal{S}))\right) \otimes \mathbb{H}^{-r}\left(Tot(\widetilde{BC}^{**}(\mathcal{Y}/\mathcal{S}))\right) \rightarrow \mathbb{H}^{-q-r}\left(Tot(\widetilde{BC}^{**}(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}))\right). \tag{38}$$

Now, the outcome of (29) directly follows from the description of cyclic homology with regard to S.

Theorem (3-4):

A morphism of schemes over a $\mathcal{K}$-algebra, $f: \mathcal{X} \to \mathcal{S}$, shall exist. When this happens, the structure of a graded algebra is carried by $\otimes_{r\in\mathbb{Z}} \ \mathbb{HC}_r(\mathcal{X} \ \ /\mathcal{S})$.

Proof:

It follows from theorem (3-3) that we have a multiplication:

$$\mathbb{HC}_q(\mathcal{X} \ \ /\mathcal{S}) \otimes \mathbb{HC}_r(\mathcal{X} \ \ /\mathcal{S}) \to \mathbb{HC}_{q+r}(\mathcal{X} \times_{\mathcal{S}} \mathcal{X}/\mathcal{S}) \quad (39)$$

for every $q, r \in \mathbb{Z}$, the diagonal map $\Delta_{X/S} : X \to X \times_S X$ also induces the following morphism:

$$\Delta^*_{\mathcal{X}/\mathcal{S}}: \mathbb{HC}_{q+r}(\mathcal{X} \times_{\mathcal{S}} \mathcal{X}/\mathcal{S}) \to \mathbb{HC}_{q+r}(\mathcal{X} \ \ /\mathcal{S}). \quad (40)$$

The morphisms in (39) and (40), when combined, give us a product structure (41). This formula turns $\otimes_{r\in\mathbb{Z}} \ \mathbb{HC}_r(\mathcal{X}/\mathcal{S})$ into a graded algebra:

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \otimes \mathbb{HC}_r(\mathcal{X}/\mathcal{S}) \to \mathbb{HC}_{q+r}(\mathcal{X}/\mathcal{S}), \ \forall q, r \in \mathbb{Z}. \quad (41)$$

Theorem (3-5):

Let morphisms of schemes over a $\mathcal{K}$-algebra be $f: \mathcal{X} \to \mathcal{S}$ and $t: \mathcal{S}' \to \mathcal{S}$. Following that, we have a product structure by setting $\mathcal{X}_{\mathcal{S}'} = \mathcal{X} \times_{\mathcal{S}} \mathcal{S}''$

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \otimes \mathbb{HC}_r(\mathcal{S}'/\mathcal{S}) \to \mathbb{HC}_{q+r}(\mathcal{X}_{\mathcal{S}'}/\mathcal{S}'), \forall q, r \in \mathbb{Z}. \quad (42)$$

Proof:

It follows from theorem (3-4) that for every $q, r \in \mathbb{Z}$, we have a product "$\mathbb{HC}_q(\mathcal{X}/\mathcal{S})$" and "$\mathbb{HC}_r(\mathcal{S}'/\mathcal{S})$": The equation is

$$\mathbb{HC}_q(\mathcal{X}/\mathcal{S}) \otimes \mathbb{HC}_r(\mathcal{S}'/\mathcal{S}) \to \mathbb{HC}_{q+r}(\mathcal{X}_{\mathcal{S}'} \ \ /\mathcal{S}') = \mathbb{HC}_{q+r}(\mathcal{X}_{\mathcal{S}'} \ \ /\mathcal{S}). \quad (43)$$

Additionally, we observe that the natural morphisms

$$CC_h^{-q-r}(\mathcal{X}_{\mathcal{S}'} \ \ /\mathcal{S}) = O_{\mathcal{X}_{\mathcal{S}'}}^{\otimes_{\Gamma(\mathcal{S},O_{\mathcal{S}})}q+r+1} \to O_{\mathcal{X}_{\mathcal{S}'}}^{\otimes_{\Gamma(\mathcal{S}',O_{\mathcal{S}'})}q+r+1} = CC_h^{-q-r}(\mathcal{X}_{\mathcal{S}'} \ \ /\mathcal{S}') \quad (44)$$

create complicated morphisms $Tot(\widetilde{BC}^{**}(\mathcal{X}_{\mathcal{S}'}/\mathcal{S})) \to Tot(\widetilde{BC}^{**}(\mathcal{X}_{\mathcal{S}'}/\mathcal{S}))$ and, as a result, a morphism of hypercohomologies

$$\mathbb{HC}_{q+r}(\mathcal{X}_{\mathcal{S}'}/\mathcal{S}) = \mathbb{H}^{-q-r}(Tot(\widetilde{BC}^{**}(\mathcal{X}_{\mathcal{S}'}/\mathcal{S}))) \to \mathbb{H}^{-q-r}(ot(\widetilde{BC}^{**}(\mathcal{X}_{\mathcal{S}'} \ \ /\mathcal{S}'))) \ = \mathbb{HC}_{q+r}(\mathcal{X}_{\mathcal{S}'}/\mathcal{S}') \quad (45)$$

(43) and (44) together produce (42).

5. Some Properties of the Cyclic Homology in Schemes

During this section, the relations between dihedral and cyclic homology for Scheme algebra are obtained. We will discuss the trace and inclusion map of cyclic homology for Scheme algebra.

Theorem (4-1):

For any two cyclic modules $\mathcal{U}$ and $\mathcal{V}$, and let $\mathcal{V}$ and $H_*(\mathcal{V})$ be projective over $\mathcal{K}$-algebra. Next comes a chronological long exact sequence:

$$\cdots \to \mathbb{HC}_n(\mathcal{U} \times \mathcal{V}) \xrightarrow{i} \bigoplus_{r+s=n} \mathbb{HC}_r(\mathcal{U}) \otimes \mathbb{HC}_s \ \ B(\mathcal{V}) \xrightarrow{S\otimes -1-\otimes S} \bigoplus_{p+q=n-2} \mathbb{HC}_p(\mathcal{U}) \otimes \mathbb{HC}_q(\mathcal{V}) \xrightarrow{\partial} \mathbb{HC}_{n-1}(\mathcal{U} \times \mathcal{V}) \to \cdots. \quad (46)$$

Proof:

First, we shall demonstrate that the shuffle product and the cyclic shuffle product create a canonical

$$sh: \ \mathbb{HC}_*(\mathcal{U} \otimes \mathcal{V}) \cong \mathbb{HC}_*(\mathcal{U} \times \mathcal{V}). \quad (47)$$

It commutes with the precise sequence of the morphisms B, I, and S of cones' exact sequence. As following:

Since B does not commute with sh, the shuffle map $sh: \mathcal{U} \otimes \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ is the map of b -complexes where $[b, sh] = 0$ but not a map of mixed complexes. The standard Eilenberg-Zilber theorem states that sh is a quasi-isomorphism on Hochschild homology. In order to show the declared isomorphism, it is sufficient to provide a degree 0 map $\mathrm{sh}: \mathrm{Tot} \ \ \mathrm{B}(\mathcal{U} \otimes \mathcal{V}) \to \mathrm{Tot} \ \ \mathrm{B}(\mathcal{U} \times \mathcal{V})$ such that the following diagram commutes.

$$\begin{array}{ccccccc} 0 \to \mathcal{U} \otimes \mathcal{V} & \to & Tot \ \ B(\mathcal{U} \otimes \mathcal{V}) & \to & Tot \ \ B(\mathcal{U} \otimes \mathcal{V})[2] \to 0 \\ \downarrow sh & & \downarrow sh & & \downarrow sh[2] \\ 0 \to \mathcal{U} \times \mathcal{V} & \to & Tot \ \ B(\mathcal{U} \times \mathcal{V}) & \to & Tot \ \ B(\mathcal{U} \times \mathcal{V})[2] \to 0 \end{array} \quad (48)$$

Remember that $Tot \ \ B(\mathcal{U} \times \mathcal{V})$ is equivalent to

$$(Tot \ \ B(\mathcal{U} \otimes \mathcal{V}))_n = (\mathcal{U} \otimes \mathcal{V})_n \oplus (\mathcal{U} \otimes \mathcal{V})_{n-2} \oplus \cdots,. \quad (49)$$

Thus, sh can be a matrix. Our selection for sh is

$$sh = \begin{bmatrix} sh & sh' & 0 & & \\ & sh & sh' & 0 & \\ & & sh & sh' & \\ & & & . & \end{bmatrix}, \quad (50)$$

where

$$sh': (\mathcal{U} \otimes \mathcal{V})_n = \bigoplus_{p+q=n} C_p \otimes C_q \to (\mathcal{U} \times \mathcal{V})_{n+2}. \quad (51)$$

Due to the fact that the maps b, B, sh, and sh' meet the following formulas in the normalised setting, $[b, sh] = 0$ and $[B, sh] + [b, sh'] = 0$, $[B, sh'] = 0$. Let's identify $Tot \ \ B(\mathcal{U})$ with $\mathcal{K}[u] \otimes \mathcal{U}$ where $|u| = 2$ and $Tot \ \ B(\mathcal{V})$ with $\mathcal{K}[u'] \otimes \mathcal{V}$ where $|v| = 2$ in order to compute the kernel. Since $S: \mathcal{K}[u] \otimes \mathcal{U} \to \mathcal{K}[u] \otimes \mathcal{U}$ is given by $S(u^n \otimes x) = u^{n-1} \otimes x$, it can be shown that the elements produced by $\sum_{p+q=n} u^p u'^q x x'$ are what make up the kernel of $S \otimes 1 - 1 - \otimes S$.

As a result,

$$\triangle: k[v] \otimes (\mathcal{U} \otimes \mathcal{V}) \to k[u] \otimes \mathcal{U} \otimes k[u'] \otimes \mathcal{V} \cong k[u, u'] \otimes (\mathcal{U} \otimes \mathcal{V}) \quad (52)$$

is caused by $\mathrm{v}^{\mathrm{n}} \to \sum_{\mathrm{p+q=n}} \mathrm{u}^{\mathrm{p}} \mathrm{u}'^{\mathrm{q}}$. We check that the boundary of $Tot \ \ B(\mathcal{U}) \otimes Tot \ \ B(\mathcal{V})$ restricted to the image of matches with $Tot\ \ B(\mathcal{U}\otimes\mathcal{V})$ is a simple process. According to that, $Tot\ \ B(\mathcal{U})\otimes Tot\ \ B(\mathcal{V})$ homology is exactly $\mathbb{HC}_*(\mathcal{U})\otimes\mathbb{HC}_*(\mathcal{V})$. Consequently, the homology exact sequence is the precise sequence of:

$$0\to Tot\ \ B(\mathcal{U}\otimes\mathcal{V})\overset{\Delta}{\to} Tot\ \ B(\mathcal{U})\otimes Tot\ \ B(\mathcal{V}) \xrightarrow{S\otimes 1-1-\otimes S}(Tot\ \ B(\mathcal{U})\otimes Tot\ \ B(\mathcal{V}))[2]\to 0. \quad (53)$$

Thus we can obtain the exact long sequence

$$\cdots\to\mathbb{HC}_n(\mathcal{U}\times\mathcal{V})\overset{i}{\to}\bigoplus_{r+s=n}\mathbb{HC}_r(\mathcal{U})\otimes\mathbb{HC}_s\ \ B(\mathcal{V})\xrightarrow{S\otimes-1-\otimes S}$$

$$\bigoplus_{p+q=n-2}\mathbb{HC}_p(\mathcal{U})\otimes\mathbb{HC}_q(\mathcal{V})\overset{\partial}{\to}\mathbb{HC}_{n-1}(\mathcal{U}\times\mathcal{V})\to\cdots. \quad (54)$$

Theorem (4-2):

For any algebra $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$, we can write a commutative diagram of brief precise sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{U} & \overset{f}{\to} & \mathcal{V} & \overset{g}{\to} & \mathcal{W} & \to & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{U}' & \overset{f'}{\to} & \mathcal{V}' & \overset{g'}{\to} & \mathcal{W}' & \to & 0\end{array}, \quad (55)$$

Then we get a commutative diagram of long exact sequences:

$$\begin{array}{c} \cdots\overset{\delta}{\to}\mathbb{HC}_n(\mathcal{U})\overset{f}{\to}\mathbb{HC}_n(\mathcal{V})\overset{g}{\to}\mathbb{HC}_n(\mathcal{W})\overset{\delta}{\to}\mathbb{HC}_{n-1}(\mathcal{U})\overset{f}{\to}\cdots\\ \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow\\ \cdots\overset{\delta}{\to}\mathbb{HC}_n(\mathcal{U}')\overset{f'}{\to}\mathbb{HC}_n(\mathcal{V}')\overset{g'}{\to}\mathbb{HC}_n(\mathcal{W}')\overset{\delta}{\to}\mathbb{HC}_{n-1}(\mathcal{U}')\overset{f'}{\to}\cdots\end{array}. \quad (56)$$

Proof:

If we use the [7], [8], and [9], then the next commutative diagram which the bottom and top rows is exact;

$$\begin{array}{c} \mathcal{U}\to\mathcal{V}\overset{p}{\to}\mathcal{W}\to 0\\ a\downarrow\ \ b\downarrow\ \ d\downarrow\\ 0\to\mathcal{U}'\overset{i}{\to}\mathcal{V}'\to\mathcal{W}'\end{array}, \quad (57)$$

There exists an exact sequence

$$ker\ a\to ker\ b\to ker\ d\overset{\delta}{\to} coker\ \ a\to coker\ \ b\to coker\ \ d, \quad (58)$$

with δ is homomorphism

$$\delta(x) \coloneqq (i^{-1}\circ b\circ p^{-1})(x), \qquad \forall\ \ x\in ker\,\mathcal{W}. \quad (59)$$

If we take the commutative diagram

$$\begin{array}{ccccccc} \mathcal{U}_n/im\ \ d^{\mathcal{A}}_{n+1} & \to & \mathcal{V}_n/im\ \ d^{\mathcal{V}}_{n+1} & \overset{p}{\to} & \mathcal{W}_n/im\ \ d^{\mathcal{W}}_{n+1} & \to 0\\ d^{\mathcal{U}}\downarrow & & d^{\mathcal{V}}\downarrow & & d^{\mathcal{W}}\downarrow & \\ 0\to ker\,d^{\mathcal{U}}_{n+1} & \overset{i}{\to} & ker\,d^{\mathcal{V}}_{n+1} & \to & ker\,d^{\mathcal{W}}_{n+1} & \end{array}. \quad (60)$$

The diagram suggests that we must check that the top and bottom rows are exact;

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0\to & & ker\,d^{\mathcal{U}}_n & \to & ker\,d^{\mathcal{V}}_n & \to & ker\,d^{\mathcal{W}}_n & & \to 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0\to & & \mathcal{U}_n & \to & \mathcal{V}_n & \to & \mathcal{W}_n & & \to 0\\ & & d^{\mathcal{U}}_n\downarrow & & d^{\mathcal{V}}_n\downarrow & & d^{\mathcal{W}}_n\downarrow & & \\ 0\to & & \mathcal{U}_{n-1} & \to & \mathcal{V}_{n-1} & \to & \mathcal{W}_{n-1} & & \to 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathcal{U}_{n-1}/im\ \ d^{\mathcal{U}}_n & & \mathcal{V}_{n-1}/im\ \ d^{\mathcal{V}}_n & & \mathcal{W}_{n-1}/im\ \ d^{\mathcal{W}}_n & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \quad (61)$$

In the commutative diagram (60), we have

$$ker(\mathcal{U}_n/imd^{\mathcal{U}}_{n+1}\to ker\,d^{\mathcal{U}}_n)\ \ =\mathbb{HC}_n(\mathcal{U}) \quad (62)$$

$$coker(\mathcal{U}_n/imd^{\mathcal{U}}_{n+1}\to ker\,d^{\mathcal{U}}_n)\ \ =\mathbb{HC}_{n-1}(\mathcal{U}), \quad (63)$$

likewise with regard to the other two columns. We get

$$\mathbb{HC}_n(\mathcal{U}_*)\overset{f_*}{\to}\mathbb{HC}_n(\mathcal{V}_*)\overset{g_*}{\to}\mathbb{HC}_n(\mathcal{W}_*)\overset{\delta}{\to}\mathbb{HC}_{n-1}(\mathcal{U}_*)\overset{f_*}{\to}\mathbb{HC}_{n-1}(\mathcal{V}_*)\overset{g_*}{\to}\mathbb{HC}_{n-1}(\mathcal{W}_*). \quad (64)$$

Theorem (4-3):

Suppose that $\mathcal{U}$ is Scheme algebra and let $\mathcal{V}$ be H-untial over $\mathcal{K}$. Then we have two maps

$$inc\colon\mathbb{HC}_n(\mathcal{U},\mathcal{V})\to\mathbb{HC}_n(M_m(\mathcal{U}),M_m(\mathcal{V})), \quad (65)$$

and

$$tr\colon\mathbb{HC}_n\big(M_m(\mathcal{U}),M_m(\mathcal{V})\big)\to\mathbb{HC}_n(\ \ \mathcal{U},\mathcal{V}\ \ ), \quad (66)$$

as each of them is inverse to the other and isomorphisms.

Proof:

A commutative diagram is the extension morphism:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{V} & \to & \mathcal{V}_+ & \to & \mathcal{K} & \to & 0\\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & M_m(\mathcal{V}) & \to & M_m(\mathcal{V}_+) & \to & M_m(\mathcal{K}) & \to & 0\end{array}, \quad (67)$$

where the horizontal and vertical mappings are implemented, the rows are precise, and isomorphism may be seen in the left vertical arrow by virtue of the algebraic extension property. Using [10], [11] and the cyclic homology of the upper picture, we can obtain

$$\begin{array}{ccccccc} 0\to & \mathbb{HC}_n(\mathcal{V}) & \to & \mathbb{HC}_n(\mathcal{V}_+) & \to & \mathbb{HC}_n(\mathcal{K}) & \to 0\\ & \uparrow & & \uparrow & & \uparrow & \\ 0\to & \mathbb{HC}_n(M_m(\mathcal{V})) & \to & \mathbb{HC}_n(M_m(\mathcal{V}_+)) & \to & \mathbb{HC}_n(M_m(\mathcal{K})) & \to 0\end{array}. \quad (68)$$

Consequently, the right vertical arrow is isomorphic, and we get a commutative diagram:

$$\begin{array}{ccc} \mathbb{HC}_n(M_m(\mathcal{U}),M_m(\mathcal{V})) & \overset{\cong}{\to} & \mathbb{HC}_n(M_m(\mathcal{U}),M_m(\mathcal{V}_+))\\ inc\uparrow & & inc\uparrow\\ \mathbb{HC}_n(\mathcal{U},\mathcal{V}) & \overset{\cong}{\to} & \mathbb{HC}_n(M_m(\mathcal{U}),M_m(\mathcal{V}_+))\end{array}. \quad (69)$$

If the right inc is an isomorphism according to Morita invariance for unital algebra, then it is an isomorphism:

$$inc : \mathbb{HC}_n\big(M_m(\mathrm{g},\mathcal{M})\otimes M_m(\mathfrak{I},\mathcal{M})\big) \to \mathbb{HC}_n(\mathrm{g}\otimes\mathfrak{I},\mathcal{M}). \quad (70)$$

Thus, tr is a right inverse of inc.

Theorem (4-4):

For any $\mathcal{U}$ scheme algebra, we obtain a relation the dihedral homology $\mathbb{HD}(\mathcal{U})$ and the cyclic homology $\mathbb{HC}(\mathcal{U})$ is defined as:

$$\mathbb{HC}_n(\mathcal{U}) = {}^{-}\mathbb{HD}_{n+1}(\mathcal{U})\oplus {}^{+}\mathbb{HD}_n(\mathcal{U}). \quad (71)$$

Proof:

By using [12], [13], and [14], let $\mathcal{U}$ be scheme algebra over k, and take the total complex and tricomplex of $\mathcal{CC}\ (\mathcal{U})$. We get the following short exact sequence:

$$0 \to Tot\mathcal{C}\ (\mathcal{U}) \to Tot^{\alpha}D\ (\mathcal{U}) \to Tot^{-\alpha}D\ (\mathcal{U})[-4] \to 0, \quad (72)$$

where $\mathsf{D}\ (\mathcal{U})$ is tricomplex. We can represent the short sequence by the long sequence as follows:

$$\cdots \to {}^{-}\mathbb{HD}_{n+1}(\mathcal{U}) \to {}^{+}\mathbb{HD}_n(\mathcal{U}) \xrightarrow{j^*} \mathbb{HC}_n(\mathcal{U}) \xrightarrow{i^*} {}^{-}\mathbb{HD}_n(\mathcal{U}) \to \cdots, \quad (73)$$

where j^* is a connecting homomorphism, where the long sequence is a relationship between cyclic homology and dihedral homology, we get the following exact sequence:

$$0 \to {}^{+}\mathbb{HD}_n(\mathcal{U}) \to \mathbb{HC}_n(\mathcal{U}) \to {}^{-}\mathbb{HD}_n(\mathcal{U}) \to 0. \quad (74)$$

Remark (4-5):

For any two modules $\mathcal{U}$ and $\mathcal{V}$, we get the long exact sequence:

$$\cdots \to \mathbb{HD}_n(\mathcal{U}\times\mathcal{V}) \xrightarrow{i} \bigoplus_{r+s=n} \mathbb{HD}_r(\mathcal{U})\otimes\mathbb{HD}_s\ B(\mathcal{V}) \xrightarrow{S\otimes-1-\otimes S} \bigoplus_{p+q=n-2} \mathbb{HD}_p(\mathcal{U})\otimes\mathbb{HD}_q(\mathcal{V}) \xrightarrow{\partial} \mathbb{HD}_{n-1}(\mathcal{U}\times\mathcal{V}) \to \cdots. \quad (75)$$

Remark (4-6):

For all $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$ schemes, we give the long exact sequence of dihedral homology as:

$$\cdots \to \mathbb{HD}_n(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HD}_n(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HD}_n(\mathcal{W}_*) \xrightarrow{\delta} \mathbb{HD}_{n-1}(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HD}_{n-1}(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HD}_{n-1}(\mathcal{W}_*) \to \cdots. \quad (76)$$

6. Conclusions

The aim of this study was to extend the standard cyclic homology $\mathcal{HC}_*$ of $\mathcal{K}$-algebra by proving the existence of the cyclic homology theory $\mathbb{HC}_*$ of scheme on $\mathcal{K}$ - algebra. The aim was to give the product structures on $\mathbb{HC}_*(\mathcal{X}/\mathcal{S})$ cyclic homology groups. Particularly, we show that $\mathbb{HC}_*(\mathcal{X}/\mathcal{S}) = \otimes_{n\in\mathbb{Z}} \mathbb{HC}_n(\mathcal{X}/\mathcal{S})$ from the algebra. We investigated the relationship between dihedral homology $(\mathbb{HD}(\mathcal{U}))$ and cyclic homology $(\mathbb{HC}(\mathcal{U}))$ as follows:

$$\mathbb{HC}_n(\mathcal{U}) = {}^{-}\mathbb{HD}_{n+1}(\mathcal{U})\oplus {}^{+}\mathbb{HD}_n(\mathcal{U}).$$

We explained that the inverse between the trace map and the embedding map of the periodic symmetry of the algebra diagram would take the form of

$$inc: \mathbb{HC}_n(\mathcal{U},\mathcal{V}) \to \mathbb{HC}_n(M_m(\mathcal{U}), M_m(\mathcal{V}))$$

and

$$tr: \mathbb{HC}_n\big(M_m(\mathcal{U}), M_m(\mathcal{V})\big) \to \mathbb{HC}_n(\ \mathcal{U},\mathcal{V}\ ).$$

For the map $\mathrm{sh}: \mathcal{U}\otimes\mathcal{V} \to \mathcal{U}\times\mathcal{V}$, we obtained the exact long sequence of the cyclic homology of Scheme:

$$\cdots \to \mathbb{HC}_n(\mathcal{U}\times\mathcal{V}) \xrightarrow{i} \bigotimes_{r+s=n} \mathbb{HC}_r(\mathcal{U})\otimes\mathbb{HC}_s\ B(\mathcal{V}) \xrightarrow{S\otimes-1-\otimes S} \bigoplus_{p+q=n-2} \mathbb{HC}_p(\mathcal{U})\otimes\mathbb{HC}_q(\mathcal{V}) \xrightarrow{\partial} \mathbb{HC}_{n-1}(\mathcal{U}\times\mathcal{V}) \to \cdots.$$

For any three types of algebras $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$, we'll write the following long exact sequence as a commutative diagram:

$$\cdots \to \mathbb{HC}_n(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HC}_n(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HC}_n(\mathcal{W}_*) \xrightarrow{\delta} \mathbb{HC}_{n-1}(\mathcal{U}_*) \xrightarrow{f_*} \mathbb{HC}_{n-1}(\mathcal{V}_*) \xrightarrow{g_*} \mathbb{HC}_{n-1}(\mathcal{W}_*) \to \cdots.$$

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Contributions

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Ethics Approval

Not applicable.

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