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Some Convergence Properties of a Random Closed Set Sequence

Bourakadi Ahssaine[1,*], Baraka Achraf Chakir[1], Khalifi Hamid[2]

[1]Department of Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco
[2]Department of Informatics, Faculty of Sciences, Mohammed V University, Rabat, Morocco

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Abstract In this article, we have discussed the properties of the probability law "T" called functional capacity and other closely related functionals "Q and C" pertaining to random closed sets. We are interested in the most widely used functional in random set theory "T". We have established the belonging of "T" to the interval [0,1], and proven that it is increasing in the sense of inclusion, and its sub-additivity property through probability techniques. Moreover, we have explored the various types of convergences of a sequence of random closed sets, such as weak convergence, strong convergence (almost surely in the sense of Hausdorff), convergence in the sense of Painlevé-Kuratowski and Wijsman-Mosco, as well as convergence in probability. In the second part of our work, we have proven a new corollary which states that the strong convergence in the sense of Hausdorff implies the convergence in probability of a sequence of random closed sets at infinity. Our proof involves the definition of mathematical expectation for a discrete variable and the indicator variable, which is a random variable that takes two possible values, 0 or 1.

Keywords Random Closed Set, Convergence in Probability, Weak Convergence, Convergence in the Sense of Hausdorff

1. Introduction

It is interesting to note that sequences of random closed sets [1] also have properties of convergence at infinity, similar to sequences of random variables and vectors. In the literature, two types of convergence are commonly discussed. The first is weak convergence of a sequence of random closed set X_n to a random closed set X, and we generally prove this type of convergence by using the convergence of the probability law of sequence X_n at infinity to the probability law of X. The second type of convergence is strong convergence, or convergence in the sense of Hausdorff. This happens when the sequence X_n converges almost surely to X as $n \to \infty$.

2. Definition of a Random Closed Set

In random set theory [2, 3], X is called a random closed set in the Euclidean space IR^d, if X is an application from (Ω, T, P) to the closed set family F in IR^d and $X^-(K) = \{\omega \in \Omega: X(\omega) \cap K \neq \emptyset\}$ belongs to the σ-algebra T on Ω for every compact K in IR^d.

2.1. Definition of Functional Capacity

The functional capacity of a random set X [4,5] is an application starting from Κ the family of compact sets from IR^d to $[0,1]$, defined by the following formula:

$$T_X(K) = P(X \cap K \neq \emptyset) \tag{1}$$

with $K \in$ Κ.

If $X = \{x\}$, then:

$$T_X(K) = P(\{x\} \cap K \neq \emptyset) = P(x \in K)$$

and in this case, T_X becomes the probability distribution of the random variable x.

There are the other functionals generated by the random set X called Avoidance Functional Q and Containment Functional C.

The Avoidance Functional Q verifies the following property:

For each $K \in Ƙ$, we have:

$$Q(K) = 1 - T_X(K) \quad (2)$$

Following the equation (2), the Avoidance Functional Q can be written as follows:

$$Q(K) = P\{X \cap K = \emptyset\} = P(X \subset K^C) \quad (3)$$

where K^C is the complementary of K.

On the other hand, the Containment Functional C is defined on the family of all closed sets F and gives the capacity function on the open sets $G = F^C$ as follows:

$$T_X(G) = P\{X \cap G \neq \emptyset\} = 1 - P(X \subseteq G^C) = 1 - C(F) \quad (4)$$

2.2. Property of the functional Capacity T_X

Among the properties of the functional capacity, we mention the following assertions that we have proven:

1. $T_X(\emptyset) = 0$
2. $0 \leq T_X(K) \leq 1, K \in Ƙ$
3. $T_X(K_1) \leq T_X(K_2)$ if $K_1 \subset K_2$
4. $T_X(K_1 \cup K_2) \leq T_X(K_1) + T_X(K_2)$
5. $T(K_1 \cup K_2 \cup K) + T(K) \leq T(K_1 \cup K) + T(K_2 \cup K)$

Proof:

We write T instead of T_X

1. $T(\emptyset) = 0$

We have: $T(\emptyset) = P\{X \cap \emptyset \neq \emptyset\} = 0$

2. $0 \leq T(K) \leq 1, K \in Ƙ$

We have $0 \leq P\{X \cap K \neq \emptyset\} \leq 1$, then:
$0 \leq T(K) \leq 1$.

3. $T(K_1) \leq T(K_2)$ if $K_1 \subset K_2$

For K_1, K_2 in Ƙ if $K_1 \subset K_2$, then:
$X \cap K_1 \subset X \cap K_2$ so if $X \cap K_2 = \emptyset$
Then: $X \cap K_1 = \emptyset$.
And the event
$\{X(\omega) \cap K_2 = \emptyset\} \subset \{X(\omega) \cap K_1 = \emptyset\}$ then: $P(\{X(\omega) \cap K_2 = \emptyset\}) \leq P(\{X(\omega) \cap K_1 = \emptyset\})$
That is

$$P(X \cap K_2 = \emptyset) \leq P(X \cap K_1 = \emptyset)$$

Therefore:

$$1 - P(X \cap K_1 = \emptyset) \leq 1 - P(X \cap K_2 = \emptyset)$$

This implies $P(X \cap K_1 \neq \emptyset) \leq P(X \cap K_2 \neq \emptyset)$
And we conclude that $T(K_1) \leq T(K_2)$ hence property 3.

4. $T(K_1 \cup K_2) \leq T(K_1) + T(K_2)$

We have:

$$T(K_1 \cup K_2) = P(X \cap K_1 \cup K_2 \neq \emptyset) = P((X \cap K_1 \neq \emptyset) \cup (X \cap K_2 \neq \emptyset))$$

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$ for two events $A, B \subset \Omega$.
Then:

$$P((X \cap K_1 \neq \emptyset) \cup (X \cap K_2 \neq \emptyset)) \leq P(X \cap K_1 \neq \emptyset) + P(X \cap K_2 \neq \emptyset).$$

So we conclude that:
$T(K_1 \cup K_2) \leq T(K_1) + T(K_2)$ therefore the sub addivity of the functional capacity.

5. $T(K_1 \cup K_2 \cup K) + T(K) \leq T(K_1 \cup K) + T(K_2 \cup K)$

For all compact sets K, K_1, K_2 in Ƙ we use:

$$\begin{aligned} 0 &\leq P\{X \cap K = \emptyset, X \cap K_1 \neq \emptyset, X \cap K_2 \neq \emptyset\} \\ &= P\{X \cap K = \emptyset, X \cap K_1 \neq \emptyset\} - P\{X \cap (K \cup K_2) = \emptyset, X \cap K_1 \neq \emptyset\} \\ &= P\{X \cap K = \emptyset\} - P\{X \cap (K \cup K_1) = \emptyset\} - P\{X \cap (K \cup K_2) = \emptyset\} + P\{X \cap (K \cup K_1 \cup K_2) = \emptyset\} \\ &= T(K \cup K_2) + T(K \cup K_1) - T(K) - T(K \cup K_1 \cup K_2) \end{aligned}$$

3. Convergences of Sequences of Random Sets

3.1. Convergence in the Sense of Painlevé-Kuratowski

Let (E, τ) be a metrisable topological space, given a sequence of sets $X_1, X_2, \ldots, X_n$ of the topological space E [6], then the upper and lower (topological) bounds of the sequence $\{X_n, n \in IN\}$ are defined by the following formulas:

$$LiX_n = \{x \in X: \exists (x_n) \rightarrow x \ with \ x_n \ X_n, \ \forall \ n \in IN\} \quad (5)$$

$$LsX_n = \{x \in X: \exists \ n_1, n_2, \ldots n_k \ldots and \ \forall k \in IN, \ x_n \in X_n(k) \ with \ a_n \rightarrow x\} \quad (6)$$

In other words, LiX_n is the set formed by all possible limits of sequences $\{x_n, n \in IN\}$ with $x_n \in X_n$ for all $n \in IN$, while LsX_n is formed by all adherence values of such sequences.

Given a sequence $\{X_n, n \in IN\}$, we can always define those sets LiX_n and LsX_n which can be possibly empty, and we have the inclusion:

$$LiX_n \subset LsX_n$$

When equality takes place, the sequence $\{X_n, n \in IN\}$ is said to converge, or more precisely, to converge in the Painlevé-Kuratowski sense, and we note:

$$LimX_n = LiX_n = LsX_n \quad (7)$$

3.2. Convergence in the Sense of Wijsman-Mosco

When the topology τ is metrizable, noting d a distance inducing this topology, and by posing:

$$d(x, X) = inf_{y \in X} \ d(x, y) \quad (8)$$

We can reformulate the topological limits of sequences of sets:

$$LiX_n = \{x \in X: limsup_{n\to\infty} d(x, X_n) = 0\} = \{x \in X: lim_{n\to\infty} d(x, X_n)) = 0\} =$$

$$LsX_n = \{x \in X \ liminf_{n\to\infty} \ d(x, X_n) = 0. \quad (9)$$

The above formulas naturally suggest the following notion of convergence:

The sequence $\{X_n, n \in IN\}$ converges in the Wijsman-Mosco sense to X ([7]), then we note:

$W - limX_n = X$, if for each $x \in E$, we have

$$lim_{n\to\infty} \ d(x, X_n) = d(x, X) \quad (10)$$

Convergence in the Wijsman sense involves convergence in the Kuratowski-Painlevé sense. It is in general, strictly stronger than the latter. This convergence allows us to topologize the convergence of sets in a simple way: identifying a set A with the distance function d (., A), and we consider the topology provided by the simple convergence of distance functions.

Wijsman's topology proves to be of delicate use in infinite dimensions because it depends on the choice of the metric. When we restrict ourselves to closed convexes of a Hilbert space and take for d the distance associated with the metric, we obtain the Mosco-convergence.

Let $X_n, n \in IN$ be a sequence of closed sets of a normed space E. We say that the sequence X_n Mosco-converges to X, and we write $X = M - limX_n$, if we have the following two properties:

- For any x in X there exists a sequence x_n converges strongly to x such that x_n belongs to X_n for all $n \in IN$.
- For any sub-sequence $n(1) < n(2) < n(3) \ldots$ and for each k in IN, and $x_k \in X_{n(k)}$, with x_k converges weakly to x, we have $x \in X$.

Generally, the types of convergences seen previously are based on the elements of the set sequence studied, on the other hand, convergence in the sense of Hausdorff takes into consideration the mathematical structure of the whole set sequence, while dealing with the behavior of the Hausdorff distance to infinity, and this is why Hausdorff sence convergence remains the strongest and the most reliable in the treatment of sequences of random sets.

3.3. Convergence in the Sense of Hausdorff

The space $(E, \|.\|)$ denoted a normed space of closed unit ball B.

Let $C, D \subset E$, we note the excess of the set C over D defined by: $e(C, D) = sup_{x \in C} \ d(x, D)$, with the convention, $e = 0$ if $C = \emptyset$.

The Hausdorff distance between C and D is defined by:

$$d_H(C, D) = max \ \{e(C, D)), e(D, C))\} \quad (11)$$

According to the formula (11), we obtain the following formula:

$$d_H(C, D) = max \ \{sup_{x \in C} \ d(x, D), sup_{x \in D} \ d(x, C)\} \quad (12)$$

This distance can also be defined by the following formula:

$$d_H(C, D) = \inf\{r: X \subset Y^r, Y \subset X^r\} \quad (13)$$

With:

$X^r = \{a \in E, d(a, X) \leq r\}$, $Y^r = \{b \in E, d(b, Y) \leq r\}$, and d verifies the property:

$d(x, X) = inf_{y \in X} \ d(x, y)$, and $\|X\| = d_H(X, \{0\})$

This metric which plays a fundamental role in analysis is often too strong when one has a menu of unbounded sets such as epigraphs, cones, graphs...

Then we call a sequence $X_n \subset E, n \in IN$ of parts of E converges in the sense of Hausdorff distances to a set X [8], if $lim_{n\to\infty} \ d_H(X_n, X) = 0$, and we denote X_n converges almost surely to X (strong convergence).

3.4. Weak Convergence

Consider the sequence of random sets $X_n, n \in IN$, of distribution P_n. The sequence X_n is said to be weakly convergent or converges in distribution (d) to a random closed set X with a distribution P, if P_n converges weakly to P.

That is to say for any K in Ꝁ, knowing $(\partial K) = 0$ (∂K the boundary of K which is defined with respect to the topology on F on the σ-Effros algebra).

We have: $P_n(K) \rightarrow_{n\infty} P(K)$, and we note: $X_n \rightarrow^d X$ as $n \rightarrow \infty$.

3.5. The Weak Convergence Using Capacity Functional

In a locally compact and countable Hausdorff space E, the sequence of a random closed set $X_n, n \in IN$ converges weakly to a random closed set X, if and only if for any K in family of compact set Ꝁ in E, we have:

$$T_{X_n}(K) \rightarrow_{n\infty} T_X(K).$$

3.6. Convergence in Probability

The sequence of random closed sets $X_n, n \in IN$ in a locally compact and separable space E is said to converge in probability (P) with respect to the Hausdorff measure to a random closed set X, if and only if:

For any $\varepsilon > 0, P(d_H(X_n, X) > \varepsilon) \to_{n\infty} 0$ with ε is an infinitesimal number, then we note:

$$X_n \to^P X \text{ as } n \to \infty$$

After having mentioned the different types of convergences of a sequence of random closed sets, we end this article with the proof of the following corollary demonstrated by the authors, which says that if X_n converges almost surely to X (that is to say X_n converges in the sense of Hausdorff to X) then X_n converges in probability to X as $n \to \infty$.

3.7. Corollary

If X_n converges almost surely to X with respect to the Hausdorff metric d_H, then X_n converges in probability (P) to X with respect to the same metric.

Proof:

Suppose that $X_n \to^{d_H} X$ then $d_H(X_n, X) \to 0$ almost surely as $n \to \infty$.

For a discrete random variable X which takes two possible values 0 or 1, we have:

$$E(X) = \sum_{x \in \{0,1\}} xP(X = x) \quad (14)$$

where E is the mathematical expectation of the random variable X.

We define the indicator variable $1_{d_H(X_n,X)>\varepsilon}$ which takes two values 0 or 1 by:

$$1_{d_H(X_n,X)>\varepsilon} = \begin{cases} 1 \ \ if \ \ d_H(X_n, X) > \varepsilon \\ 0 \ \ if \ \ d_H(X_n, X) \le \varepsilon \end{cases} \quad (15)$$

We have:

$$P(d_H(X_n, X) > \varepsilon) = P\left(1_{d_H(X_n,X)>\varepsilon} = 1\right) \quad (16)$$

And:

$$E\left(1_{d_H(X_n,X)>\varepsilon}\right) = \sum_{x \in \{0,1\}} xP\left(1_{d_H(X_n,X)>\varepsilon} = x\right) \quad (17)$$

Then:

$$E\left(1_{d_H(X_n,X)>\varepsilon}\right) =$$
$$1 \times P\left(1_{d_H(X_n,X)>\varepsilon} = 1\right) + 0 \times P\left(1_{d_H(X_n,X)>\varepsilon} = 0\right)$$
$$= P\left(1_{d_H(X_n,X)>\varepsilon} = 1\right)$$

Therefore following the two equations (16) and (17), we obtain:

$$P(d_H(X_n, X) > \varepsilon)$$
$$= E\left(1_{d_H(X_n,X)>\varepsilon}\right) = E\left(1_{\frac{d_H(X_n,X)}{\varepsilon}>1}\right) \quad (18)$$

On the other hand, we have $\frac{d_H(X_n,X)}{\varepsilon} \ge 1$, and the indicator $1_{\frac{d_H(X_n,X)}{\varepsilon}>1}$ is always positive then:

$$1_{\frac{d_H(X_n,X)}{\varepsilon}>1} \le \frac{d_H(X_n, X)}{\varepsilon} \times 1_{\frac{d_H(X_n,X)}{\varepsilon}>1}$$

we know that the mathematical expectation E is always increasing, then:

$$E\left(1_{\frac{d_H(X_n,X)}{\varepsilon}>1}\right) \le E\left(\frac{d_H(X_n, X)}{\varepsilon} \times 1_{\frac{d_H(X_n,X)}{\varepsilon}>1}\right)$$

Then:

$$P(d_H(X_n, X) > \varepsilon) \le E\left(\frac{d_H(X_n, X)}{\varepsilon} \times 1_{\frac{d_H(X_n,X)}{\varepsilon}>1}\right)$$

That is:

$$P(d_H(X_n, X) > \varepsilon) \le \frac{d_H(X_n, X)}{\varepsilon} \times E\left(1_{\frac{d_H(X_n,X)}{\varepsilon}>1}\right)$$

Also we have:

$1_{\frac{d_H(X_n,X)}{\varepsilon}>1} = \begin{cases} 1 \quad if \quad d_H(X_n, X) > \varepsilon \\ 0 \ \ elseif \end{cases}$ that implies $E\left(1_{\frac{d_H(X_n,X)}{\varepsilon}>1}\right) \le 1$.

Then we obtain the inequality below which is the result of this demonstration:

$$P(d_H(X_n, X) > \varepsilon) < \frac{d_H(X_n,X)}{\varepsilon} \quad (19)$$

And as a final result if X_n converges almost surely to X with respect to the Hausdorff metric d_H, then:

$d_H(X_n, X) \to 0$ almost surely as $n \to \infty$, and following the inequality demonstrated above we find that:

$$P(d_H(X_n, X) > \varepsilon) \quad \to 0, \text{ as } n \to \infty.$$

Which implies the convergence in probability of the set sequence X_n to X.

And finally, we note that $X_n \to^P X$ if $X_n \to^{d_H} X$ as $n \to \infty$ almost surely.

4. Conclusions

The study of the properties of different types of convergences of sequences of random sets is very important, because it allows us to have an idea of their behavior at infinity. In this article, we have tried to prove some properties of the functional capacity that represents the law of a random closed set, (properties 1,2,3,4 and 5), and thus present the different types of convergence of the sequences of random closed sets, in particular the weak convergence and the strong convergence in the sense of Hausdorff and the convergence in probability, and as a result of this work, we succeeded in proving a new corollary that says that the strong convergence in the sense of Hausdorff entrains to the convergence in probability.

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