Mathematics and Statistics 11(5): 827-833, 2023 http://www.hrpub.org
DOI: 10.13189/ms.2023.110509

Neutrosophic Generalized Pareto Distribution

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Received February 6, 2023; Revised August 16, 2023; Accepted September 14, 2023

Cite This Paper in the Following Citation Styles

(a): *[1] Nahed I. Eassa, Hegazy M. Zaher, Noura A. T. Abu El-Magd , "Neutrosophic Generalized Pareto Distribution," Mathematics and Statistics, Vol. 11, No. 5, pp. 827 - 833, 2023. DOI: 10.13189/ms.2023.110509.*

(b): *Nahed I. Eassa, Hegazy M. Zaher, Noura A. T. Abu El-Magd (2023). Neutrosophic Generalized Pareto Distribution. Mathematics and Statistics, 11(5), 827 - 833. DOI: 10.13189/ms.2023.110509.*

Abstract The purpose of this paper is to present a neutrosophic form of the generalized Pareto distribution (NGPD) which is more flexible than the existing classical distribution and deals with indeterminate, incomplete and imprecise data in a flexible manner. In addition to this, NGPD will be obtained as a generalization of the neutrousophic Pareto distribution. Also, the paper introduces its special cases as neutrosophic Lomax distribution. The mathematical properties of the proposed distributions, such as mean, variance and moment generating function are derived. Additionally, the analysis of reliability properties, including survival and hazard rate functions, is mentioned. Furthermore, neutrosophic random variable for Pareto distribution was presented and recommended using it when data in the interval form follow a Pareto distribution and have some sort of indeterminacy. This research deals the statistical problems that have inaccurate and vague data. The proposed model NGPD is widely used in finance to model low probability events. So, it is applied to a real-world data set to modelling the public debt in Egypt for the purpose of dealing with neutrosophic scale and shape parameters, finally the conclusions are discussed.

Keywords Lomax Distribution, Neutrosophic Logic, Generalized Pareto Distributionwith Tow Parameters, Neutrosophic Pareto Distribution, Neutrosophic Exponential Distribution, Neutrosophic Uniform

1. Introduction

The Generalized Pareto distribution (GPD) is widely used in extreme value theory, engineering, industrial and finance. GPD is related to several distributions such as Exponential, Lomax, and Uniform distributions. To add flexibility to this model, neutrosophic Logic has been used. Neutrosophy was presented by Smarandache in 1995, as a generalization for the fuzzy logic and intuitionist fuzzy logic [1]. Fuzzy logic which is the special case of neutrosophic logic gives information only about the measures of truth and falseness. The neutrosophic logic gives information about the measure of indeterminacy additionally. The neutrosophic logic used the set analysis, where any type of set can be used to capture the data inside the intervals.

Neutrosophic statistics which utilize the idea of neutrosophic logic are found to be more efficient than classical statistics [1]. Neutrosophic statistics deal with the data having imprecise, interval, and uncertain observations. Neutrosophic statistics reduce to classical statistics when no indeterminacy is found in the data or the parameters of statistical distribution. Various applications of neutrosophic logic can be read in [2, 3].

Many researchers have introduced neutrosophic logic as an extended and generalized approach to the classical distributions such as neutrosophic Weibull distribution [4] and its several families, neutrosophic binomial distribution and neutrosophic normal distribution [5], neutrosophic multinomial distribution, neutrosophic Poisson [6], neutrosophic exponential [7,8], neutrosophic uniform distribution, neutrosophic gamma distribution [9], neutrosophic beta distribution [10], and neutrosophic Rayleigh model [11]. The neutrosophic Pareto distribution (NPD), generalization of the Pareto distribution is developed by Zahid khan et al. [12]. Almarashet and aslam [13] presented a repetitive sampling control chart for the gamma distribution under the indeterminate environment. This paper proposes neutrosophic Pareto distribution with neutrosophic random variables, Neutrosophic Lomax distribution and neutrosophic Generalized Pareto distributions.

The paper is organized as follows: The next section describes the neutrosophic generalized Pareto with two parameters and some special cases. In section 3, it studies the probability density function (pdf), cumulative density function (cdf), and hazard rate function of the neutrosophic Lomax distribution model. The mathematical statistics are studied in the subsequent section such as mean and variance.

Then, we will introduce the neutrosophic Pareto distribution model in section 4. Finally, section 5 concludes the research outcomes.

2. The Neutrosophic Generalized Pareto Distribution (NGPD)

The neutrosophic generalized Pareto distribution is a family of continuous probability distributions. It is often used to model the tails of another distribution. NGPD will be obtained as a generalization of the neutrousophic Pareto distribution given by Zahed Khan et al. [12]. The NGPD is related to several distributions such as neutrosophic Exponential, Uniform distribution which is introduced by Carlos Granadosa et al. [14] and neutrosophic Lomax distribution, as it will be shown below.

We can have defined the NGPD with neutrosophic parameters as follows:

- The probability density function (*pdf*) is given by:

$$f(x,\alpha_N,\beta_N)=\frac{1}{\beta_N}(1+\frac{\alpha_N x}{\beta_N})^{-\frac{1}{\alpha_N}-1}\,; x>0\,,\alpha_N,\beta_N>0 \quad (1)$$

when shape parameter $\alpha_N=0$, the density is:

$$f(x,0,\beta_N)=\frac{1}{\beta_N}e^{-(x)/\beta_N}\quad ; x>0\quad ,\beta_N>0 \quad (2)$$

(Exponential distribution)

- The cumulative distribution function (cdf):

$$F(x,\alpha_N,\beta_N)=1-\left(1+\frac{\alpha_N x}{\beta_N}\right)^{\frac{-1}{\alpha_N}};\quad \alpha_N\neq 0,\beta_N>0 \quad (3)$$

$$F(x,\alpha_N=0,\beta_N)=1-e^{-\frac{x}{\beta_N}};\ \alpha_N=0,\beta_N>0$$

$$x\geq 0\,, when\alpha_N\geq 0\ and\ 0\leq x\leq -\frac{\beta_N}{\alpha_N}when\alpha_N<0 \quad (4)$$

- The hazard rate function:

$$h(x,\alpha_N,\beta_N)=(\beta_N x+\alpha_N)^{-1};\ x>0,\alpha_N,\ \beta_N>0 \quad (5)$$

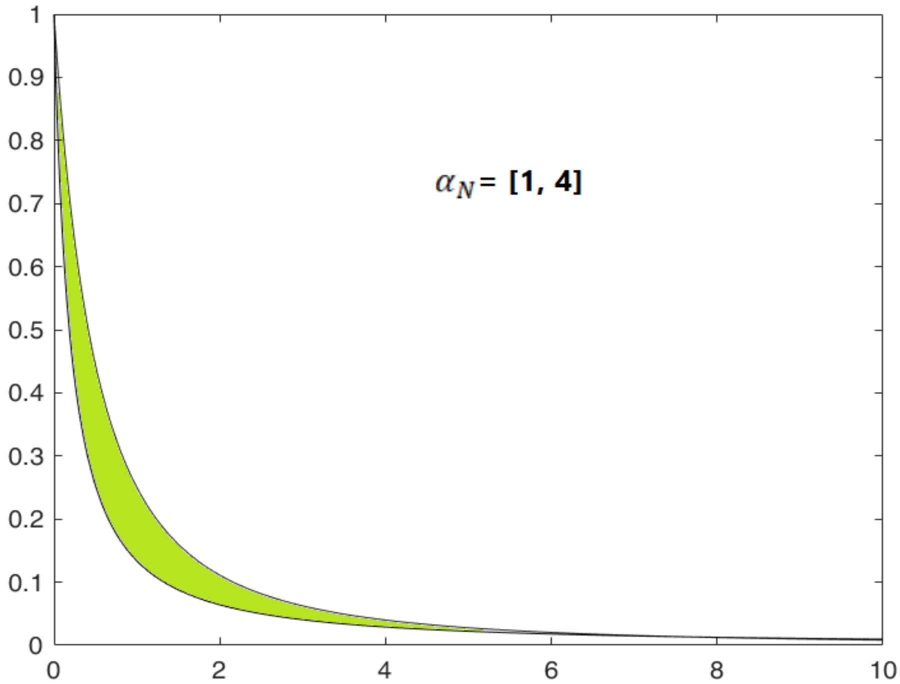
- The survival function:

$$s(x,\alpha_N,\beta_N)=\left(\frac{\alpha_N}{\beta_N x+\alpha_N}\right)^{\frac{1}{\beta_N}}; x>0\ and\alpha_N,\beta_N>0 \quad (6)$$

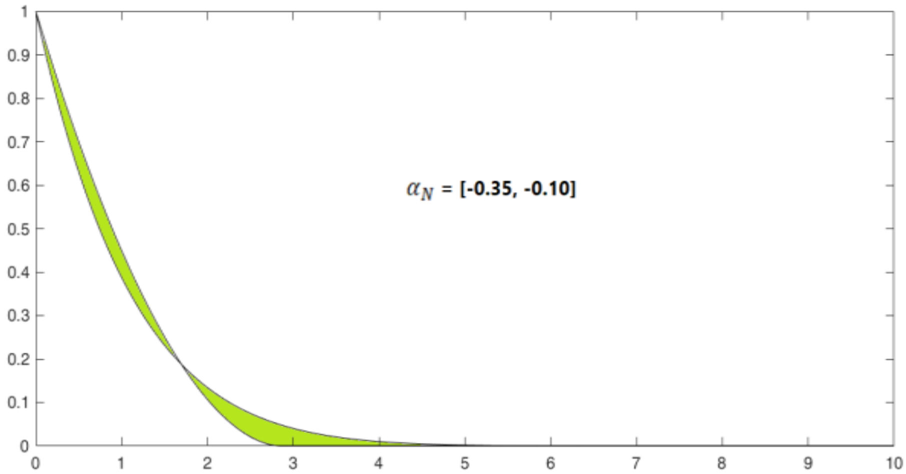
Where α_N is neutrosophic scale parameter and β_N is a neutrosophic shape parameter $\alpha_N\in(\alpha_L,\alpha_U)$ and $\beta_N\in(\beta_L,\beta_U)$.

2.1. Figures

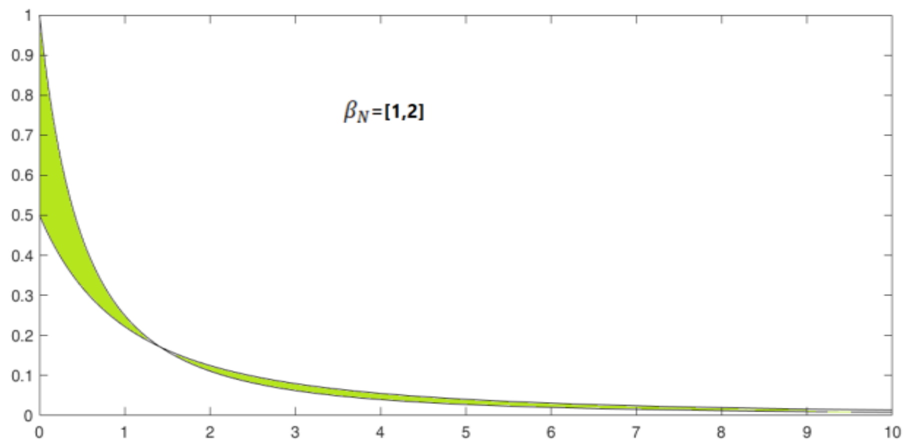
The pdf curves of the NGPD are presented in Figure 1. The graphical expression of the $f(x,\alpha_N,\beta_N)$ of the neutrosophic generalized Pareto with imprecise parameters.



(A) The pdf of the NGPD with neutrosophic Shape parameters



(B) The pdf of the NGPD with neutrosophic shape parameters $\alpha_N < 0$



(C) The pdf of the NGPD with neutrosophic Scale parameters

Figure 1. Theneutrosophic shape parameter and the neutrosophic scale parameter

Figure 1-A shows the pdf curve of the distribution with the neutrosophic shape parameter α_N =[1, 4] if the data are believed to be NGPD with $\beta_N = 1$. Figure 1-B shows the neutrosophic shape parameter if $\alpha_N < 0$. Also, Figure 1-C shows the NGPD with neutrosophic scale parameters β_N = [1, 2] and $\alpha_N = 1$.

2.2. The Properties of NGPD

We will introduce some properties of NGPD, such as mean, variance and special cases.

1- The Mean of NGPD

$$\mu_N = E(x) = \int_0^\infty x \frac{1}{\beta_N}(1+\frac{\alpha_N x}{\beta_N})^{-\frac{1}{\alpha_N}-1} dx$$

$$= \frac{1}{\beta_N}\int_0^\infty x(1+\frac{\alpha_N x}{\beta_N})^{-\frac{1}{\alpha_N}-1} dx$$

$$\mu_N = \frac{\beta_N}{1-\alpha_N} \quad ; \quad \alpha_N < 1$$

$$\mu_N = (\frac{\beta_l}{1-\alpha_l}, \frac{\beta_u}{1-\alpha_u}) \tag{7}$$

2- The Variance:

$$V(x) = E(x^2) - (\mu_N)^2$$

$$E(x^2) = \int_0^\infty x^2(1+\frac{\alpha_N x}{\beta_N})^{-\frac{1}{\alpha_N}-1} dx$$

$$= \frac{1}{1-2\alpha_N} \quad ; \quad \alpha_N < \frac{1}{2}$$

$$var(x) = \frac{{\beta_N}^2}{(1-2\alpha_N)(1-\alpha_N)^2} \quad ; \alpha_N < \frac{1}{2} \tag{8}$$

3- Special Cases:

1) If the neutrosophic shape parameter $\alpha_N = 0$ in equation 1, the NGPD will be equivalent to neutrosophic exponential distribution as shown in equation 2.
2) If $\alpha_N > 0$, the NGPD will be neutrosophic Lomax distribution as shown in equation (9).

3) If $\alpha_N = -1$, then the NGPD will be neutrosophic uniform $(0,\beta_N)$.

$$f(x,-1,\beta_N) = \frac{1}{\beta_N}(1+\frac{-x}{\beta_N})^{1-1} = \frac{1}{\beta_N} \approx Uniform(0,\beta_N)$$

3. The Neutrosophic Lomax Distribution (NLD)

The Lomax distribution, also called the Pareto Type II distribution was presented by Lomax in the mid of the last century, and it is a heavy-tail probability distribution used in economics and business failure data. To improve the flexibility of the existing model, there are some methods as increasing the number of parameters, making some transformation [15] and proper mixing of two distributions [16]. In this paper, we present a new model (neutrosophic logic) to add more flexibility with incomplete data and indeterminacy data.

The probability density function of a Lomax distribution with neutrosophic shape parameter $\alpha_N > 0$ and neutrosophic scale parameter $\beta_N > 0$ is given by

$$f_N(x) = \frac{\alpha_N}{\beta_N}\left(1+\frac{x}{\beta_N}\right)^{-(\alpha_N+1)}; x \geq 0 \; and \beta_N, \alpha_N > 0 \quad (9)$$

where $\alpha_N = (\alpha_L, \alpha_U)$ and $\beta_N = (\beta_L, \beta_U)$. Note that the NGPD differs from the classical distribution, when the indeterminate part with shape and scale parameters is considered zero in the neutrosophic Lomax distribution, that is, $\alpha_L = \alpha_U = \alpha$ and $\beta_L = \beta_U = \beta$, it tends to classical Lomax distribution.

3.1. The Properties of the Neutrosophic Lomax Distribution (NLD)

We will introduce here the properties of the neutrosophic Lomax distribution (NLD), such as the cumulative distribution function, the mean, variance, the survivor function, the hazard function and the inverse distribution function.

1- The Cumulative Distribution Function (cdf) of the NLD:

$$F_N(x) = 1-\left(1+\frac{x}{\beta_N}\right)^{-\alpha_N}; x \geq 0 \; and \beta_N, \alpha_N > 0 \quad (10)$$

2- The Mean of the NLD:

$$E(x_N) = \frac{\beta_N}{\alpha_N - 1} for \alpha_N > 1 \quad (11)$$

3- The Variance of the NLD:

$$v(x_N) = \frac{(\beta_N)^2\alpha_N}{(\alpha_N-1)^2(\alpha_N-2)} for \alpha_N > 2 \quad (12)$$

4- The Survivor Function of the NLD:

$$S(x) = (1+\beta_N x)^{-\alpha_N} \quad (13)$$

5- The HAzard Function of the NLD:

$$H(x) = \alpha_N \ln(1+\beta_N x) \quad (14)$$

6- The Inverse Distribution Function of the NLD:

$$F^{-1}(u) = \frac{(1-u)^{-1/\beta_N \frac{-1}{\alpha_N}}}{\alpha_N} \quad (15)$$

4. Neutrosophic Pareto Distribution (NPD) Model

The Pareto distribution, is the power-law probability distribution that is used in description of social, quality control, scientific, geophysical, actuarial and many other areas. A neutrosophic Pareto distribution is a classical Pareto distribution but its parameters nor its variables are unclear or imprecise. Zahed Khan et al. [12] introduced the neutrosophic Pareto distribution model in the neutrosophic parameter and studied its properties. In this paper we will investigate the neutrosophic random variable for Pareto distribution.

The neutrosophic random variable x follows the NPD model with the following neutrosophic density function:

$$f_N(x) = \frac{\alpha_N \beta_N^{\alpha_N}}{x_N^{\alpha_N+1}}(1+I_N); \; for x_N > \beta_N and \beta_N, \alpha_N > 0 \quad (16)$$

where $\alpha_N = (\alpha_L, \alpha_U)$ is neutrosophic shape parameter, $\beta_N = (\beta_L, \beta_U)$ is neutrosophic scale parameter and $x_N \in (x_L, x_U)$ is neutrosophic random variable, where $x_N = x_L + x_U I_N$ and $I_N \in (I_L, I_U)$ is an indeterminacy interval where N is the neutrosophic statistical number. The netrosophic Pareto distribution tends to the classical distribution when $I_N = 0$.

The corresponding neutrosophic cumulative distribution is:

$$F_N(x) = \left(1-\left(\frac{\beta_N}{x_N}\right)^{\alpha_N}\right)(1+I_N), \; x_N > \beta_N and \beta_N, \alpha_N > 0 \quad (17)$$

4.1. Statistical Properties of Neutrosophic Pareto Distribution

The main properties of mathematical statistics like mean, variance, quantile and moment generating functions have been studied in this section.

1- The Mean of NPD:

$$E(x_N) = (1+I_N)\int_{\beta_N}^{\infty} x_N \frac{\alpha_N \beta_N^{\alpha_N}}{x_N^{\alpha_N+1}} d\,x_N$$

$$= (1+I_N)\alpha_N \beta_N^{\alpha_N}\int_{\beta_N}^{\infty} x_N^{-\alpha_N} dx_N$$

$$= \frac{\alpha_N \beta_N}{\alpha_N - 1}(1+I_N) \quad \text{for } \alpha_N > 1$$

$$\mu_N = \left[\frac{\alpha_l \beta_l}{\alpha_l - 1}(1+I_l), \frac{\alpha_u \beta_u}{\alpha_u - 1}(1+I_u)\right] \quad (18)$$

2- The Variance of the NPD:

$$V(x_N) = E(x_N^2) - \mu^2$$

$$E(x_N^2) = (1 + I_N)\int_{\beta_N}^{\infty}(x_N)^2\frac{\alpha_N\beta_N^{\alpha_N}}{x_N^{\alpha_N+1}}d\,x_N$$

$$= \frac{\alpha_N\beta_N^2}{\alpha_N - 2}(1 + I_N) for \alpha_N > 2$$

$$V(x_N) = \frac{\alpha_N\beta_N^2}{\alpha_N - 2}(1 + I_N) - \left[\left(\frac{\alpha_N\beta_N}{\alpha_N - 1}\right)(1 + I_N)\right]^2$$

$$for \alpha_N > 2 \qquad (19)$$

3- The Moment Generating Function of the NPD:

$$\mu_{x_N}(t) = E(e^{tx_N}) = (1 + I_N)\int_{\beta_N}^{\infty} e^{tx_N}\frac{\alpha_N\beta_N^{\alpha_N}}{x_N^{\alpha_N+1}}d\,x_N$$

$$= (1 + I_N)\alpha_N(-\beta_N t)^{-\alpha_N} \qquad (20)$$

To add more flexibility to the neutrosophic Pareto distribution, various families and generalization of the neutrosophic distribution have been derived including the neutrosophic generalized Pareto distribution with tow parameter.

$$s(x, \alpha_N, \beta_N) = \left(\frac{\alpha_N}{\beta_N x + \alpha_N}\right)^{\frac{1}{\beta_N}};\ x > 0\ and \alpha_N, \beta_N > 0 \qquad (21)$$

where α_N is a neutrosophic scale parameter and β_N is a neutrosophic shape parameter. $\alpha_N \in (\alpha_L, \alpha_U)$ and $\beta_N \in (\beta_L, \beta_U)$.

5. Real Application

In this part, a practical application using a real-world data set has been used to assess the interest in the NGPD model.

The public debt in Egypt is increasing at an alarming rate. This study has applied the extreme value theory in modelling the public debt where NGPD has been used. In Figure 2, the PDF-plot demonstrates that the GPD will fit public debt in Egypt.

The data under consideration includes a set of public debt in Egypt covering the period from 2000 to 2022. The data reported by central bank of Egypt (https://www.cbe.org.eg/ar/economic-research) are shown in Table 1. The observations in this dataset represent the public debt (in millions dollar). The data from sources are crisp values; for the purpose of illustration we treat the data set such as shown in Table 1 and the graphical summary of crisp data is shown in Figure 2.

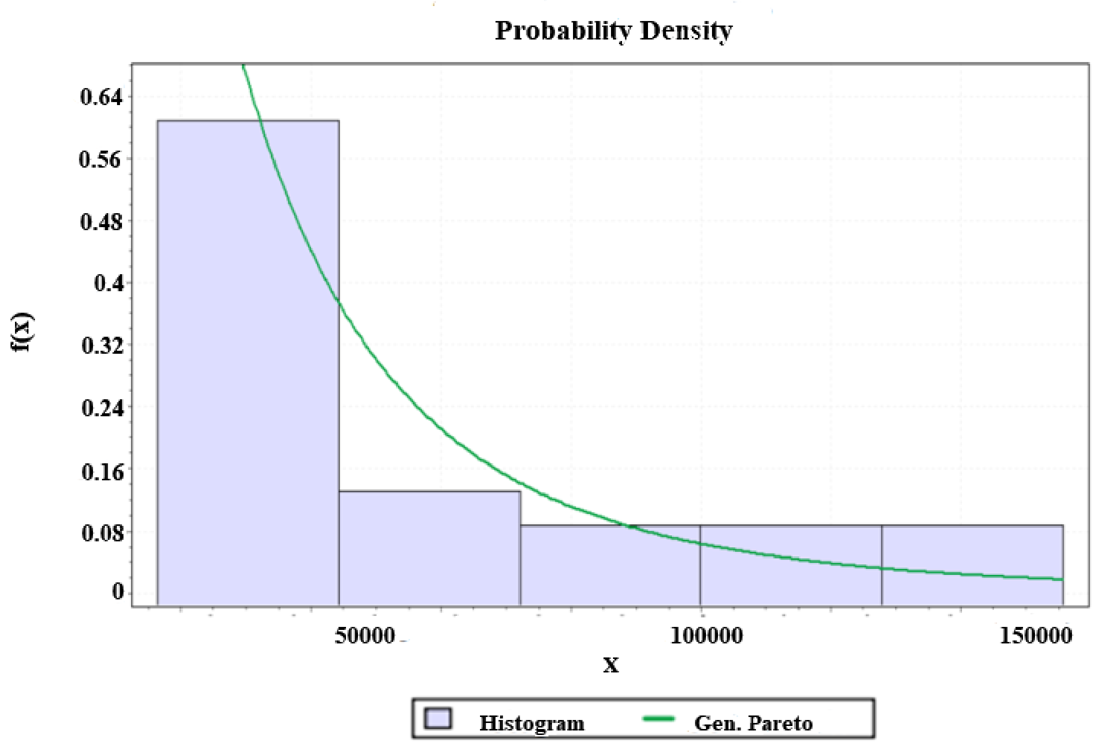


Figure 2. The graphical summary of crisp data

Table 1. Sample from data of Public debit in Egypt

[27780.1, 27783.3]	26560	28660	29396.2
16384.8	29592.6	29898	
[33850 ,33892]	31531.1	[46060,46067.1]	29871.8
33694.2	34905.7	34384.5	43233.4
48062.9	55764.4	[79032.8,79033]	[92613.9, 92643.9]
108699.1	123490.5	137859.6	[155608.9,155708.9]

The data aren't precisely reported but are provided in intervals. These uncertainties in the sample render the classical generalized Pareto models inapplicable. On the contrary, the NGP may effectively be used to investigate the properties of the neutrosophic data set. The descriptive statistics of the public debt data using the NGP model are given in Table 2. The following Table 3 is the result of fitting NGPD:

Table 2. Parameters Estimation

parameters	estimate
Shape parameter $\boldsymbol{\alpha_N}$	[0.25397,0.2536]
Scale parameter $\boldsymbol{\beta_N}$	[26528,26543]
threshold	[19968,19955]

Table 3. Neutrosophic statistics of public debt data by using NGP

Descriptive measures	
Mean	[55516,55527]
Variance	[2.5662E+9,2.5697E+09]
Mode	[19955,19968]
Skewness	[7.3581,7.3890]

Table 2 and Table 3 show the estimated neutrosophic measures based on the NGPD. All the estimated values are expressed as intervals because of indeterminacies inherent in the analyzed dataset.

To estimate the Generalized Pareto Distribution (GPD) model parameters, we find that the shape parameter α_N= [0.25397,0.2536], the scale parameter, β_N=[26528,26543] and the Threshold is [19968, 19955] as shown in Table 2.

5.1. Goodness of Fit Test for Neutrosophic Data

Table 4. Goodness-of fit test for the data by NK-S tests

Model	NK-S Critical value $\boldsymbol{D_{0.05,23}}$	$\boldsymbol{\alpha}$=0.05 Statistic($\boldsymbol{D_N}$)
NGPD	0.28358	[0.15589, 0.15601]

For the public debt in Egyptdata, we interested in testing the assumption that the data follows a neutrosouphic generalized Pareto distribution. To test the assumption, we applied the neutrosophic Kolmogrov-Smirnov (NK-S) test, which are the generalization of the Kolmogrov-Smirnov [17]. The results are shown in Table 4.

The neutrosophic null hypothesis that the sample coming from the NGPD is accepted when $\boldsymbol{D_N} \in [\boldsymbol{D_L}, \boldsymbol{D_U}] < \boldsymbol{D_{\alpha,N}}$.where $\boldsymbol{D_{\alpha,N}}$ is a neutrosophic critical value. Note here that the$\boldsymbol{D_N}$ ∈[0.15589, 0.15601]<0.28358 then the data follow the NGPD.

6. Conclusions

The NGPD plays an important role in modelling extreme value when data is incomplete or indeterminate. This paper discussed the new neutrosophic distribution (neutrosophic Generalized Pareto distribution). The neutrosophic Lomax distribution and neutrosophic Pareto distribution are a special case study from the NGPD. Properties of mathematical characteristics of the proposed distributions under an indeterminacy environment are described. The effectiveness of the proposed model NGPD has been demonstrated by using a real dataset on Public debit data.

In the future, the estimation of parameters for the NGPD can be studied and also other neutrosophic families can be presented.

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