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Convergence of the Jordan Neutrosophic Ideal in Neutrosophic Normed Spaces

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Abstract In the context of the Neutrosophic Norm, the essay explores the challenge of constructing precise sequence spaces whose elements' convergence is a generalised form of the Cauchy convergence. It has proven to be a crucial tool, opening the door to the theory of functions and the law of large numbers applications. Numerous authors, including those who investigated the Euler totient matrix operator, have studied the strategy for building new sequence spaces that are specified as the domain of matrix operators. Recently, the Jordan totient function $\mathcal{T}_r$. generalised the Euler totient function ϕ. In the context of neutrosophic Norm spaces, we establish some sequence spaces, specifically $c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$, $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$, $\ell^I_{\infty(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ and $\ell_{\infty(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ as a domain of the triangular Jordan totient matrix operator, and investigate the ideal convergence of these sequences. These concepts serve as an introduction to a new sort of convergence that Fast and Steinhaus presented as more general than normal convergence and statistical convergence. According to Kostyrko et al., this form is known as ideal convergence. In order to arrive at a finite limit, the Jordan totient operator, an infinite matrix operator, is used. We also construct a number of inclusion connections between the spaces as we explain various topological and algebraic properties.

The Jordan totient operator, an infinite matrix operator, is used to accomplish the task of reaching a finite limit. As we discuss various topological and algebraic features, we also create several inclusion relations between the spaces.

Keywords Jordan I-convergence, Compactness, Completeness, Hausdorff, Neutrosophic Sets

1 Introduction

Fast [1] and Schoenberg [2] were the first to independently introduce the idea of statistical convergence. Salat et al. [3] established the concept of I-convergence, a statistical convergence generality. Later, the concept of statistical convergence for double sequences was independently developed by Edely and Mursaleen [4] and Tripathy[5], and for fuzzy numbers by Mursaleen and Saves [6]. In connection with this, there are two very distinct types of convergence for double sequences, namely I and I^*-convergence [7].

Converged triple sequences were introduced by Gurdal, Sahiner, and Duden [8] in 2007. Numerous authors have further explored this idea; see [9, 10, 11]. The I-convergence of triple sequences in probabilistic normed spaces was a concept used by Tripathy and Goswami[12].

A generalization of fuzzy set theory, intuitionistic fuzzy set theory was first proposed by Atanassov[13] in 1986. Fuzzy set theory is a powerful tool for modeling uncertainty and vagueness because it assigns the degree of membership to the components so that distinct individuals can be identified in a given set. The notion of fuzzy sets has curiously evolved into the present standard for young scientists or researchers, according to a large body of research that has lately arisen in the scientific discipline. The idea of fuzzy topology has become a highly important tool for many writers' works.

After some time, Smarandache [14], by introducing an intermediate membership function, introduced the idea of Neutrosophic Sets [$\mathcal{NS}$], which is a unique sort of notation for classical set theory. This set is a formal setting designed to gauge the veracity, ambiguity, and falsity of statements. Converged triple sequences were introduced by Gurdal, Sahiner, and Duden in 2007. Numerous authors have further explored this idea . The concept of triple sequences I-convergent in probabilistic normed spaces is familiar to Tripathy and Goswami. Tripathy and Shiner examined the I-convergence qualities in triple sequence spaces and presented some insightful findings.

The Jordan totient matrix operator, represented by $\mathcal{M}^r$, is one such definite matrix operator. It was first described in [15] via the Jordan totient $\mathcal{T}_r$ function, whose domain and codomain are $\mathbb{N}$. The number of r tuples is the function $(\mathscr{f}_1, \mathscr{f}_2, \cdots, \mathscr{f}_r)$ like that $gcd(\mathscr{f}_1, \mathscr{f}_2, \cdots, \mathscr{f}_r, \grave{\jmath}) = 1$ and $1 \leq \mathscr{f}_i \leq \grave{\jmath}$. It has the following definition: $\mathcal{T}_r(\grave{\jmath}) = \grave{\jmath}^r \prod_{\mathscr{p}|\grave{\jmath}} \left(1 - \frac{1}{\mathscr{p}^r}\right)$.
Since $\vartheta \geq 1$ is the prime decomposition of $\grave{\jmath}$,
$\mathscr{f} = \mathscr{p}_1^{\vartheta_1}, \mathscr{p}_2^{\vartheta_2}, \cdots, \mathscr{p}_{\mathscr{k}}^{\vartheta_{\mathscr{k}}}$. Consequently, the definition of the Jordan totient matrix operator $\mathcal{M}^r = (\rho^r_{\grave{\jmath}\mathscr{k}})$ is as follows:

$$\rho^r_{\grave{\jmath}\mathscr{k}} = \begin{cases} \dfrac{\mathcal{T}_r(\mathscr{k})}{\grave{\jmath}^r} & \text{if } \mathscr{k}|\grave{\jmath}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

and its inverse $(\mathcal{M}^r)^{-1}$ is given by

$$(\mathcal{M}^r)^{-1} = \begin{cases} \dfrac{\mathscr{J}\left(\dfrac{\grave{\jmath}}{\mathscr{k}}\right)}{\mathcal{T}_r(\grave{\jmath})}\mathscr{k}^r & \text{if } \mathscr{k}|\grave{\jmath}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where the Mӧbius function $\mathscr{J}$ is represented by the expression:

$$\mathscr{J}(\grave{\jmath}) = \begin{cases} 0 & \text{if } \mathscr{p}^2|\grave{\jmath} \text{ for a few prime } \mathscr{p}, \\ 1 & \text{if } \grave{\jmath} = 1, \\ (-1)^n & \text{if } \grave{\jmath} = \prod_{\mathscr{k}=1}^{n} \mathscr{p}_{\mathscr{k}} \text{ where } \mathscr{p}_{\mathscr{k}}s \text{ are vary.} \end{cases}$$

Later, using the operator to look at sequence spaces, Khan, Ilkhan, and Kara ([16], [17], and [18], respectively) presented some fascinating results. Also, we discussed Statistical Δ^m-Convergence [19] and Lacunary $\Im$-Statistical Convergence [20] in Neutrosophic Normed Spaces. In this article, we construct sequence spaces, investigate the ideal convergence of these sequences, offer strong reasons against them, and analyze the algebraic and topological properties of these spaces within the framework of Neutrosophic Normed Spaces. Consider an open ball with a radius of $r > 0$, a center of ψ, and a fuzziness parameter of $\grave{\epsilon} \in (0, 1)$.

2 Preliminaries

For the two sequence spaces $\mathfrak{S}, \mathfrak{R}$ and an non finite matrix $\mathfrak{P} = (\mathscr{p}_{\grave{\jmath}\mathscr{k}})$, the $\mathfrak{P}$ transform of $\mathscr{D} = (\mathscr{D}_{\mathscr{k}})$ provided by $\mathfrak{P}\mathscr{D} = \{\mathfrak{P}_{\grave{\jmath}}(\mathscr{D})\}_{\grave{\jmath}=1}^{\infty} \in \mathfrak{R}$, where

$$\mathfrak{P}_{\grave{\jmath}}(\mathscr{D}) = \sum_{\mathscr{k}=1} \mathscr{p}_{\grave{\jmath}\mathscr{k}} \cdot \mathscr{D}_{\mathscr{k}}, \quad \grave{\jmath} \in \mathbb{N}.$$

In this article, the Jordan totient infinite matrix operator will be used $\mathcal{M}^r = (\rho^r_{\grave{\jmath}\mathscr{k}})$ which is defined as:

$$\rho^r_{\grave{\jmath}\mathscr{k}} = \begin{cases} \frac{\mathcal{T}_r(\mathscr{k})}{\grave{\jmath}^r} & \text{if } \mathscr{k}|\grave{\jmath}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Tantamountly,

$$\mathcal{M}^r = \begin{bmatrix} \frac{\mathcal{T}_r(1)}{1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{\mathcal{T}_r(1)}{2^r} & \frac{\mathcal{T}_r(2)}{2^r} & 0 & 0 & 0 & 0 & \cdots \\ \frac{\mathcal{T}_r(1)}{3^r} & 0 & \frac{\mathcal{T}_r(3)}{3^r} & 0 & 0 & 0 & \cdots \\ \frac{\mathcal{T}_r(1)}{4^r} & \frac{\mathcal{T}_r(2)}{4^r} & 0 & \frac{\mathcal{T}_r(4)}{4^r} & 0 & 0 & \cdots \\ \frac{\mathcal{T}_r(1)}{5^r} & 0 & 0 & 0 & \frac{\mathcal{T}_r(5)}{5^r} & 0 & \cdots \\ \frac{\mathcal{T}_r(1)}{6^r} & \frac{\mathcal{T}_r(2)}{6^r} & \frac{\mathcal{T}_r(3)}{6^r} & 0 & 0 & \frac{\mathcal{T}_r(6)}{6^r} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The $\mathcal{M}^r$ transform of $(\mathscr{D}_{\mathscr{k}}) \in \Xi$ is defined as
$\mathcal{M}^r_{\grave{\jmath}}(\mathscr{D}) := \frac{1}{\grave{\jmath}^r} \sum \mathcal{T}_r(\mathscr{k})\mathscr{D}_{\mathscr{k}}$.

Definition 2.1 *[21]. A assemblage of subsets $\mathfrak{I}$ of a void set $\Im$ is known as an ideal in $\Im$ if:*

(i) $\emptyset \in \mathfrak{I}$,

(ii) $\breve{\mathcal{U}}, \breve{\mathcal{V}} \in \mathfrak{I} \Rightarrow \breve{\mathcal{U}} \cup \breve{\mathcal{V}} \in \mathfrak{I}$,

(iii) $\breve{\mathcal{U}} \in \mathfrak{I}, \breve{\mathcal{V}} \subseteq \breve{\mathcal{U}} \Rightarrow \breve{\mathcal{V}} \in \mathfrak{I}$.

An ideal where $\mathfrak{I} \subseteq 2^{\Im}$ and $\mathfrak{I} \neq 2^{\Im}$ are nontrivial ideals. It becomes a maximal ideal if no nontrivial ideal $\mathfrak{T} \neq \mathfrak{I}$ exists like that $\mathfrak{I} \subset \mathfrak{T}$ and becomes an admissible ideal if $\mathfrak{I}$includes every singleton subset of $\Im$.

Definition 2.2 *[21]. A assemblage of subsets $\mathfrak{F}$ of a void set $\Im$ is known as a filter in $\Im$ if:*

(i) $\emptyset \notin \mathfrak{F}$,

(ii) $\breve{\mathcal{U}}, \breve{\mathcal{V}} \in \mathfrak{F} \Rightarrow \breve{\mathcal{U}} \cap \breve{\mathcal{V}} \in \mathfrak{F}$,

(iii) $\breve{\mathcal{U}} \in \mathfrak{F}$ *and* $\breve{\mathcal{V}} \supset \breve{\mathcal{U}}$ *implying* $\breve{\mathcal{V}} \in \mathfrak{F}$.

$\mathfrak{F}(\mathfrak{I}) = \{\mathscr{K} \subset \Im : \mathscr{K}^c \in \mathfrak{I}\}$ *is the filter connected to the ideal $\mathfrak{I}$. Consider $\mathfrak{I}$ as an admissible ideal in $\mathbb{N}$.*

Definition 2.3 *[20] The 7-tuple $(\Im, \mathscr{J}, \mathscr{G}, \mathscr{H}, *, \odot, \Diamond)$ is known as $\mathcal{NNS}$ if $\Im$ is a linear space, $*$ is a continuous t-norm, $\odot$ and $\Diamond$ are continuous t-conorm, $\mathscr{J}, \mathscr{G}$ and $\mathscr{H}$ are fuzzy sets on $\Im \times (0, \infty)$ fulfils the coming after conditions:*
For every one $\grave{x}, \grave{y} \in \Im$ and $\grave{\jmath}, \grave{\varpi} > 0$;

(a) $0 \leq \mathscr{J}(\grave{x}, \grave{\varpi}) \leq 1; 0 \leq \mathscr{G}(\grave{x}, \grave{\varpi}) \leq 1; 0 \leq \mathscr{H}(\grave{x}, \grave{\varpi}) \leq 1$,

(b) $\mathscr{J}(\grave{x}, \grave{\varpi}) + \mathscr{G}(\grave{x}, \grave{\varpi}) + \mathscr{H}(\grave{x}, \grave{\varpi}) \leq 3$,

(c) $\mathscr{J}(\grave{x}, \grave{\varpi}) > 0$,

(d) $\mathscr{J}(\grave{x}, \grave{\varpi}) = 1$ *if and only if* $\grave{x} = 0$,

(e) $\mathcal{J}(\alpha\dot{x},\dot{\varpi}) = \mathcal{J}\left(\dot{x},\frac{\dot{\varpi}}{|\alpha|}\right)$ *for every* $\alpha \neq 0$*,*

(f) $\mathcal{J}(\dot{x},\dot{\varpi}) * \mathcal{J}(\dot{y},\dot{z}) \leq \mathcal{J}(\dot{x}+\dot{y},\dot{\varpi}+\dot{z})$*,*

(g) $\mathcal{J}(\dot{x},\dot{\varpi}) : (0,\infty) \to [0,1]$ *is continuous,*

(h) $\lim_{\dot{\varpi}\to\infty} \mathcal{J}(\dot{x},\dot{\varpi}) = 1$ *and* $\lim_{\dot{\varpi}\to 0} \mathcal{J}(\dot{x},\dot{\varpi}) = 0$*,*

(i) $\mathcal{G}(\dot{x},\dot{\varpi}) < 1$*,*

(j) $\mathcal{G}(\dot{x},\dot{\varpi}) = 0$ *if and only if* $\dot{x} = 0$*,*

(k) $\mathcal{G}(\alpha\dot{x},\dot{\varpi}) = \mathcal{G}\left(\dot{x},\frac{\dot{\varpi}}{|\alpha|}\right)$ *for each* $\alpha \neq 0$*,*

(l) $\mathcal{G}(\dot{x},\dot{\varpi}) \odot \mathcal{G}(\dot{y},\dot{z}) \geq \mathcal{G}(\dot{x}+\dot{y},\dot{\varpi}+\dot{z})$*,*

(m) $\mathcal{G}(\dot{x},\dot{\varpi}) : (0,\infty) \to [0,1]$ *is continuous,*

(n) $\lim_{\dot{\varpi}\to\infty} \mathcal{G}(\dot{x},\dot{\varpi}) = 0$ *and* $\lim_{\dot{\varpi}\to 0} \mathcal{G}(\dot{x},\dot{\varpi}) = 1$*,*

(o) $\mathcal{H}(\dot{x},\dot{\varpi}) < 1$*,*

(p) $\mathcal{H}(\dot{x},\dot{\varpi}) = 0$ *if and only if* $\dot{x} = 0$*,*

(q) $\mathcal{H}(\alpha\dot{x},\dot{\varpi}) = \mathcal{H}\left(\dot{x},\frac{\dot{\varpi}}{|\alpha|}\right)$ *for each* $\alpha \neq 0$*,*

(r) $\mathcal{H}(\dot{x},\dot{\varpi})\Diamond\mathcal{G}(\dot{y},\dot{z}) \geq \mathcal{H}(\dot{x}+\dot{y},\dot{\varpi}+\dot{z})$

(s) $\mathcal{H}(\dot{x},\dot{\varpi}) : (0,\infty) \to [0,1]$ *is continuous,*

(t) $\lim_{\dot{\varpi}\to\infty} \mathcal{H}(\dot{x},\dot{\varpi}) = 0$ *and* $\lim_{\dot{\varpi}\to 0} \mathcal{H}(\dot{x},\dot{\varpi}) = 1$.

Definition 2.4 *[20] Let* $(\Im,\mathcal{J},\mathcal{G},\mathcal{H},*,\odot,\Diamond)$ *be an* $\mathcal{NNS}$*. A sequence* $\dot{x} = (\dot{x}_k)$ *is known as* $\mathcal{I}$*-convergent to* $\ell \in \Im$ *in regard to Neutrosophic Norms(*$\mathcal{NN}$*)* $(\mathcal{J},\mathcal{G},\mathcal{H})$*, if every* $\epsilon > 0$ *and* $\dot{\varpi} > 0$*, the set*

$$\left\{\begin{array}{c} \mathcal{J}(\dot{x}_k - l,\dot{\varpi}) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\dot{x}_k - l,\dot{\varpi}) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\dot{x}_k - l,\dot{\varpi}) \geq \grave{\epsilon} \end{array}\right\} \in \mathcal{I}.$$

The notation $\mathcal{I}_{(\mathcal{J},\mathcal{G},\mathcal{H})} - \lim \dot{x}_k = l$ *will be used to indicate the ideal convergence of the sequence in the article* $(\dot{x}_k)$ *to* l *in regard to the* $\mathcal{NN}$ $(\mathcal{J},\mathcal{G},\mathcal{H})$*.*

Definition 2.5 *[20] Let* $(\Im,\mathcal{J},\mathcal{G},\mathcal{H},*,\odot,\Diamond)$ *be an* $\mathcal{NNS}$*. A sequence* $\dot{x} = (\dot{x}_k)$ *is known as* $\mathcal{I}$*-Cauchy sequence in regard to* $(\mathcal{J},\mathcal{G},\mathcal{H})$*, if every* $\epsilon > 0$ *and* $\dot{\varpi} > 0$*, there exists* $\mathscr{p} = \mathscr{p}(\epsilon) \in \mathbb{N}$ *like that the set*

$$\left\{\begin{array}{c} \mathcal{J}(\dot{x}_k - \dot{x}_{\mathcal{N}},\dot{\varpi}) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\dot{x}_k - \dot{x}_{\mathcal{N}},\dot{\varpi}) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\dot{x}_k - \dot{x}_{\mathcal{N}},\dot{\varpi}) \geq \grave{\epsilon} \end{array}\right\} \in \mathcal{I}.$$

Definition 2.6 *Let* $(\Im,\mathcal{J},\mathcal{G},\mathcal{H},*,\odot,\Diamond)$ *be an* $\mathcal{NNS}$*. Then* $(\Im,\mathcal{J},\mathcal{G},\mathcal{H},*,\odot,\Diamond)$ *is known as complete if each Cauchy sequence is convergent in regard to the* $\mathcal{NN}$ $(\mathcal{J},\mathcal{G},\mathcal{H})$*.*

3 Main results

For the purposes of this section, we'll assume that the ideal I is a non-trivial admissible ideal of a subset of $\mathbb{N}$. Following sequence spaces are arranged as follows:

$$c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r) = \left\{\left\{\begin{array}{c} n\in\mathbb{N} : \text{for a few } l \in \mathbb{C}, \\ \mathcal{J}(\mathcal{M}^r_n(\mathscr{p}) - l,\dot{\varpi}) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\mathscr{p}) - l,\dot{\varpi}) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\mathscr{p}) - l,\dot{\varpi}) \geq \grave{\epsilon} \end{array}\right\} \in I\right\}, \quad (3.1)$$

$$c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r) = \left\{\left\{\begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\mathscr{p}) - l,\dot{\varpi}) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\mathscr{p}) - l,\dot{\varpi}) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\mathscr{p}) - l,\dot{\varpi}) \geq \grave{\epsilon} \end{array}\right\} \in I\right\}, \quad (3.2)$$

$$\ell^I_{\infty(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r) = \left\{\left\{\begin{array}{c} \text{there exists } \grave{\epsilon} \in (0,1), \\ \mathcal{J}(\mathcal{M}^r_n(\mathscr{p}),\dot{\varpi}) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\mathscr{p}),\dot{\varpi}) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\mathscr{p}),\dot{\varpi}) \geq \grave{\epsilon} \end{array}\right\} \in I\right\}, \quad (3.3)$$

$$\ell_{\infty(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r) = \left\{\left\{\begin{array}{c} n\in\mathbb{N} : \text{ there exists } \grave{\epsilon} \in (0,1), \\ \mathcal{J}(\mathcal{M}^r_n(\mathscr{p}),\dot{\varpi}) \geq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\mathscr{p}),\dot{\varpi}) \leq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\mathscr{p}),\dot{\varpi}) \leq \grave{\epsilon} \end{array}\right\} \in I\right\}. \quad (3.4)$$

With regard to the fuzziness parameter $\grave{\epsilon} \in (0,1)$, with a center at $\mathscr{p}$ and a radius of $r > 0$, we offer the definitions of the open ball and closed ball as follows:

$$\mathfrak{B}^I_{\mathscr{p}}(r,\epsilon)(\mathcal{M}^r) = \left\{\begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\mathscr{p}) - \mathcal{M}^r_n(\varphi), r) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\mathscr{p}) - \mathcal{M}^r_n(\varphi), r) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\mathscr{p}) - \mathcal{M}^r_n(\varphi), r) \geq \grave{\epsilon} \end{array}\right\} \in I \quad (3.5)$$

$$\mathfrak{B}^I_{\mathscr{p}}(r,\epsilon)(\mathcal{M}^r) = \left\{\begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\mathscr{p}) - \mathcal{M}^r_n(\varphi), r) < 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\mathscr{p}) - \mathcal{M}^r_n(\varphi), r) > \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\mathscr{p}) - \mathcal{M}^r_n(\varphi), r) > \grave{\epsilon} \end{array}\right\} \in I \quad (3.6)$$

Theorem 3.1 *As of the Spaces* $c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ *and* $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ *are linear spaces.*

Proof:
The space $c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$'s linearity is demonstrated. On the basis of comparisons to $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ may be drawn with certainty.
Given arbitrary sequences $\eta = (\eta_k), \chi = (\chi_k) \in c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ indicate that there is $\eta_0, \chi_0 \in \mathbb{C}$ so that (η_k) and (χ_k) I-converge to η_0 and χ_0, respectively.
For $\dot{\varpi} > 0, \grave{\epsilon} \in (0,1)$ and $\gamma, I \in \mathbb{R}$, think about the following

sets:

$$\mathfrak{A} = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \leq 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \geq \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \geq \grave{\epsilon} \end{array} \right\} \in I,$$

$$\mathfrak{A}^c = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) < \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I),$$

$$\mathfrak{B} = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \leq 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \geq \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \geq \grave{\epsilon} \end{array} \right\} \in I,$$

$$\mathfrak{B}^c = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) < \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I),$$

The set $\mathscr{C} = \mathfrak{A}^c \cap \mathfrak{B}^c$ being a void set lies in $\mathfrak{F}(I)$, so consider $n \in \mathscr{C}$, then

$$\begin{aligned}
&\mathscr{J}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) \\
&\geq \mathscr{J}\left(\gamma\mathcal{M}_n^r(\eta) - \gamma\eta_0, \frac{\dot{\varpi}}{2}\right) * \mathscr{J}\left(I\mathcal{M}_n^r(\chi) - I\chi_0, \frac{\dot{\varpi}}{2}\right) \\
&= \mathscr{J}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) * \mathscr{J}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \\
&> (1 - \grave{\epsilon}) * (1 - \grave{\epsilon}) = 1 - \grave{\epsilon} \\
&\Rightarrow \mathscr{J}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) > 1 - \grave{\epsilon} \\
&\mathscr{G}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) \\
&\leq \mathscr{G}\left(\gamma\mathcal{M}_n^r(\eta) - \gamma\eta_0, \frac{\dot{\varpi}}{2}\right) \odot \mathscr{G}\left(I\mathcal{M}_n^r(\chi) - I\chi_0, \frac{\dot{\varpi}}{2}\right) \\
&= \mathscr{G}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \odot \mathscr{G}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \\
&< \grave{\epsilon} \odot \grave{\epsilon} = \grave{\epsilon} \\
&\Rightarrow \mathscr{G}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) < \grave{\epsilon} \\
&\mathscr{H}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) \\
&\leq \mathscr{H}\left(\gamma\mathcal{M}_n^r(\eta) - \gamma\eta_0, \frac{\dot{\varpi}}{2}\right) \Diamond \mathscr{H}\left(I\mathcal{M}_n^r(\chi) - I\chi_0, \frac{\dot{\varpi}}{2}\right) \\
&= \mathscr{H}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \Diamond \mathscr{H}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \\
&< \grave{\epsilon} \Diamond \grave{\epsilon} = \grave{\epsilon} \\
&\Rightarrow \mathscr{H}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) < \grave{\epsilon}
\end{aligned}$$

Then, we draw a conclusion

$$\mathscr{C} \subset \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) < \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) < \grave{\epsilon} \end{array} \right\}.$$

By utilising the attributes of $\mathfrak{F}(I)$, we have

$$\left\{ \begin{array}{l} \mathscr{J}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) < \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I),$$

which implies that the sequence $(\gamma\eta_k + I\chi_k)I$-converges to $\gamma\eta_0 + I\chi_0$.
Thus, $(\gamma\eta_k + I\chi_k) \in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.
Therefore, the linear space is $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.

Theorem 3.2 *The inclusion relation* $c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) \subset c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) \subset c^I_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ *holds.*

Proof:
The inclusion of $c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ in $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ is fairly obvious.
We provide evidence for $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) \subset c^I_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.
Consider the sequence $\mathscr{p} = (\mathscr{p}_k) \in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.
After it, there is $l \in \mathscr{C}$ like that
$I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) - \lim(\mathscr{p}_k) = l$, and for every $\grave{\epsilon} \in (0,1)$ and $\dot{\varpi} > 0$, the set

$$\Delta = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \end{array} \right\} \in F(I).$$

Choose $\mathscr{J}\left(l, \frac{\dot{\varpi}}{2}\right) = \mathfrak{p}, \mathscr{G}\left(l, \frac{\dot{\varpi}}{2}\right) = \mathfrak{q}$ and $\mathscr{H}\left(l, \frac{\dot{\varpi}}{2}\right) = \mathfrak{s}$ where $\mathfrak{p}, \mathfrak{q}, \mathfrak{s} \in (0,1), \dot{\varpi} > 0$ and $\grave{\epsilon} \in (0,1)$, there exist $c, d, e \in (0,1)$ like that $(1 - \grave{\epsilon}) * \mathfrak{p} > 1 - c, \grave{\epsilon} \odot \mathfrak{q} < d$ and $\grave{\epsilon} \Diamond \mathfrak{s} < e$.
For this reason $n \in \Im$, there are

$$\begin{aligned}
\mathscr{J}\left(\mathcal{M}_n^r(\mathscr{p}), \dot{\varpi}\right) &= \mathscr{J}\left(\mathcal{M}_n^r(\mathscr{p}) - l + l, \dot{\varpi}\right) \\
&\geq \mathscr{J}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \frac{\dot{\varpi}}{2}\right) * \mathscr{J}\left(l, \frac{\dot{\varpi}}{2}\right) \\
&> (1 - \grave{\epsilon}) * \mathfrak{p} > 1 - c, \\
\mathscr{G}\left(\mathcal{M}_n^r(\mathscr{p}), \dot{\varpi}\right) &= \mathscr{G}\left(\mathcal{M}_n^r(\mathscr{p}) - l + l, \dot{\varpi}\right) \\
&\leq \mathscr{G}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \frac{\dot{\varpi}}{2}\right) \odot \mathscr{G}\left(l, \frac{\dot{\varpi}}{2}\right) \\
&< (1 - \grave{\epsilon}) \odot \mathfrak{q} < d, \\
\mathscr{H}\left(\mathcal{M}_n^r(\mathscr{p}), \dot{\varpi}\right) &= \mathscr{H}\left(\mathcal{M}_n^r(\mathscr{p}) - l + l, \dot{\varpi}\right) \\
&\leq \mathscr{H}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \frac{\dot{\varpi}}{2}\right) \Diamond \mathscr{H}\left(l, \frac{\dot{\varpi}}{2}\right) \\
&< (1 - \grave{\epsilon}) \Diamond \mathfrak{s} < e.
\end{aligned}$$

Choose $\hbar = \max\{c, d, e\}$, there are

$$\left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \dot{\varpi}\right) > 1 - \hbar \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \dot{\varpi}\right) < \hbar \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\mathscr{p}) - l, \dot{\varpi}\right) < \hbar \end{array} \right\} \in \mathfrak{F}(I)$$
$$\Rightarrow \mathscr{p} = (\mathscr{p}_k) \in c^I_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r).$$

The inclusion relation's inverse is not true. To prove our point, we offer the coming after examples.

Example 3.3 *Let the Normed space* $(\mathbb{R}, \|\cdot\|)$ *equipped with supremum norm,* $\mathscr{r} * \mathscr{p} = \min\{\mathscr{r}, \mathscr{p}\}, \mathscr{r} \odot \mathscr{p} = \max\{\mathscr{r}, \mathscr{p}\}$ *and* $\mathscr{r} \Diamond \mathscr{p} = \max\{\mathscr{r}, \mathscr{p}\}$ *for every* $\mathscr{p}, \mathscr{r} \in (0,1)$. *Consider the norms* $(\mathscr{J}, \mathscr{G}, \mathscr{H})$ *on* $\Im^2 \times (0, \infty)$ *as follows:* $\mathscr{J}(\mathscr{p}, \dot{\varpi}) = \frac{\dot{\varpi}}{\dot{\varpi} + \|\mathscr{p}\|}$, $\mathscr{G}(\mathscr{p}, \dot{\varpi}) = \frac{\|\mathscr{p}\|}{\dot{\varpi} + \|\mathscr{p}\|}$, *and* $\mathscr{H}(\mathscr{p}, \dot{\varpi}) = \frac{\|\mathscr{p}\|}{\dot{\varpi}}$. $(\mathbb{R}, \mathscr{J}, \mathscr{G}, \mathscr{H}, *, \odot, \Diamond)$ *is then a standard* $\mathcal{NNS}$. *Regarding to the sequence* $(\mathscr{p}_k) = \left\{\frac{1}{k} + \mathscr{p}_0\right\}$ *where* $\mathscr{p}_0 \in \mathbb{R} - \{0\}$. *The sequence* $(\mathscr{p}_k)$ *distinctly lying in* $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) \setminus c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.

Example 3.4 *Assume that $(\mathbb{R}, \|\cdot\|)$ is the normed space outfitted with the $\mathcal{NN}(\mathcal{J}, \mathcal{G}, \mathcal{H})$ as previously mentioned. Regarding the sequence $(p_k) = \sin\left(\frac{1}{k}\right)$.*
Then, $(p_k) \in c^I_{\infty(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r) \setminus c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$.

Theorem 3.5 *Each open ball has centre at ψ and has positive radius r in regard to parameter of fuzziness $\grave{\epsilon}$ lying between 0 and 1, i.e., $\mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r)$ is an unclosed set in $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$.*

Proof:
Consider the open ball with ψ as a center and a positive r as a radius with the parameter of fuzziness $\grave{\epsilon}$ lies between 0 and 1,

$$\mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r_n) = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \leq 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \geq \grave{\epsilon} \end{array} \right\} \in I$$

$$\mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r_n) = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

Think about the element $\Upsilon = (\Upsilon_k) \in \mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r_n)$.
Then its matching set

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

For $\mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) > 1 - \grave{\epsilon}$, $\mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) < \grave{\epsilon}$ and $\mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) < \grave{\epsilon}$ there exists r_0 lying between 0 and r like that

$$\begin{array}{l} \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r_0) > 1 - \grave{\epsilon}, \\ \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r_0) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r_0) < \grave{\epsilon}. \end{array}$$

Setting $\grave{\epsilon}_0 = \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r_0)$ we get $\grave{\epsilon}_0 > 1 - \grave{\epsilon}$ which further proves the element's existence $s \in (0,1)$ like that $\epsilon_0 > 1 - s > 1 - \grave{\epsilon}$.
For a certain $\grave{\epsilon}_0 > 1 - s$, we can locate $\grave{\epsilon}_1, \grave{\epsilon}_2, \grave{\epsilon}_3 \in (0,1)$ like that $\grave{\epsilon}_0 * \grave{\epsilon}_1 > 1 - s, (1 - \grave{\epsilon}_0) \odot (1 - \grave{\epsilon}_2) < s$ and $(1 - \grave{\epsilon}_0) \lozenge (1 - \grave{\epsilon}_3) < s$.
Assume $\grave{\epsilon}_4 = \max\{\grave{\epsilon}_1, \grave{\epsilon}_2, \grave{\epsilon}_3\}$.
Consider the open ball $\mathfrak{B}^I_\Upsilon(r - r_0, 1 - \grave{\epsilon}_4)(\mathcal{M}^r_n)$.
The restraint of $\mathfrak{B}^I_\Upsilon(r - r_0, 1 - \grave{\epsilon}_4)(\mathcal{M}^r_n)$ in $\mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r_n)$ will provide the outcome we want.
Let $\phi = (\phi_k) \in \mathfrak{B}^I_\Upsilon(r - r_0, 1 - \grave{\epsilon}_4)(\mathcal{M}^r_n)$, then

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\Upsilon) - \mathcal{M}^r_n(\phi), r - r_0) > \grave{\epsilon}_4 \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\Upsilon) - \mathcal{M}^r_n(\phi), r - r_0) < 1 - \grave{\epsilon}_4 \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\Upsilon) - \mathcal{M}^r_n(\phi), r - r_0) < 1 - \grave{\epsilon}_4 \end{array} \right\} \in \mathfrak{F}(I).$$

Therefore,

$$\begin{array}{l} \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) \\ \geq \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r_0) * \mathcal{J}(\mathcal{M}^r_n(\Upsilon) - \mathcal{M}^r_n(\phi), r - r_0) \\ \geq \grave{\epsilon}_0 * \grave{\epsilon}_3 \geq \grave{\epsilon}_0 * \grave{\epsilon}_1 > (1 - s) > (1 - \grave{\epsilon}) \\ \Rightarrow \{n \in \mathbb{N} : \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) > 1 - \grave{\epsilon}\} \in \mathfrak{F}(I), \\ \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) \\ \leq \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r_0) \odot \mathcal{G}(\mathcal{M}^r_n(\Upsilon) - \mathcal{M}^r_n(\phi), r - r_0) \\ \leq (1 - \grave{\epsilon}_0) \odot (1 - \grave{\epsilon}_4) \leq (1 - \grave{\epsilon}_0) \odot (1 - \grave{\epsilon}_2) < s < \grave{\epsilon} \\ \Rightarrow \{n \in \mathbb{N} : \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) < \grave{\epsilon}\} \in \mathfrak{F}(I), \end{array}$$

and correspondingly

$$\begin{array}{l} \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) \\ \leq \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r_0) \lozenge \mathcal{H}(\mathcal{M}^r_n(\Upsilon) - \mathcal{M}^r_n(\phi), r - r_0) \\ \leq (1 - \grave{\epsilon}_0) \lozenge (1 - \grave{\epsilon}_4) \leq (1 - \grave{\epsilon}_0) \lozenge (1 - \grave{\epsilon}_2) < s < \grave{\epsilon} \\ \Rightarrow \{n \in \mathbb{N} : \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) < \grave{\epsilon}\} \in \mathfrak{F}(I), \end{array}$$

Hence, the set

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), r) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I)$$
$$\Rightarrow \mathfrak{B}^I_\Upsilon(r - r_0, 1 - \grave{\epsilon}_3)(\mathcal{M}^r_n) \subset \mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r_n).$$

Remark 3.6 *The spaces $c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ and $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ are $\mathcal{NNS}$ in regard to $\mathcal{NN}$ $(\mathcal{J}, \mathcal{G}, \mathcal{H})$.*

Remark 3.7 *$\tau^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r) = \Big\{\mathfrak{A} \subset c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$: for each $\psi = (\psi_k) \in \mathfrak{A}$, a thing exists $r > 0$ and $\grave{\epsilon} \in (0,1)$ like that $\mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r_n) \subset \mathfrak{A}\Big\}$. Following that $\tau^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ establishes a topology in the space of sequences $\tau^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$. The group is cited by*

$$\mathfrak{B} = \Big\{\mathfrak{B}^I_\psi(r, \epsilon) : \psi \in c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r), r > 0 \, and \, \grave{\epsilon} \in (0,1)\Big\}$$

a foundation for the topology $\tau^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ on the space $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$.

Theorem 3.8 *The spaces $c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ and $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ are Hausdorff spaces.*

Proof:
Let $\varphi = (\varphi_k)$ and $\phi = (\phi_k) \in c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ such that $\varphi \neq \phi$. Then every $n \in \mathbb{N}$ and $r > 0$, suggests

$$\begin{array}{l} 0 < \mathcal{J}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r) < 1, \\ 0 < \mathcal{G}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r) < 1 \text{ and} \\ 0 < \mathcal{H}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r) < 1. \end{array}$$

Putting $\grave{\epsilon}_1 = \mathcal{J}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r), \grave{\epsilon}_2 = \mathcal{G}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r), \grave{\epsilon}_3 = \mathcal{H}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r)$ and $\grave{\epsilon} = \max\{\grave{\epsilon}_1, 1 - \grave{\epsilon}_2, 1 - \grave{\epsilon}_3\}$.

Then for every $\grave{\epsilon}_0 > \grave{\epsilon}$, there exist $\grave{\epsilon}_4, \grave{\epsilon}_5, \grave{\epsilon}_6 \in (0,1)$ like that $\grave{\epsilon}_4 * \grave{\epsilon}_4 \geq \grave{\epsilon}_0, (1-\grave{\epsilon}_5) \odot (1-\grave{\epsilon}_5) \leq (1-\grave{\epsilon}_0)$ and $(1-\grave{\epsilon}_6)\Diamond(1-\grave{\epsilon}_6) \leq (1-\grave{\epsilon}_0)$.
Assigning $\grave{\epsilon}_7 = \max\{\grave{\epsilon}_4, \grave{\epsilon}_5, \grave{\epsilon}_6\}$, consider the open balls $\mathfrak{B}^I_\varphi\left(1-\grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r_n)$ and $\mathfrak{B}^I_\phi\left(1-\grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r_n)$ centered at φ and ϕ respectively.
We demonstrate that

$$\mathfrak{B}^I_\varphi\left(1-\grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r_n) \cap \mathfrak{B}^I_\phi\left(1-\grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r_n) = \mathcal{J}.$$

Ideally,
$\psi = (\psi_k) \in \mathfrak{B}^I_\varphi\left(1-\grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r) \cap \mathfrak{B}^I_\phi\left(1-\grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r_n)$.
For the set, next $\{n \in \mathbb{N}\} \in \mathfrak{F}(I)$, we have

$$\begin{aligned}\grave{\epsilon}_1 &= \mathcal{J}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r)\\ &\geq \mathcal{J}\left(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\psi), \frac{r}{2}\right) * \mathcal{J}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), \frac{r}{2}\right) \quad (3.7)\\ &> \grave{\epsilon}_7 * \grave{\epsilon}_7 \geq \grave{\epsilon}_3 * \grave{\epsilon}_3 \geq \grave{\epsilon}_0 > \grave{\epsilon}_1\\ \grave{\epsilon}_2 &= \mathcal{G}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r)\\ &\leq \mathcal{G}\left(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\psi), \frac{r}{2}\right) \odot \mathcal{G}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), \frac{r}{2}\right) \quad (3.8)\\ &< \grave{\epsilon}_7 \odot \grave{\epsilon}_7 \leq \grave{\epsilon}_4 \odot \grave{\epsilon}_4 \leq \grave{\epsilon}_0 > \grave{\epsilon}_2\\ \grave{\epsilon}_3 &= \mathcal{H}(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\phi), r)\\ &\leq \mathcal{H}\left(\mathcal{M}^r_n(\varphi) - \mathcal{M}^r_n(\psi), \frac{r}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\phi), \frac{r}{2}\right) \quad (3.9)\\ &< \grave{\epsilon}_7 \Diamond \grave{\epsilon}_7 \leq \grave{\epsilon}_5 \Diamond \grave{\epsilon}_5 \leq \grave{\epsilon}_0 > \grave{\epsilon}_3\end{aligned}$$

Equation (3.7) brings about contradiction.
Therefore,
$\mathfrak{B}^I_\varphi\left(1-\grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r_n) \cap \mathfrak{B}^I_\phi\left(1-\grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r_n) = \mathcal{J}$.
Hence, the space $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ is a Hausdorff space.

4 Jordan Neutrosophic Ideal Convergence

Theorem 4.1 *If a sequence $\psi = (\psi_k) \in \xi$ is Jordan Neutrosophic Ideal Convergent ($\mathcal{JNIC}$) then the $I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r_n)$-limit is unique.*

Proof:
Assuming that the $\mathcal{JNIC}$ sequence $\psi = (\psi_k)$ has non-identical ideal limits l_1 and l_2. There exists a $\grave{\epsilon}_1 \in (0,1)$ with the given $\grave{\epsilon} \in (0,1)$ like that
$(1-\grave{\epsilon}_1) * (1-\grave{\epsilon}_1) > 1-\grave{\epsilon}, \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon}$ and $\grave{\epsilon}_1 \Diamond \grave{\epsilon}_1 < \grave{\epsilon}$.
Hence, the sets

$$\mathfrak{S} = \left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \leq 1-\grave{\epsilon}_1 \text{ or}\\ \mathcal{G}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1 \text{ and}\\ \mathcal{H}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1\end{array}\right\} \in I$$

$$\mathfrak{S}^c = \left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) > 1-\grave{\epsilon}_1 \text{ or}\\ \mathcal{G}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1 \text{ and}\\ \mathcal{H}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1\end{array}\right\} \in \mathfrak{F}(I)$$

$$\mathfrak{R} = \left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \leq 1-\grave{\epsilon}_1 \text{ or}\\ \mathcal{G}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1 \text{ and}\\ \mathcal{H}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1\end{array}\right\} \in I$$

$$\mathfrak{R}^c = \left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) > 1-\grave{\epsilon}_1 \text{ or}\\ \mathcal{G}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1 \text{ and}\\ \mathcal{H}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1\end{array}\right\} \in \mathfrak{F}(I)$$

Then $\mathfrak{S}^c \cap \mathfrak{R}^c \neq \mathcal{J}$. Taking $n \in \mathfrak{S}^c \cap \mathfrak{R}^c$, we have

$$\begin{aligned}&\mathcal{J}(l_1 - l_2, \dot{\varpi})\\ &\geq \mathcal{J}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) * \mathcal{J}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right)\\ &> (1-\grave{\epsilon}_1) * (1-\grave{\epsilon}_1) > (1-\grave{\epsilon}),\\ &\mathcal{G}(l_1 - l_2, \dot{\varpi})\\ &\leq \mathcal{G}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \odot \mathcal{G}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right)\\ &< \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon} \text{ and}\\ &\mathcal{H}(l_1 - l_2, \dot{\varpi})\\ &\leq \mathcal{H}\left(\mathcal{M}^r_n(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}^r_n(\psi) - l_2, \frac{\dot{\varpi}}{2}\right)\\ &< \grave{\epsilon}_1 \Diamond \grave{\epsilon}_1 < \grave{\epsilon}\end{aligned}$$

$\grave{\epsilon} \in (0,1)$ being arbitrary; $l_1 = l_2$. That is $I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r_n)$-limit is individual.

Theorem 4.2 *A sequence of $\psi = (\psi_k) \in \xi$ is $\mathcal{JNIC}$ in regard to $\mathcal{NN}(\mathcal{J},\mathcal{G},\mathcal{H})$ if it is Jordan Neutrosophic Ideal Cauchy ($\mathcal{JNIC}a$) in relation to the same norms.*

Proof:
Let $\psi = (\psi_k) \in \xi$ be $\mathcal{JNIC}$ in regard to $\mathcal{NN}(\mathcal{J},\mathcal{G},\mathcal{H})$ like that
$I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r_n) - \lim(\psi_k) = l$ and there exists $\grave{\epsilon}_1 \in (0,1)$ like that
$(1-\grave{\epsilon}_1) * (1-\grave{\epsilon}_1) > 1-\grave{\epsilon}, \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon}$ and $\grave{\epsilon}_1 \Diamond \grave{\epsilon}_1 < \grave{\epsilon}$ to a certain $\grave{\epsilon} \in (0,1)$.
Thus every $\dot{\varpi} > 0$,

$$\mathfrak{P} = \left\{\begin{array}{c}\mathcal{J}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) \leq 1-\grave{\epsilon}_1 \text{ or}\\ \mathcal{G}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) \geq \grave{\epsilon}_1 \text{ and}\\ \mathcal{H}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) \geq \grave{\epsilon}_1\end{array}\right\} \in I$$

$$\mathfrak{P}^c = \left\{\begin{array}{c}\mathcal{J}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) > 1-\grave{\epsilon}_1 \text{ or}\\ \mathcal{G}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) < \grave{\epsilon}_1 \text{ and}\\ \mathcal{H}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) < \grave{\epsilon}_1\end{array}\right\} \in \mathfrak{F}(I).$$

For $n \in \mathfrak{P}^c$,
$\mathcal{J}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) > 1-\grave{\epsilon}_1$,
$\mathcal{G}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) < \grave{\epsilon}_1$ and
$\mathcal{H}(\mathcal{M}^r_n(\psi) - l_1, \dot{\varpi}) < \grave{\epsilon}_1$.
For a certain $k \in \mathfrak{P}^c$, we may state

$$\mathbb{Q} = \left\{\begin{array}{c}\mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_k(\psi), \dot{\varpi}) \leq 1-\grave{\epsilon} \text{ or}\\ \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_k(\psi), \dot{\varpi}) \geq \grave{\epsilon} \text{ and}\\ \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_k(\psi), \dot{\varpi}) \geq \grave{\epsilon}\end{array}\right\}.$$

Let $n \in \mathbb{Q} \Rightarrow \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_k(\psi), \dot{\varpi}) \leq 1-\grave{\epsilon}$ or
$\mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_k(\psi), \dot{\varpi}) \geq \grave{\epsilon}$ and

$\mathscr{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon}$.
On the contrary, let $\mathscr{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) > 1 - \grave{\epsilon}$. Then

$$\begin{aligned} 1 - \grave{\epsilon} &\geq \mathscr{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\geq \mathscr{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) * \mathscr{J}\left(\mathcal{M}_k^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &> (1 - \grave{\epsilon}_1) * (1 - \grave{\epsilon}_1) > (1 - \grave{\epsilon}), \end{aligned}$$

this is incongruous.
Likewise, think about
$\mathscr{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon}, \mathscr{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon}$ such that
$\mathscr{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1$ and $\mathscr{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1$.
Contrarily, let $\mathscr{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1$,
$\mathscr{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1$. Hence

$$\begin{aligned} \epsilon &\leq \mathscr{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\leq \mathscr{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \odot \mathscr{G}\left(\mathcal{M}_k^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon}, \\ \epsilon &\leq \mathscr{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\leq \mathscr{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \lozenge \mathscr{H}\left(\mathcal{M}_k^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \lozenge \grave{\epsilon}_1 < \grave{\epsilon}, \end{aligned}$$

it also contradicts itself.
Therefore, $n \in \mathbb{Q}$, we have
$\mathscr{J}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) \leq 1 - \grave{\epsilon}_1, \mathscr{G}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) \geq \grave{\epsilon}_1$ and $\mathscr{H}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) \geq \grave{\epsilon}_1$, may suggest $n \in \mathfrak{P}$.
Therefore, $\mathbb{Q} \subset \mathfrak{P}$ and $\mathbb{Q} \in I$.
As a result, the sequence $\psi = (\psi_k)$ is $\mathscr{JNIC}a$ in regard to norms $(\mathscr{J}, \mathscr{G}, \mathscr{H})$.
Conversely, let $\psi = (\psi_k)$ be $\mathscr{JNIC}a$ in regard to the norms $(\mathscr{J}, \mathscr{G}, \mathscr{H})$ and not $\mathscr{JNIC}$. Consequently, there is $k \in \mathbb{N}$ like that

$$\mathfrak{A} = \left\{ \begin{array}{c} \mathscr{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \leq 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon} \end{array} \right\} \in I$$

and

$$\mathfrak{B} = \left\{ \begin{array}{c} \mathscr{J}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon} \end{array} \right\} \in I$$

$$\begin{aligned} \Rightarrow 1 - \grave{\epsilon} &\geq \mathscr{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\geq \mathscr{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) * \mathscr{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &> (1 - \grave{\epsilon}_1) * (1 - \grave{\epsilon}_1) > 1 - \grave{\epsilon} \text{ and} \\ \grave{\epsilon} &\leq \mathscr{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\leq \mathscr{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \odot \mathscr{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon}, \end{aligned}$$

Simulteneously,

$$\begin{aligned} \grave{\epsilon} &\leq \mathscr{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\leq \mathscr{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \lozenge \mathscr{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \lozenge \grave{\epsilon}_1 < \grave{\epsilon}. \end{aligned}$$

This resulted in conflict.
Therefore, $\mathfrak{B} \in \mathfrak{F}(I)$ and hence, $\psi = (\psi_k)$ is $\mathscr{JNIC}$.

Theorem 4.3 *Consider $\mathcal{NNS}$ $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ and $\tau^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ be the topology on $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$. Let $(\psi_j) = (\psi_k^j)_{j=1}^{\infty}$ be a sequence in $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$. The sequence $\psi_j \to \psi$ as $j \to \infty$ if and only if $\mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 1, \mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 0$ and $\mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 0$ as $n \to \infty$.*

Proof:
Let $\psi_j \to \psi$ as j tends to ∞. Fix a specific $r > 0$ and $\grave{\epsilon} \in (0, 1)$, there exists the natural number $n \in \mathbb{N}$ like that $(\psi_j) \in \mathfrak{B}^I_{\psi}(r, \grave{\epsilon})(\mathcal{M}_n^r)$ every $j \geq k$.
Then,

$$\mathfrak{S} = \left\{ \begin{array}{c} \mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \leq 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \geq \grave{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \geq \grave{\epsilon} \end{array} \right\} \in I,$$

or equivalently,

$$\mathfrak{S}^c = \left\{ \begin{array}{c} \mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I)$$

For $\{n \in \mathbb{N}\} \subseteq \mathfrak{S}^c$,
$\mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) > 1 - \grave{\epsilon}$
$\Rightarrow 1 - \mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon}$,
$\mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon}$ and
$\mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon}$.
Therefore, for $n \to \infty$,
$1 - \mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$,
$\mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$ and
$\mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$.
This implies that
$\mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 1$,
$\mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$ and
$\mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$ as $n \to \infty$.
On the other hand, imagine for every $\dot{\varpi} > 0$,
$\mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 1$,
$\mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 0$ and
$\mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 0$ as n tends to ∞.
Then every $\grave{\epsilon} \in (0, 1)$, a thing exists the natural number $k \in \mathbb{N}$ like that
$1 - \mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$,
$\mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$ and
$\mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$ for every $n \geq k$
$\Rightarrow \mathscr{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) > 1 - \grave{\epsilon}$,
$\mathscr{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$ and
$\mathscr{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$ for each $n \geq k$.

Think about the ideal I produced by the set $\{n \in \mathbb{N} : n < k\}$, it implies that the set's family of sets contains $\{n \in \mathbb{N} : n \geq k\}$ which relates to $\mathfrak{F}(I)$. Therefore,

$$\left\{\begin{array}{c}\mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}\end{array}\right\} \in \mathfrak{F}(I)$$

$$\Rightarrow (\psi_j) \in \mathfrak{B}^I_{\psi_k}(r, \grave{\epsilon})(\mathcal{M}_n^r), \text{ for every } n \geq k.$$

Thus, $\psi_j \to \psi \;\text{ as }\; j \to \infty$.

Theorem 4.4 *Let $\psi = (\psi_k) \in c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$. Then for anyone $l \in \mathscr{C}, \psi_k \xrightarrow{I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)} l$ if for every $\grave{\epsilon} \in (0,1)$ and $\dot{\varpi} > 0$, positive integers exist $\mathcal{N} = \mathcal{N}(\psi, \grave{\epsilon}, \dot{\varpi})$ like that*

$$\left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}\end{array}\right\} \in \mathfrak{F}(I).$$

Proof:
Suppose $\psi_k \xrightarrow{I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)} l$ for anyone $l \in \mathscr{C}$.
For given that $\grave{\epsilon} \in (0,1)$, there exists a decimal $r \in (0,1)$ like that $(1-\grave{\epsilon}) * (1-\grave{\epsilon}) > 1 - r, \grave{\epsilon} \odot \grave{\epsilon} < r$ with $\grave{\epsilon} \Diamond \grave{\epsilon} < r$.
Since $\psi_k \xrightarrow{I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)} l$, for every $\dot{\varpi} > 0$,

$$\Delta = \left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \leq 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}\end{array}\right\} \in I;$$

which implies that

$$\Delta^c = \left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}\end{array}\right\} \in \mathfrak{F}(I).$$

On the other hand, let us pick $\mathcal{N} \in \Delta^c$.
Then

$$\left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}\end{array}\right\}.$$

We demonstrate that a non negative integer exists $\mathcal{N} = \mathcal{N}(\psi, \epsilon, \dot{\varpi})$ like that

$$\mathfrak{P} = \left\{\begin{array}{c}\mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \leq 1 - r \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \geq r \\ \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \geq r\end{array}\right\} \in I.$$

We'll demonstrate that $\mathfrak{P} \subseteq \Delta$. Contrarily, let $\mathfrak{P} \not\subseteq \Delta$, that is, there exists $n \in \mathfrak{P}$ like that n not in Δ.
Then $\mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \leq 1 - r$ and $\mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon}$.
Particularly, $\mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon}$.
Therefore, we have

$$\begin{aligned}1 - r &\geq \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) * \mathcal{J}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &\geq (1-\grave{\epsilon}) * (1-\grave{\epsilon}) > 1 - r\end{aligned}$$

This is incongruous. Also,

$$\mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \geq r \text{ and } \mathcal{G}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon}.$$

Particularly, $\mathcal{G}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon}$.
Therefore,

$$\begin{aligned}r &\leq \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \\ &\leq \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \odot \mathcal{G}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &\leq \grave{\epsilon} \odot \grave{\epsilon} < r,\end{aligned}$$

which again is in conflict. Similarly,

$$\mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \geq r \text{ and } \mathcal{H}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon}.$$

Particularly, $\mathcal{H}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon}$.
Therefore,

$$\begin{aligned}r &\leq \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \\ &\leq \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &\leq \grave{\epsilon} \Diamond \grave{\epsilon} < r,\end{aligned}$$

Therefore, $\mathfrak{P} \subseteq \Delta$ and since $\Delta \in I \Longrightarrow \mathfrak{P} \in I$.

Definition 4.5 *The void set $\mathfrak{S} \subset c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ is compact if each open cover of $\mathfrak{S}$ specified by the open set of $\tau^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ has a non infinite subcover.*

Theorem 4.6 *Each non infinite subset $\mathfrak{S}$ of $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ is compact.*

Proof:
Let $\mathfrak{S} = \{\psi_1, \psi_2, \psi_3, \ldots, \psi_n\}$ be the finite subset of $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$. For $r > 0$ and $0 < \grave{\epsilon} < 1$.
Let $\left\{\mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}_n^r) : \psi \in \mathfrak{S}\right\}$ be an open cover of $\mathfrak{S}$.
Following that $\mathfrak{S} \subseteq \bigcup_{\psi \in \mathfrak{S}} \mathfrak{B}^I_\psi(r, \epsilon)(\mathcal{M}_n^r)$.
Now for each $\psi_i \in \mathfrak{S}, i = 1, 2, 3, \ldots n$, we have $\psi_i \in \bigcup_{\psi_i \in s} \mathfrak{B}^I_{\psi_i}(r, \grave{\epsilon})(\mathcal{M}_n^r)$.
That suggests $\psi_i \in \mathfrak{B}^I_{\psi_j}(r, \grave{\epsilon})(\mathcal{M}_n^r)$ for anyone $j \in \{1, 2, 3, \ldots, n\}$.
Following that $\left\{\mathfrak{B}^I_{\psi_i}(r, \grave{\epsilon})(\mathcal{M}_n^r) : i = 1, 2, 3, \ldots, n\right\}$ is a non infinite subcover of $\mathfrak{S}$.

Theorem 4.7 *The set $\mathfrak{S} \subseteq c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ is compact if each sequence in $\mathfrak{S}$ has a convergent subsequence.*

Proof:
A compact subset of $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ would be $\mathfrak{S}$, assuming that let $(\psi_k^j) = (\psi_j)_{j=1}^\infty$ be a sequence in $\mathfrak{S}$.
Given $0 < \grave{\epsilon} < 1$ and $r > 0$,
let $\left\{\mathfrak{B}^I_\psi\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r) : \psi = (\psi_k) \in \mathfrak{S}\right\}$ be an open cover of $\mathfrak{S}$.
This suggests, $(\psi_j) \in \bigcup_{\psi \in \mathfrak{S}} \left\{\mathfrak{B}^I_\psi\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r)\right\}$.
Next, there are some $\psi = (\psi_k) \in \mathfrak{S}$ such that

$(\psi_j) \in \mathfrak{B}_{\psi}^{I}\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r)$.
Therefore, the set

$$\Delta = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) < \grave{\epsilon} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

A finite subcover exists $\left\{\mathfrak{B}_{\psi_i}^{I}\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r) : \psi_i \in \mathfrak{S} \text{ and } i = 1, 2, 3, \ldots, m\right\}$ since $\mathfrak{S}$ is compact like that $\mathfrak{S} \subseteq \bigcup_{i=1}^{m} \mathfrak{B}_{\psi_i}^{I}\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r)$.
Let (ψ^{j_p}) be a subsequence of (ψ_j).
Then $(\psi^{j_p}) \in \bigcup_{i=1}^{m} \mathfrak{B}_{\psi_i}^{I}\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r)$, implies $(\psi^{j_p}) \in \mathfrak{B}_{\psi_i}^{I}\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r)$, for some $\psi_i \in \mathfrak{S}$.
Therefore, the set

$$\nabla = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

For $n \in \Delta \cap \nabla$,

$$\begin{aligned} &\mathcal{J}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) * \mathcal{J}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \\ &* \mathcal{J}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) \\ &> (1-\grave{\epsilon}) * (1-\grave{\epsilon}) * (1-\grave{\epsilon}) = (1-\grave{\epsilon}). \end{aligned}$$

Also

$$\begin{aligned} &\mathcal{G}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r) \\ &\leq \mathcal{G}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \odot \mathcal{G}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \\ &\odot \mathcal{G}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) \\ &< \grave{\epsilon} \odot \grave{\epsilon} \odot \grave{\epsilon} = \grave{\epsilon}. \end{aligned}$$

Simultaneously,

$$\begin{aligned} &\mathcal{H}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r) \\ &\leq \mathcal{H}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \Diamond \mathcal{H}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \\ &\Diamond \mathcal{H}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) \\ &< \grave{\epsilon} \Diamond \grave{\epsilon} \Diamond \grave{\epsilon} = \grave{\epsilon}. \end{aligned}$$

Take $\grave{\epsilon} = \frac{1}{n}$. Then

$$\begin{aligned} &\lim_{n\to\infty} \mathcal{J}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r\right) = \lim_{n\to\infty} 1 - \frac{1}{n} = 1, \\ &\lim_{n\to\infty} \mathcal{G}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r) = \lim_{n\to\infty} \frac{1}{n} = 0 \text{ and} \\ &\lim_{n\to\infty} \mathcal{H}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r) = \lim_{n\to\infty} \frac{1}{n} = 0. \end{aligned}$$

Thus by theorem (4.3), $\psi^{j_p} \to \psi$, as $p \to \infty$.
In contrast, imagine (ψ^{j_p}) be the subsequence of a sequence (ψ_j) in $\mathfrak{S}$ like that $(\psi^{j_p}) \to \psi$ in $\mathfrak{S}$.
Let $\mathfrak{S}$ not be a compact subset of $c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^{I}(\mathcal{M}^r)$. On the other hand, let $\left\{\mathfrak{B}_{\psi}^{I}\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r)\right\}$ be an open cover of $\mathfrak{S} \Longrightarrow \mathfrak{S} \subseteq \bigcup_{\psi\in\mathfrak{S}} \mathfrak{B}_{\psi}^{I}\left(\frac{r}{3}, \grave{\epsilon}\right)(\mathcal{M}_n^r)$.
Therefore, the set

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

Due to $\mathfrak{S}$ not being compact, there exists a non infinite subcover $\left\{\mathfrak{B}_{\psi_i}^{I}(r, \grave{\epsilon})(\mathcal{M}_n^r) : \psi_i \in \mathfrak{S}, i = 1, 2, 3 \ldots, m\right\}$ like that $\mathfrak{S} \nsubseteq \bigcup_{\psi_i\in\mathfrak{S}} \mathfrak{B}_{\psi_i}^{I}(r, \grave{\epsilon})(\mathcal{M}_n^r)$, it suggests the set

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), r) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), r) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), r) < \grave{\epsilon} \end{array} \right\} \notin \mathfrak{F}(I)$$

$\Rightarrow$ for anyone $\grave{\epsilon} \in (0, 1)$ and a positive r, $(\psi^{j_p}) \notin \mathfrak{B}_{\psi}^{I}(r, \grave{\epsilon})$.
Hence, $(\psi^{j_p}) \nrightarrow z$. This is incongruous.
Thus, $\mathfrak{S}$ is compact.

Theorem 4.8 *Consider the $\mathcal{NNS}$ $c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^{I}(\mathcal{M}^r)$. Choose a positive r and $0 < \grave{\epsilon}, \grave{\epsilon}' < 1$ like that $(1-\epsilon') \leq (1-\grave{\epsilon}) * (1-\grave{\epsilon})$, $\grave{\epsilon} \odot \grave{\epsilon} \leq \grave{\epsilon}'$ and $\grave{\epsilon} \Diamond \grave{\epsilon} \leq \grave{\epsilon}'$. Then for some one $\psi = (\psi_k) \in c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^{I}(\mathcal{M}^r)$, $\overline{\mathfrak{B}_{\psi}^{I}\left(\frac{r}{2}, \grave{\epsilon}\right)(\mathcal{M}_n^r)} \subseteq \mathfrak{B}_{\psi}^{I}(r, \grave{\epsilon}')(\mathcal{M}_n^r)$.*

Proof:
Let $q = (q_k) \in \mathfrak{B}_{\psi}^{I}\left(\frac{r}{2}, \grave{\epsilon}\right)(\mathcal{M}_n^r)$ and $\mathfrak{B}_{q}^{I}\left(\frac{r}{2}, \grave{\epsilon}\right)(\mathcal{M}_n^r)$ be an open ball which has centre at q and has radius $\grave{\epsilon}$.
Thus, $\mathfrak{B}_{\psi}^{I}\left(\frac{r}{2}, \grave{\epsilon}\right)(\mathcal{M}_n^r) \cap \mathfrak{B}_{q}^{I}\left(\frac{r}{2}, \grave{\epsilon}\right)(\mathcal{M}_n^r) \neq \mathcal{J}$.
Suppose $\varphi = (\varphi_k) \in \mathfrak{B}_{q}^{I}\left(\frac{r}{2}, \grave{\epsilon}\right)(\mathcal{M}_n^r) \cap \mathfrak{B}_{\psi}^{I}\left(\frac{r}{2}, \grave{\epsilon}\right)(\mathcal{M}_n^r)$.
The sets follow

$$\Delta = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) > 1 - \grave{\epsilon}_1 \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) < \grave{\epsilon}_1 \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) < \grave{\epsilon}_1 \end{array} \right\} \in \mathfrak{F}(I),$$

$$\nabla = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) < \grave{\epsilon}_1 \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) < \grave{\epsilon}_1 \end{array} \right\} \in \mathfrak{F}(I).$$

Consider $n \in \Delta \cap \nabla$. Then

$$\begin{aligned} &\mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(q), r) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) * \mathcal{J}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) \\ &> (1-\grave{\epsilon}) * (1-\grave{\epsilon}) \geq (1-\grave{\epsilon}'), \\ &\mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(q), r) \\ &\leq \mathcal{G}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) \odot \mathcal{G}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) \\ &< \grave{\epsilon} \odot \grave{\epsilon} \leq \grave{\epsilon}' \text{ and} \\ &\mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(q), r) \\ &\leq \mathcal{H}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) \\ &< \grave{\epsilon} \Diamond \grave{\epsilon} \leq \grave{\epsilon}' \end{aligned}$$

Therefore, the set

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(q), r) > 1 - \grave{\epsilon}' \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(q), r) < \grave{\epsilon}' \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(q), r) < \grave{\epsilon}' \end{array} \right\} \in \mathfrak{F}(I)$$

$$\Rightarrow q = (q_k) \in \mathfrak{B}_{\psi}^{I}(r, \grave{\epsilon}')(\mathcal{M}_n^r).$$

Thus, $\overline{\mathfrak{B}^I_\psi\left(\frac{r}{2}, \grave{\epsilon}\right)(\mathcal{M}^r_n)} \subseteq \mathfrak{B}^I_\psi\left(\frac{r}{2}, \grave{\epsilon}'\right)(\mathcal{M}^r_n)$.

Theorem 4.9 *Let $\psi = (\psi_k) \in \xi$. If a sequence exists, $\psi' = (\psi_k^{'}) \in c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ such that $\mathcal{M}^r_n(\psi) = \mathcal{M}^r_n(\psi')$ for every n relative to I, then $\psi \in c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$.*

Proof:
Suppose $\mathcal{M}^r_n(\psi) = \mathcal{M}^r_n(\psi')$ for every n relative to I.
Following that $\{n \in \mathbb{N} : \mathcal{M}^r_n(\psi) \neq \mathcal{M}^r_n(\psi')\} \in I$.
This implies $\{n \in \mathbb{N} : \mathcal{M}^r_n(\psi) = \mathcal{M}^r_n(\psi')\} \in \mathfrak{F}(I)$.
Whereas, because $n \in \mathfrak{F}(I)$ for every $\dot{\varpi} > 0$,
$\mathcal{J}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\psi'), \frac{\dot{\varpi}}{2}\right) = 1$,
$\mathcal{G}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\psi'), \frac{\dot{\varpi}}{2}\right) = 0$ and
$\mathcal{H}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\psi'), \frac{\dot{\varpi}}{2}\right) = 0$.
Since $(\psi_k^{'}) \in c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$,
let $I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r) - \lim(\psi_k^{'}) = l$.
Then, each $\grave{\epsilon} \in (0,1)$ and $\dot{\varpi} > 0$,

$$\Delta = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}^r_n(\psi') - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}^r_n(\psi') - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}^r_n(\psi') - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

Think about the set

$$\nabla = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}^r_n(\psi) - l, \dot{\varpi}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}^r_n(\psi) - l, \dot{\varpi}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}^r_n(\psi) - l, \dot{\varpi}\right) < \grave{\epsilon} \end{array} \right\}.$$

We show that $\Delta \subset \nabla$. So for $n \in \Delta$, we have

$$\begin{aligned}
&\mathcal{J}(\mathcal{M}^r_n(\psi) - l, \dot{\varpi}) \\
&\geq \mathcal{J}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\psi'), \frac{\dot{\varpi}}{2}\right) * \mathcal{J}\left(\mathcal{M}^r_n(\psi') - l, \frac{\dot{\varpi}}{2}\right) \\
&> 1 * (1 - \grave{\epsilon}) = 1 - \grave{\epsilon}, \\
&\mathcal{G}(\mathcal{M}^r_n(\psi) - l, \dot{\varpi}) \\
&\leq \mathcal{G}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\psi'), \frac{\dot{\varpi}}{2}\right) \odot \mathcal{G}\left(\mathcal{M}^r_n(\psi') - l, \frac{\dot{\varpi}}{2}\right) \\
&< 0 \odot \grave{\epsilon} = \grave{\epsilon} \text{ and} \\
&\mathcal{H}(\mathcal{M}^r_n(\psi) - l, \dot{\varpi}) \\
&\leq \mathcal{H}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\psi'), \frac{\dot{\varpi}}{2}\right) \lozenge \mathcal{H}\left(\mathcal{M}^r_n(\psi') - l, \frac{\dot{\varpi}}{2}\right) \\
&< 0 \lozenge \grave{\epsilon} = \grave{\epsilon}
\end{aligned}$$

This suggests that $n \in \nabla$ and thus, $\Delta \subset \nabla$.
Hence $\nabla \in \mathfrak{F}(I)$ because $\Delta \in \mathfrak{F}(I)$.
Thus, $\psi = (\psi_k) \in c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$.

Theorem 4.10 *The closed ball $\mathfrak{B}^I_\psi[r, \grave{\epsilon}](\mathcal{M}^r)$ is a closed set in $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$.*

Proof:
Let $\Upsilon = (\Upsilon_k) \in \xi$ be like that $\Upsilon \in \mathfrak{B}^I_\psi[r, \grave{\epsilon}](\mathcal{M}^r)$.
Consequently, a sequence exists
$(\Upsilon^j) = (\Upsilon^j_k) \in \mathfrak{B}^I_\psi[r, \grave{\epsilon}](\mathcal{M}^r)$ like that Υ^j converges to Υ when $j \to \infty$. Thus

$$\Delta = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\psi), r) \geq 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\psi), r) \leq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\psi), r) \leq \grave{\epsilon} \end{array} \right\}.$$

Since $\Upsilon^j \to \psi$ as $j \to \infty$, by Theorem (4.3),
$\mathcal{J}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\Upsilon), r) \to 1, \mathcal{G}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\Upsilon), r) \to 0$ and $\mathcal{H}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\Upsilon), r) \to 0$
for every $\dot{\varpi} > 0$ as $n \to \infty$.
Thus, $n \in \Delta$,

$$\begin{aligned}
&\mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), \dot{\varpi} + r) \\
&\geq \lim_{n\to\infty} \mathcal{J}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\Upsilon), \dot{\varpi}) * \mathcal{J}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\psi), r) \\
&\geq 1 * (1 - \grave{\epsilon}) = 1 - \grave{\epsilon}, \\
&\mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), \dot{\varpi} + r) \\
&\leq \lim_{n\to\infty} \mathcal{G}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\Upsilon), \dot{\varpi}) \odot \mathcal{G}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\psi), r) \\
&\leq 0 \odot \grave{\epsilon} = \grave{\epsilon} \text{ and} \\
&\mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), \dot{\varpi} + r) \\
&\leq \lim_{n\to\infty} \mathcal{H}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\Upsilon), \dot{\varpi}) \lozenge \mathcal{H}(\mathcal{M}^r_n(\Upsilon^j) - \mathcal{M}^r_n(\psi), r) \\
&\leq 0 \lozenge \grave{\epsilon} = \grave{\epsilon}.
\end{aligned}$$

A particular $k \in \mathbb{N}$, take $\dot{\varpi} = \frac{1}{k}$. Then

$$\begin{aligned}
&\mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \\
&\quad = \lim_{k\to\infty} \mathcal{J}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r + \frac{1}{k}\right) \geq 1 - \grave{\epsilon}, \\
&\mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \\
&\quad = \lim_{k\to\infty} \mathcal{G}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r + \frac{1}{k}\right) \leq \grave{\epsilon} \text{ and} \\
&\mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \\
&\quad = \lim_{k\to\infty} \mathcal{H}\left(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r + \frac{1}{k}\right) \leq \grave{\epsilon}
\end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \geq 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \leq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}^r_n(\psi) - \mathcal{M}^r_n(\Upsilon), r) \leq \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I)$$

$\Rightarrow \Upsilon \in \mathfrak{B}^I_\psi[r, \grave{\epsilon}](\mathcal{M}^r)$. Therefore, $\mathfrak{B}^I_\psi[r, \grave{\epsilon}](\mathcal{M}^r)$ is a closed set.

Theorem 4.11 *Consider the compact subset $\mathfrak{S}$ of $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ like that $\psi = (\psi_k) \notin \mathfrak{S}$. Next, there are two open sets $\mathfrak{J}, \mathfrak{K}$ in $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ like that $\mathfrak{S} \subseteq \mathfrak{K}, \psi \in \mathfrak{J}$ and $\mathfrak{J} \cap \mathfrak{K} = \emptyset$.*

Proof:
Consider a compact subset $\mathfrak{S}$ of $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ and $\psi \notin \mathfrak{S}$.
Next, for some one $s \in \mathfrak{S}$ we have $\psi \neq s$.
$c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ being a Hausdorff space, then for any one positive r and $\grave{\epsilon} \in (0,1)$ two open balls are available $\mathfrak{J} = \mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r)$ and $\mathfrak{K} = \mathfrak{B}^I_s(r, \grave{\epsilon})(\mathcal{M}^r)$ like that $\mathfrak{J} \cap \mathfrak{K} = \mathcal{J}$.
Think of the open cover $\mathfrak{K}_s = \left\{\mathfrak{B}^I_s(r, \grave{\epsilon})(\mathcal{M}^r) : s \in \mathfrak{S}\right\}$ of $\mathfrak{S}$ and $\mathfrak{S}$ is compact,
consequently, a non infinite subcover exists
$\mathfrak{S}_{s_i} = \left\{\mathfrak{B}^I_{s_i}(r, \grave{\epsilon})(\mathcal{M}^r) : s_i \in \mathfrak{S} \text{ and } i = 1, 2, 3, \ldots, j\right\}$ like that $\mathfrak{S} \subseteq \bigcup_{i=1}^j \mathfrak{K}_{s_i}$.
Taking $\mathfrak{K} = \bigcap_{i=1}^j \mathfrak{K}_{s_i}$ we have $\psi \notin \mathfrak{S}$.
Hence, $\mathfrak{J}, \mathfrak{K}$ are open sets such that $\mathfrak{S} \subseteq \mathfrak{K}$ and $\mathfrak{J} \cap \mathfrak{K} = \mathcal{J}$.

5 Conclusions

The article examines the convergence of the sequences created by running the regular Jordan totient operator through a set of finite subsets of $\mathbb{N}$ in the setting of $\mathcal{N}\mathcal{N}$. In order to reach a finite limit, it then applies the idea of a regular matrix to an initially non-convergent sequence. We design unique sequence spaces $c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$, $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$, $\ell^I_{\infty(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ and $\ell_{\infty(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ and research their connections. Future research objectives might include the creation and study of function spaces employing a generalized infinite operator.

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